Föreläsning 10/9-13

Triangelolikheten för integraler:

$g : [a,b] \to \mathbb{C}$

Sats: $$\left| \int_a^b g(t)\,dt \right| \leq \int_a^b |g|\,dt$$

jfr: $|z_1 + \ldots + z_n| \leq |z_1| + \ldots + |z_n|$

Bevis: Säg $\int_a^b g(t)\,dt = w = re^{i\theta}$

$$r = \left| \int_a^b g(t)\,dt \right| = e^{-i\theta} \int_a^b g(t)\,dt = \int_a^b \underset{\uparrow \text{ Konstant}}{e^{-i\theta}} g(t)\,dt =$$

$$= \operatorname{Re} \int_a^b e^{-i\theta} g(t)\,dt = \int_a^b \operatorname{Re}(e^{-i\theta} g)\,dt \leq$$

$$\leq \int_a^b |e^{-i\theta} g|\,dt = \int_a^b |g|\,dt \qquad \blacksquare$$

Komplexa derivator

$f : \mathbb{C} \to \mathbb{C}$ $\qquad z_0 \in \mathbb{C}$

f är komplext deriverbar i z_0 om

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ existerar}$$

I så fall $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Observation: vanliga räkneregler gäller.

$(f+g)' = f' + g'$

$(af)' = af'$ (a konstant)

$(fg)' = fg' + gf'$

$\dfrac{d f(g(z))}{dz} = f'(g(z)) \cdot g'(z)$ (kedjeregeln)

ex $f(z) = z$

$\lim\limits_{z \to z_0} \dfrac{z - z_0}{z - z_0} = 1$ $f'(z) = 1$

$z^2, z^3, \dots, z^n$ OK

$\therefore$ $p(z) = a_0 + a_1 z + \dots + a_n z^n$ OK

p/q OK

ex $f(z) = \bar{z} = x - iy$ ej deriverbar

Betrakta

$\lim\limits_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = \lim\limits_{h \to 0} \dfrac{\bar{z}_0 + \bar{h} - \bar{z}_0}{h} = \lim\limits_{h \to 0} \dfrac{\bar{h}}{h}$

Men $\bar{h}/h = 1$ om $h \in \mathbb{R}$

medan $\bar{h}/h = -1$ om $h \in i\mathbb{R}$

$\therefore$ gränsvärdet existerar ej.

ex $f(z) = e^z$ är deriverbar överallt

Bevis: $\lim\limits_{h \to 0} \dfrac{e^{z_0+h} - e^{z_0}}{h} = \lim\limits_{h \to 0} e^{z_0} \dfrac{e^h - 1}{h}$

Räcker att visa att $\lim \dfrac{e^h - 1}{h} = 1$

$$\frac{e^h-1}{h}-1=\frac{e^h-1-h}{h}$$

$e^h-(1+h)=\{h=a+ib\}=e^a(\cos b+i\sin b)-(1+a+ib)=$
$=e^a\cos b-(1+a)+i[e^a\sin b-b]$

Men

$e^a\cos b-(1+a)=[e^a-(1+a)]+e^a(\cos b-1)=$
$=O(a^2)+O(b^2)\leq c\cdot|h|^2$

$\therefore \left|\frac{e^a\cos b-(1+a)}{h}\right|\longrightarrow 0$

Im-delen likuande, kolla själva!

Sats

Antag att $f=u+iv$ är deriverbar i z_0. Då uppfyller f Cauchy-Riemanns diff.-ekv.

$$\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=0$$

dvs $u_x+iv_x-v_y+iu_y=0$

$$\therefore \begin{cases} u_x=v_y \\ u_y=-v_x \end{cases}$$

ex $f=z$, $u=x$, $v=y$

$$\begin{cases} u_x=1=v_y \\ u_y=0=-v_x \end{cases}$$

$f(z)=\bar{z}$, $u=x$, $v=-y$

$u_x=1\neq v_y$

Bevis: $f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$

$z_0 = x_0 + iy_0$

① $h \in \mathbb{R}$

$$f'(z_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} =$$

$$= \frac{\partial f}{\partial x}(x_0, y_0)$$

② $h \in i\mathbb{R}$, $h = ik$

$$f'(z_0) = \lim_{k \to 0} \frac{f(z_0+ik) - f(z_0)}{ik} =$$

$$= \lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y}$$

$$\therefore \frac{\partial f}{\partial x}(z_0) = \frac{1}{i} \frac{\partial f}{\partial y}(z_0) \Longrightarrow \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

∎

Sats 2

Låt $f = u + iv$ ha kontinueriga partiella derivator, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ nära z_0.

Antag också att $\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(z_0) = 0$

Då är f komplext deriverbar i z_0

ex $f(z)=e^z=e^x(\cos y+i\sin y)$

$\frac{\partial f}{\partial x}=f \quad , \quad \frac{\partial f}{\partial y}=e^x(-\sin y+i\cos y)=if$

$\therefore \frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=f-f=0$

$\therefore e^z$ är deriverbar $\blacksquare$

Bevis av sats

$h=a+ib$

$f(z_0+h)-f(z_0)=f(x_0+a,y_0+b)-f(x_0,y_0)=$

$=a\frac{\partial f}{\partial x_0}(x_0,y_0)+b\frac{\partial f}{\partial y_0}(x_0,y_0)+o\left(\sqrt{a^2+b^2}\right)=$ ← lilla ordo

$=\left\{\begin{matrix}\text{Cauchy-} \\ \text{Riemann}\end{matrix} \quad \frac{\partial f}{\partial y}=i\frac{\partial f}{\partial x}\right\}=$

$=\frac{\partial f}{\partial x}(x_0,y_0)(a+ib)+o(|h|)$

$\therefore \frac{f(z_0+h)-f(z_0)}{h}=\frac{\partial f}{\partial x}(z_0)+\frac{o(|h|)}{h}$

$\therefore \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=\frac{\partial f}{\partial x}(z_0)$ $\blacksquare$

Harmoniska funktioner

$u(x,y)$ harmonisk om

$\Delta u=0$, dvs $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Definition

f är analytisk (holomorf) i Ω om $f'(z)$ existerar $\forall z$.

Proposition

Antag $f = u + iv$ holomorf
Då är u, v harmoniska.

Bevis

f holomorf $\Longrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$u_{xx} = v_{yx} \quad u_{yy} = -v_{xy}$
$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$ ∎

Definition

om u, v harmoniska och
$u + iv$ holomorf så är u och v
konjugerade harmoniska funktioner.

Proposition

u harmonisk i Ω.
Då finns v harmonisk så att
$f = u + iv$ holomorf

(sant om $\Omega =$  ; $\Omega \neq$ )

Bevis

u given, vill hitta v som löser

$\begin{cases} v_y = u_x \\ v_x = -u_y \end{cases}$ När kan man lösa $\begin{cases} v_y = g \\ v_x = h \end{cases}$?

Svar: när $g_x = h_y$; $v = \int h\,dx + g\,dy$

I vårt fall $g = u_x$, $h = -u_y$

$g_x = h_y \iff u_{xx} = -u_{yy} \iff u_{xx} + u_{yy} = 0$

$\iff u$ harmonisk

ex $u = x^2 - y^2$, $u_{xx} = 2$, $u_{yy} = -2$

$u_{xx} + u_{yy} = 0$

Hitta v så att $u + iv = f$ holomorf,

dvs $\begin{cases} v_y = u_x = 2x \\ v_x = -u_y = 2y \end{cases}$

$v_y = 2x \implies v = 2xy + w(x)$

$v_x = 2y + w'(x) = 2y$

Tag $w = 0$

$\therefore v = 2xy$

$f = u + iv = x^2 - y^2 + 2ixy = z^2$

ex $f(z) = \log z = \log|z| + i\arg(z)$

i termer av (x, y):

$$\left.\begin{array}{l} u = \log|z| = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2) \\ v = \arg z = \arctan(y/x) \end{array}\right\} \text{konj. harm.}$$

Koll: $u_x = \dfrac{x}{x^2+y^2}$, $u_y = \dfrac{y}{x^2+y^2}$

$$v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = -u_y$$

$$v_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = u_x$$