KOMPLEX F

MATEMATISK

ANALYS

Fö 2001 SIDOR: 53 PRIS: 25 kr

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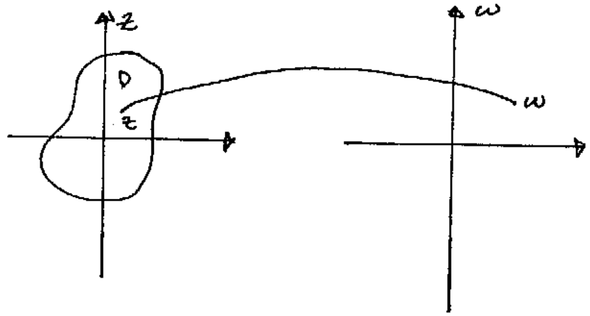
Vad handlar komplex-kursen om?



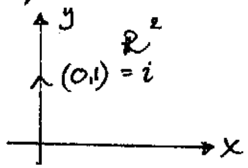
$w = f(z)$

deriverbar

$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ $\exists$ och är ett komplext tal.

"teori"-kurs. Nära besläktad med Fourier-analysen

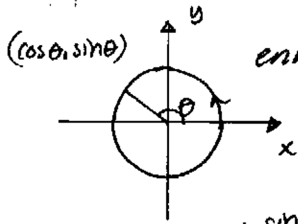
Komplexa tal



$z = (x, y)$
$w = (s, t)$
$z + w = (x, y) + (s, t) = (x+s, y+t)$
$zw = (x, y) \cdot (s, t) \overset{def}{=} (xs - yt, xt + ys)$
$i^2 = ii = (0,1)(0,1) = (-1, 0)$
skriv $(x, 0) = x$
$i^2 = -1$ $\quad z = i$ löser ekv. $z^2 + 1 = 0$

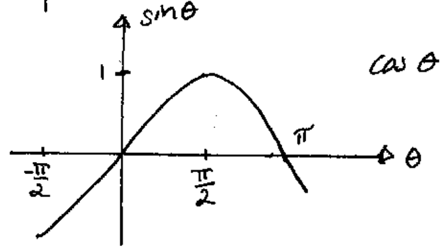
①

Polära koordinater

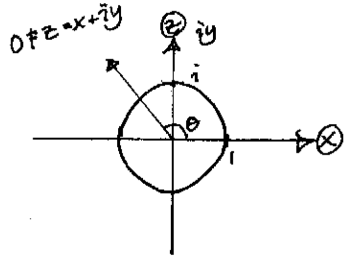


enhetscirkel

$\cos(\theta + 2\pi) = \cos\theta \quad \forall \theta \in \mathbb{R}$
$\sin(\theta + 2\pi) = \sin\theta \quad \forall \theta \in \mathbb{R}$

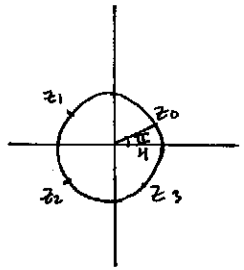


$\cos\theta = \sin(\frac{\pi}{2} - \theta)$



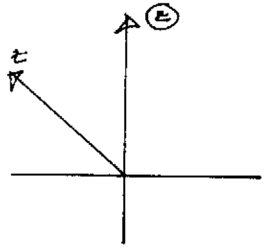
$P = \cos\theta + i\sin\theta$
$|z| = \sqrt{x^2 + y^2} \Rightarrow P = \frac{z}{|z|} \quad (z \neq 0)$
$z = |z| \cdot P$
$z = |z| (\cos\theta + i\sin\theta)$
$z = r$

$x = r\cos\theta$
$y = r\sin\theta$



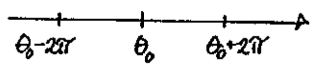
Polär framställning:
$z_0 = 1 \cdot (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
$z_1 = 1 \cdot (\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$
$z_2 = 1 (\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$
$z_3 = 1 \cdot (\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$
$= 1 \cdot (\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4})$
$= 1 \cdot (\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$

②



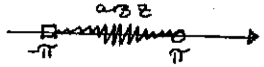
$\arg z$ är vilken vinkel som helst $\theta \in \mathbb{R}$ som uppfyller:

$z = |z|(\cos\theta + i\sin\theta)$



$\arg z$ ∞ antal tal

$\exists!\ \theta \in [-\pi, \pi[\ \cap \arg z$ Detta kallas $\operatorname{Arg} z$



Ex $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) =$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta) =$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

För 1) $\arg zw = \arg z + \arg w$

För 2) (de Moivre) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ $n \in \mathbb{Z}, \theta \in \mathbb{R}$

Binomiska ekvationer

$z^n = w \ (n \in \mathbb{Z}_+)$

$\{|z|(\cos\theta + i\sin\theta)\}^n = |w|(\cos\psi + i\sin\psi)$

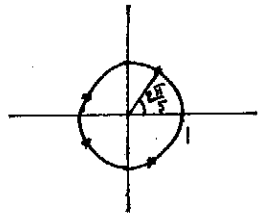
$|z|^n(\cos(n\theta) + i\sin(n\theta)) = |w|(\cos\psi + i\sin\psi)$

$|z|^n = |w|$

$$\begin{cases} n\theta = \psi + k2\pi \ (k \in \mathbb{Z}) \\ |z|^n = |w| \end{cases} \Leftrightarrow \begin{cases} |z| = |w|^{\frac{1}{n}} \\ \theta = \frac{\psi}{n} + \frac{2\pi k}{n}, \ k = 0, \dots, n-1 \end{cases}$$

$$z = |w|^{\frac{1}{n}}\left(\cos\frac{\psi}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\psi}{n} + \frac{2\pi k}{n}\right)\Big), \ k = 0, \dots, n-1$$

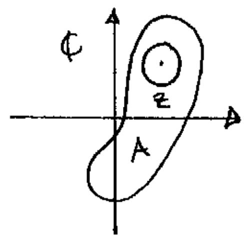
Ex $z^5 = 1$



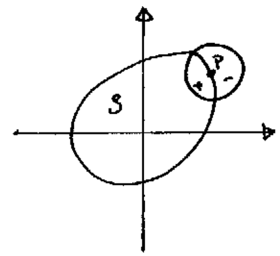
Topology

1) $B(z;r) = \{\zeta \in \mathbb{C} ; |\zeta - z| < r\}$

öppen boll med centrum z och radie r



2) z är en inre punkt i A om $\exists r > 0$ så $B(z;r) \subseteq A$



3) A är öppen om varje punkt i A är en inre punkt i A

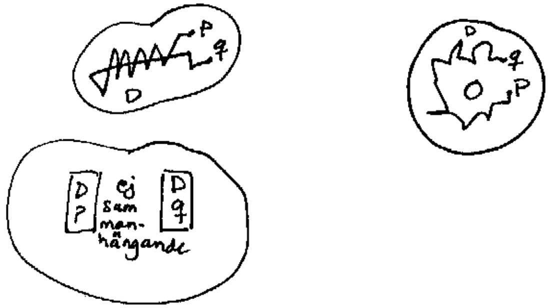
4) Antag $S \in \mathbb{C}$. En punkt $P \in \mathbb{C}$ är en randpunkt om $S \cap B(p,r) \neq 0, \forall r > 0$

$(\mathbb{C} \setminus S) \cap B(p;r) \neq 0 \quad \forall r > 0$

$\partial S = \{$alla randpunkter till $S\}$

5) Antag $F \subseteq \mathbb{C}$. Då är F sluten om $\partial F \subseteq F$

6) En öppen mängd är sammanhängande om det för $\forall\, p, q \in D\ \exists$ ett polygontåg i D som startar i p och slutar i q



7) En öppen sammanhängande mängd kallas <u>domän</u>.

Serier

$$\sum_{n=0}^{\infty} z_n$$

$$s_n = \sum_{k=0}^{n} z_k \qquad n = 0, 1, 2, \ldots$$

Den oändliga serien är konvergent om $\lim_{n\to+\infty} s_n \ \exists \in \mathbb{C}$

Gränsvärdet kallas seriens summa.

<u>Problem:</u> Är $\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)} + \frac{i}{e^k}\right)$ konvergent?

Bestäm isåfall seriens summa.

$\mathcal{L}$: $S_n = \sum_{k=1}^{n}\left(\frac{1}{k(k+1)} + \frac{i}{e^k}\right) = \sum_{k=1}^{n}\frac{1}{k(k+1)} + i\sum_{k=1}^{n} e^{-k}$ konv.

$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\frac{1}{k}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1} \to 1 \text{ då } n\to\infty$$

$$\sum_{k=1}^{n}(e^{-1})^k = e^{-1}\sum_{\nu=0}^{n-1}(e^{-1})^{\nu} = e^{-1}\,\frac{1-(e^{-1})^n}{1-e^{-1}} = e^{-1}\,\frac{1-e^{-n}}{1-e^{-1}} \to$$

$$\to \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e^{-1}-1} \text{ då } n\to+\infty$$

$\therefore\ S_n \to 1 + \frac{i}{e-1}$ då $n \to +\infty$

Svar: serien är konvergent och dess summa är $1 + \frac{i}{e-1}$

$\left[\sum_{1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}\right.$ återkommer till detta lv 5$\left.\right]$

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Om $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$

existerar säger man att f är <u>deriverbar</u> i z_0 och gränsvärdet betecknas $f'(z_0)$
Funktionen är <u>analytisk</u> om den är deriverbar i varje punkt i D. Om $D=\mathbb{C}$ och f analytisk kallas f <u>hel</u>.

<u>Ex</u>: $f(z)=z^2$;

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{z^2-z_0^2}{z-z_0} = z+z_0 \to 2z_0 \text{ då } z\to z_0$$

Får $f'(z_0)=2z_0$
Skriver också $f'(z)=2z$ eller $\frac{d}{dz}z^2=2z$

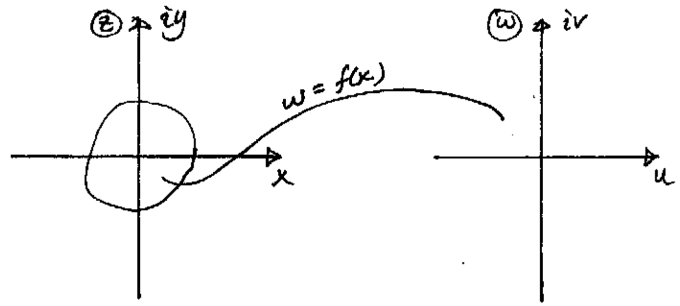
<u>Ex</u>: Antag $n\in \mathbb{Z}_+ = \mathbb{N}_+$ sätt $f(z)=z^n$

Får $\frac{f(z+h)-f(z)}{h} = \frac{(z+h)^n-z^n}{h} =$

$$= \frac{1}{h}\left\{ \cancel{z^n} + nz^{n-1}h + \dots + h^n - \cancel{z^n} \right\} =$$

$$= nz^{n-1} + \underbrace{\dots}_{\text{termer innehållande } h} + h^{n-1} \to nz^{n-1} \text{ då } h\to 0$$

$\therefore \frac{d}{dz}z^n = nz^{n-1}$



$z=x+iy$
Låt $\operatorname{Re} f(x) = u(x,y)$
$\operatorname{Im} f(x) = v(x,y)$

<u>Sats</u>: Antag att $f=u+iv$ är analytisk i D

Då:
$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{Cauchy Riemanns ekvation (CR)}$$

<u>Bevis</u>: Sätt $g(z,h) = \frac{f(z+h)-f(z)}{h}$
när $\lim_{h\to 0} g(z,h) = f'(z)$

Fall h reellt:

$$g(z,h) = \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h}$$

$$g(z,h) = \frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{v(x+h,y)-v(x,y)}{h}$$

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

Fall $h = ik$, k reellt:

$$q(z,h) = \frac{u(x,y+k) + iv(x,y+k) - u(x,y) - iv(x,y)}{ik} =$$

$$= \frac{u(x,y+k) - u(x,y)}{ik} + \frac{v(x,y+k) - v(x,y)}{k}$$

Låt $h \to 0$ $\quad f'(z) = \frac{1}{i}\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) =$

$$= \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) \qquad \boxed{\frac{1}{i} = -i}$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ och $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Detta är CR!

Kontroll av CR

Ex: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 - i2xy$

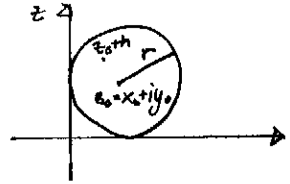
$$\therefore \begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases}$$

$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial v}{\partial y} = 2x$

$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = 2y \qquad$ CR stämmer!

Gäller omvändningen?

Sats: Antag $f = u + iv$ och antag $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ är kontinuerliga i $B(z_0, r)$. Om CR gäller i punkten z_0 så är f deriverbar i z_0



$h = \delta + i\nu \quad (\delta, \nu \in \mathbb{R})$

$|h| = \sqrt{\delta^2 + \nu^2}$

$\left|\frac{\delta}{h}\right| = \frac{|\delta|}{\sqrt{\delta^2+\nu^2}} \leq 1$ pss $\left|\frac{\nu}{h}\right| \leq 1$

Bevis: ekv.1) $u(x_0+\delta, y_0+\nu) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)\delta +$

$+ \underbrace{\frac{\partial u}{\partial y}(x_0,y_0)\nu}_{= -\frac{\partial v}{\partial x}(x_0,y_0)\nu} + \delta E_1 + \nu E_2$ där $E_1, E_2 \to 0$ då $\delta, \nu \to 0$

ekv.2) $v(x_0+\delta, y_0+\nu) = v(x_0,y_0) + \frac{\partial v}{\partial x}(x_0,y_0)\delta + \underbrace{\frac{\partial v}{\partial y}(x_0,y_0)\nu}_{= \frac{\partial u}{\partial x}(x_0,y_0)\nu} +$

$+ \delta E_3 + \nu E_4$ där $E_3, E_4 \to 0$ då $\delta, \nu \to 0$

mult ekv. 1 med 1 och ekv 2. med i och lägg ihop

$$f(z_0+h) = f(z_0) + \left\{\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)\right\}\underbrace{(\delta + i\nu)}_{h} +$$

$$+ \delta E_1 + \nu E_2 + i\delta E_3 + i\nu E_4$$

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) +$$

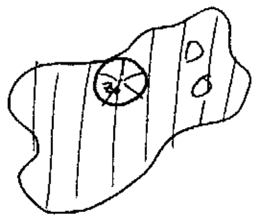
$$+ \frac{\delta}{h}E_1 + \frac{\nu}{h}E_2 + \frac{i\delta}{h}E_3 + \frac{i\nu}{h}E_4$$

Eftersom $\left|\frac{\delta}{h}\right| \leq 1$ & $\left|\frac{\nu}{h}\right| \leq 1$ så erhålles då $h \to 0$ att $f'(z_0)$ $\exists$ $= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$

Korollarium (följdsats)

Antag $f = u+iv$ och antag $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ är kontinuerliga i D samt att CR gäller i hela D. Då är f analytisk

Bevis:



Ex: $e^z \overset{def}{=} e^x(\cos y + i\sin y)$

För: $\begin{cases} u(x,y) = e^x\cos y \\ v(x,y) = e^x \sin y \end{cases}$ $\quad u, v \in C^1(D)$

Vidare $\frac{\partial u}{\partial x} = e^x\cos y$, $\frac{\partial v}{\partial y} = e^x\cos y$

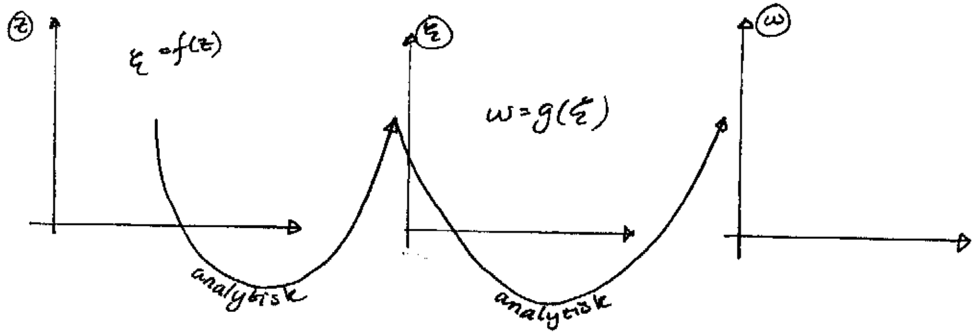
$\frac{\partial u}{\partial y} = -e^x\sin y$, $\frac{\partial v}{\partial x} = e^x\sin y$

Alltså: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ och $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\Rightarrow$ e^z är analytisk

Vi vet att

$$\frac{d}{dz}e^z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x\cos y + ie^x\sin y =$$

$$= e^x(\cos y + i\sin y) = e^z$$



$w = h(z) = g(f(z))$ är analytisk och

$h'(z) = g'(f(z))f'(z)$

Ex $\frac{d}{dz}e^{az} = ae^{az}$ $\quad (a \in \mathbb{C})$

Ex $\overset{def}{=} \frac{1}{2}(e^{iz}+e^{-iz})$, $\sin z \overset{def}{=} \frac{1}{2i}(e^{iz}-e^{-iz})$

$$\frac{d}{dz}\cos z = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = -\frac{1}{2i}\left(e^{iz}-e^{-iz}\right) = -\sin z$$

$$\cos(iy) = \underbrace{\frac{1}{2}\left(e^{-y}+e^{y}\right)}_{\cosh y} \to +\infty \text{ då } y \to \pm\infty$$

$\log z = \ln|z| + i\arg z \qquad z \neq 0$

$e^{\log z} = e^{\ln|z| + i\arg z} = |z|e^{i\arg z} = z$

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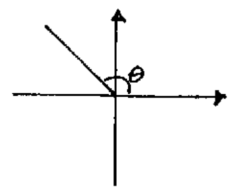
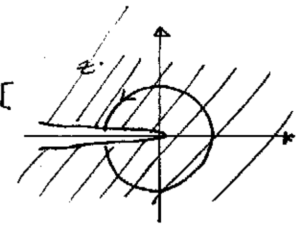
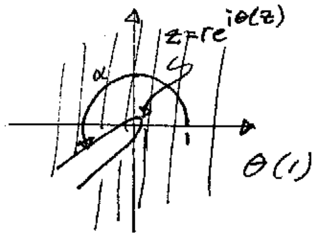


$$\begin{cases} \arg z = \theta + n2\pi,\ n \in \mathbb{Z} \\ \operatorname{Arg} z \in [-\pi, \pi[\ \cap \arg z \end{cases}$$

$[-\pi, \pi[$



$$\begin{cases} \log z = \ln|z| + i\arg z & z \neq 0 \\ \operatorname{Log} z = \ln|z| + i\operatorname{Arg} z \end{cases}$$



dåt $\theta(z)$ variera kont.

$\theta(1) = n_0 \cdot 2\pi \ (n_0 \in \mathbb{Z})$

$$\log_\alpha z = \ln|z| + i\theta(z)$$

$$e^{\log_\alpha z} = e^{\ln|z| + i\theta(z)} = |z|e^{i\theta(z)} = z$$

<u>Sats</u>:

$$\frac{d}{dz}\log_\alpha z = \frac{1}{z}$$

Bluffbevis: $\frac{d}{dw}e^w = e^w$

$$e^{\log_\alpha z} = z$$

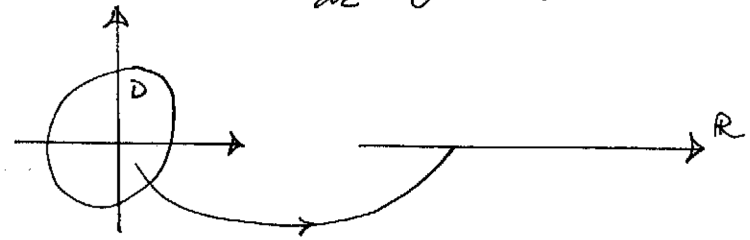
$$e^{\log_\alpha z} \frac{d}{dz}\log_\alpha z = 1$$

z

Alltså är $\frac{d}{dz}\log_\alpha z = \frac{1}{z}$

[Varför är $\log z$ deriverbar?]

<u>Koroll</u> $\qquad \frac{d}{dz}\operatorname{Log} z = \frac{1}{z}$



u är <u>harmonisk</u> om $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ i D

<u>Sats</u>: Antag att $f = u + iv$ analytisk i D. Då är u och v harmoniska.

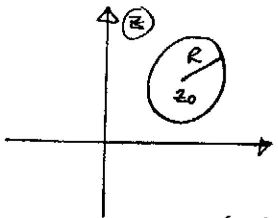
Bluffbevis:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

[ok om $u, v \in C^2(D)$]
(u, v är två gånger kontinuerligt deriverbar)

Potensserier

$e^x = \sum_0^\infty \frac{x^n}{n!}$



$\sum_{n=0}^\infty a_n (z-z_0)^n$

$\exists R \ni$ (α) serien divergerar för $|z-z_0|>R$
(β) serien konvergerar för $|z-z_0|<R$

R konvergensradie

Sätt $f(z) = \sum_{n=0}^\infty a_n (z-z_0)^n$, $|z-z_0|<R$ [a_n beror ej av z]

Får $\frac{d}{dz}f(z) = \sum_{n=0}^\infty \frac{d}{dz}(a_n(z-z_0)^n) =$

$= \sum_{n=1}^\infty n a_n (z-z_0)^{n-1}$, $|z-z_0|<R$

$\therefore$ $f(z)$ analytisk i $|z-z_0|<R$

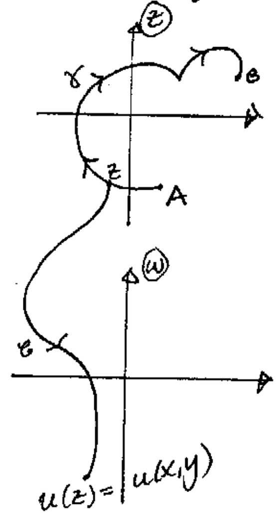
$\frac{d^2}{dz^2}f(z) = \sum_{n=2}^\infty n(n-1)a_n(z-z_0)^{n-2}$ $|z-z_0|<R$
(osv)

$\begin{cases} y'' + xy = 0 \\ y(0) = y_0 \ , \ y'(0) = y \end{cases}$

Diffekvation som endast kan lösas numeriskt med t.ex potensserie

$y(n) = \sum_0^\infty a_n x^n$

Linje integral



$\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$
$a \le t \le b$

γ är $\mathcal{C}'$ styckvis

$\int_\gamma u(z)\,dz \overset{\text{def}}{=} \int_\gamma u(z)dx + iu(z)dy =$

$= \int_a^b \{u(x(t), y(t))x'(t) +$
$+ iu(x(t), y(t))y'(t)\}\,dt$

$u(x(t), y(t)) = u(\gamma(t))$

$\gamma'(t) = x'(t) + iy'(t)$

$$\int_\gamma u(z)dz = \int_a^b u(\gamma(t))\{x'(t)+iy'(t)\}\,dt =$$

$$= \int_a^b u(\gamma(t))\gamma'(t)\,dt$$

Sats (ML-olikheten)

$$\left|\int_\gamma u(z)dz\right| \le ML$$

där $M = \max_{z\in\gamma} |u(z)|$ och $L = \ell(\gamma)$

Bevis (se sid 61):

$$\left|\int_\gamma u(z)dz\right| = \left|\int_a^b u(\gamma(t))\gamma'(t)\,dt\right| \le$$

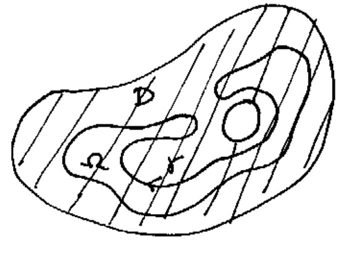
$$\le \int_a^b \underbrace{|u(\gamma(t))\gamma'(t)|}_{|u(\gamma(t))||\gamma'(t)|}dt \le \int_a^b M|\gamma'(t)|\,dt =$$

$$= M\int_a^b |\gamma'(t)|\,dt = M\int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt = ML$$

Cauchys Integralsats

Antag f analytisk i D och låt γ vara en enkel sluten kurva som innringar ett område Ω som ligger helt i D.
Då är $\int_\gamma f(z)dz = 0$

Bevis:



Ω får ej omsluta hålet

Antag $(u, v \in C^1)$

Kan anta att γ är positivt orienterad

$$\int_\gamma f(z)dz = \int_\gamma f(z)dx + i f(z)dy =$$

$$= \int \underbrace{(u+iv)}_{P}dx + \underbrace{\overset{i(u+iv)}{(iu-v)}}_{Q}dy$$

$$\iint_\Omega \left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\}dxdy = \iint_\Omega \left\{\left(i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right\}dxdy$$

$$= \iint_\Omega \left\{i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}dxdy = 0$$

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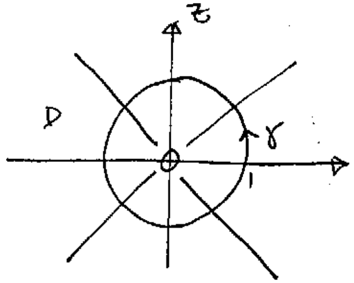
Ex:



$D = \mathbb{C} \setminus \{0\}$

$f(z) = \frac{1}{z}$

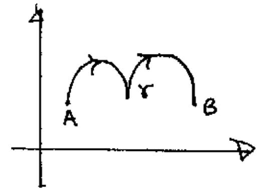
$\gamma(t) = e^{it} \quad 0 \le t \le 2\pi$;

$\int f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}} i e^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i \neq 0$

Antag $f: D \to \mathbb{C}$ är analytisk

Vi säger att $F: D \to \mathbb{C}$ är en <u>primitiv</u> till f om $F'(z) = f(z) \ \forall z \in D$

Ex:



f analytisk
F primitiv till f

$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt} F(\gamma(t))\,dt =$

$= \left[F(\gamma(t))\right]_a^b = F(\gamma(b)) - F(\gamma(a)) = F(B) - F(A)$

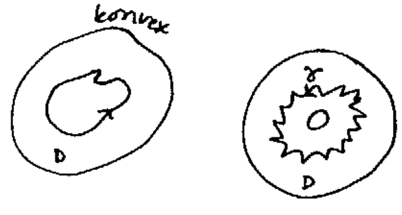
Om γ sluten är $A = B$ så $\int_\gamma f(z)\,dz = 0$

Ex: $f(z) = \frac{1}{z} \quad z \in \mathbb{C} \setminus \{0\} = D$ saknar primitiv

$\left(\text{ty } \oint_{|z|=1} f(z)\,dz = 2\pi i \neq 0\right)$

Def: En domän D är enkelt sammanhängande om varje enkel, sluten kurva i D innesluter ett område som helt ligger i D (Förkortas ES)

Ex



$\{\mathbb{C} \setminus \{0\}$ i ES$\}$

icke enkelt sammanhängande i ES

Sats: En analytisk funktion i en ES domän har en primitiv.

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Repetition av Cauchys sats:

Antag f analytisk i D och låt γ vara en enkel sluten kurva som innringar ett område Ω som ligger helt i D:

nytt Antag nu att D är ES

$\gamma = \gamma_0 + \gamma_1$

$$\int_\gamma f dz = \int_{\gamma_0} f dz + \int_{\gamma_1} f dz = 0 + 0 = 0$$

Om γ är ett slutet polygontåg i D och $f: D \to \mathbb{C}$ analytisk så $\int_\gamma f(z) dz = 0$

induktion över antalet hörn

Sats: Antag D är ES och $f: D \to \mathbb{C}$ analytisk. Då har f en primitiv i D, dvs $\exists F: D \to \mathbb{C}$ så att $F' = f$

Bevis:

$$\int_{\gamma - \gamma'} f(\xi) d\xi = 0 \qquad \int_\gamma f d\xi - \int_{\gamma'} f d\xi = 0$$

$$\int_\gamma f d\xi = \int_{\gamma'} f d\xi$$

$$F(z) = \int_\gamma f(\xi) d\xi$$

$$F(z+h) = \int_{\gamma+\alpha} f(\xi) d\xi = \int_\gamma f(\xi) d\xi + \int_\alpha f(\xi) d\xi$$

$$F(z+h) - F(z) = \int_\alpha f(\xi) d\xi = \begin{bmatrix} \alpha(t) = z + ht \\ 0 \le t \le 1 \end{bmatrix} =$$

$$= \int_0^1 f(z+ht) h \, dt = h \int_0^1 f(z+ht) dt$$

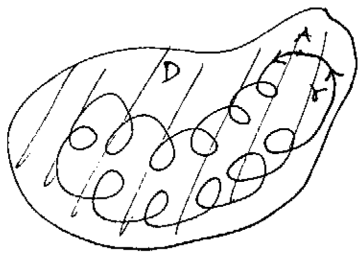
$$\frac{F(z+h)-F(z)}{h} = \int_0^1 f(z+ht)\,dt \to \int_0^1 f(z)\,dt =$$

$$= f(z)\int_0^1 1\,dt = f(z) \text{ då } h \to 0$$

$$\therefore F'(z) = f(z) \quad \boxtimes$$

<u>Koroll</u> Om D är ES och $f: D \to \mathbb{C}$ analytisk så är $\int_\gamma f(z)\,dz = 0$ för varje sluten kurva i D.

<u>Bevis</u>:

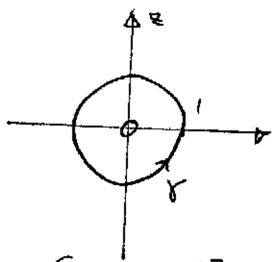


Låt $F' = f$

Då: $\int_\gamma f(z)\,dz =$

$$= \Big[F(z)\Big]_A^A = F(A) - F(A) = 0$$

<u>Ex</u>

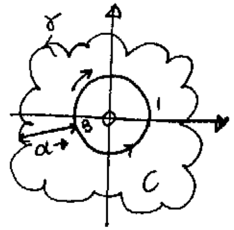


$D = \mathbb{C} \setminus \{0\}$ ej ES

$f(z) = \frac{1}{z}, \quad z \in D$

$$\int_\gamma f(z)\,dz = \begin{bmatrix} \gamma(t) = e^{it} \\ \gamma'(t) = ie^{t} \end{bmatrix} = \int_0^{2\pi} f(e^{it})\,ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i \neq 0$$

<u>Ex</u>:



$D = \mathbb{C} \setminus \{0\}$

$f(z) = \frac{1}{z}$

$\int f(z)\,dz = 0$

$\gamma + \alpha - C - \alpha$

$$\int_\gamma f\,dz + \int_\alpha f\,dz - \int_C f\,dz - \int_\alpha f\,dz = 0$$

$$\int_\gamma f(z)\,dz = \int_C f(z)\,dz = 2\pi i \qquad \int_\gamma \frac{dz}{z} = 2\pi i$$

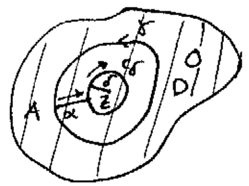
<u>Cauchys integralformel</u>

VIKTIG!

Låt $f: D \to \mathbb{C}$ vara analytisk och antag att γ är en positivt orienterad enkel sluten kurva i D som innringar ett område Ω som helt ligger i D.
Då gäller:

$$f(z) = \frac{1}{2\pi i}\int \frac{f(\xi)}{\xi - z}\,d\xi \qquad \text{om } z \in \Omega$$

Bevis



$$\int_{\gamma+\alpha-c_\delta-\alpha} \frac{f(\xi)}{\xi - z}\,dz \overset{\substack{\text{cauchys}\\ \text{integralsats}}}{=} 0$$

$\xi \to \dfrac{f(\xi)}{\xi - z}$ analytisk "mellan c_δ och γ"

$$\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi - \int \frac{f(\xi)}{\xi - z}\,d\xi = 0$$

$$\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \int_{c_\delta} \frac{f(\xi)}{\xi - z}\,d\xi$$

$$\begin{bmatrix} \xi(t) = z + \delta e^{it} & 0 \le t \le 2\pi \\ \xi'(t) = \delta i e^{it} & 0 \le t \le 2\pi \end{bmatrix}$$

$$\int_0^{2\pi} \frac{f(z + \delta e^{it})}{\delta e^{it}}\,\delta i e^{it}\,dt = i\int_0^{2\pi} f(z + \delta e^{it})\,dt \to$$

$$\to i\int_0^{2\pi} f(z)\,dt = i f(z)\int 1\,dt = 2\pi i f(z) \text{ då } \delta \to 0$$

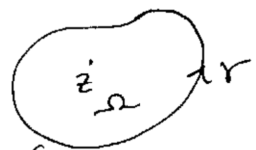
Alltså $\displaystyle\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = 2\pi i f(z)$ ☒

Koroll (samma förutsättningar som i satsen)

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}\frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^2}\,d\xi \quad z \in \Omega$$

$$f^{(n)}(z) = \frac{1}{2\pi i}\int_\gamma \left(\frac{d}{dz}\right)^n \frac{f(\xi)}{\xi - z}\,d\xi = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi \quad z \in \Omega$$

$\boxed{\text{Spec. blir } f\ \infty \text{ många gånger deriverbar}}$



$$f(z) = \frac{1}{2\pi i}\int \frac{f(\xi)}{\xi - z}\,d\xi$$

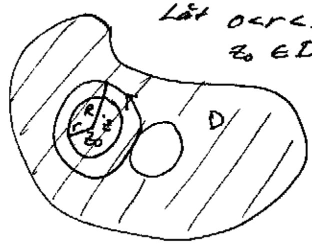
$\boxed{f \text{ känd på } \gamma \Rightarrow f \text{ känd i } \Omega}$

Potensserieutvecklingar

Antag f i $D \to \mathbb{C}$ analytisk

Då $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$

om $|z - z_0| < r$



där $a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi =$

$= \frac{f^{(k)}(z_0)}{k!}$

Bevis: $f(z) = \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z} d\xi$

$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}}$

Om $\xi \in \gamma$ så gäller att $\left| \frac{z - z_0}{\xi - z_0} \right| < 1$

och vi får $\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^k =$

$= \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}}$

Nu: $f(z) = \frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi - z)} d\xi =$

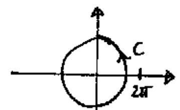
$= \frac{1}{2\pi i} \int_\gamma \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}} f(\xi) d\xi =$

$= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_\gamma \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}} f(\xi) d\xi$

$= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi \right) (z - z_0)^k =$

$= \sum_{k=0}^{\infty} a_k (z - z_0)^k$ $\boxed{}$

Ex:



Beräkna $\int_C \frac{dz}{z^2 - 4z + 3} = \int_C \frac{dz}{(z-1)(z-3)} =$

$= \int_C \frac{\frac{1}{(z-3)}}{(z-1)} dz = \int \frac{\frac{1}{\xi - 3}}{\xi - 1} d\xi =$

$= \int \frac{f(\xi)}{\xi - 1} d\xi$ om $f(\xi) = \frac{1}{\xi - 3}$

$I = 2\pi i f(1) = 2\pi i \left(-\frac{1}{2}\right) = -\pi i$

Ex: $e^z = \sum_{k=0}^{\infty} a_k z^k$, $a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}$ ty $\left(\frac{d}{dz}\right) e^z = e^z$

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Cauchys Integralformel

a) $f: D \to \mathbb{C}$ analytisk

b) γ är en enkel sluten, positivt orienterad kurva i D som inningar ett område $\Omega \subseteq D$

Slutsats:

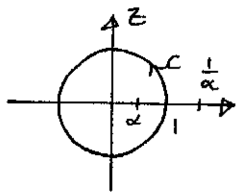
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in \Omega$$

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz \qquad a \in \Omega$$

Ex: Låt $0 < \alpha < 1$

Beräkna $\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$

L:



$z = e^{i\theta} \quad 0 \le \theta \le 2\pi$

$dz = ie^{i\theta}\,d\theta$

$\frac{dz}{iz} = d\theta$

$$\int_C g(z)\frac{dz}{iz} = \int_0^{2\pi} g(e^{i\theta})\,d\theta$$

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2} = \int \frac{d\theta}{1 - \alpha\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right) + \alpha^2} =$$

$$= \int_C \frac{1}{1 - \alpha\left(z + \frac{1}{z}\right) + \alpha^2}\,\frac{dz}{iz} = \frac{1}{i}\int_C \frac{dz}{-\alpha z^2 + (1+\alpha^2)z - \alpha} =$$

$$= \frac{i}{\alpha}\int_C \frac{dz}{z^2 - \frac{1+\alpha^2}{\alpha}z + 1} = \frac{i}{\alpha}\int \frac{dz}{(z-\alpha)\left(z - \frac{1}{\alpha}\right)} =$$

$$= \frac{i}{\alpha}\int_C \frac{\frac{1}{z - \frac{1}{\alpha}}}{z - \alpha}\,dz = \frac{i}{\alpha}\,2\pi i\,\frac{1}{\alpha - \frac{1}{\alpha}} = \frac{2\pi}{1 - \alpha^2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2} = \frac{1}{1 - \alpha^2}$$

Potensserieutvecklingar

$f: D \to \mathbb{C}$ analytisk

$$\begin{cases} f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \\ \text{om } |z - z_0| < R \leftarrow \text{se bild sid 27} \end{cases}$$

Ex: $f(z) = \cos z$; hel funktion, analytisk överallt $\Rightarrow R = +\infty$

$z_0 = 0$; $f'(z) = \cos\left(z + \frac{\pi}{2}\right)$; $f^{(n)}(z) = \cos\left(z + \frac{n\pi}{2}\right)$;

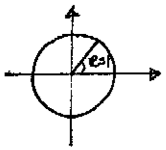
$f^{(n)}(0) = \cos\left(n\frac{\pi}{2}\right)$;

$f^{(2k)}(0) = \cos k\pi = (-1)^k$, $f^{(2k+1)}(0) = 0$

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$$

Ex $f(z) = \frac{1}{z+1}$, $z \neq 1$ antag $z_0 = 0$

$f(z) = \frac{1}{(1-(-z))} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n$

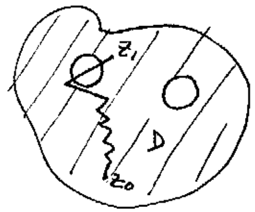


om $\begin{cases} |z-0| < R = 1 \\ \text{dvs } |z| < 1 \end{cases}$

Sats: Antag $f: D \to \mathbb{C}$ analytisk, $z_0 \in D$ och $f^{(n)}(z_0) = 0$, $\forall n \in N$

Då är $f(z) = 0$, $\forall z \in D$

Bevis: Tag z godt. i D



Har $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ z nära z_0

$\therefore f(z) = 0$ om z nära z_0

$\therefore f^{(n)}(z) = 0$ om z nära z_0 och $n \in \mathbb{N}$

Låt p vara den punkt på γ närmast z (i γ-land) sådan att $f^{(n)}(z) = 0$, $\forall z$ som tillhör γ mellan z_0 och p för varje $n \in \mathbb{N}$

Antag att $p \neq z_1$

Då $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n$ z nära p

$\therefore f(z) = 0$, $\forall z$ nära p

$\therefore f^{(n)}(z) = 0$, $\forall z$ nära p

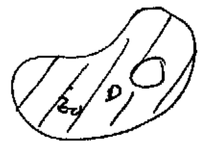
Motsägelse! $\therefore p = z$

(p finns ty sup finns)

$f = 0 \Leftrightarrow f(z) = 0$, $\forall z \in D$

Antag $f: D \to \mathbb{C}$ och $f \neq 0$

Antag $z_0 \in D$ och $f(z_0) = 0$



$\exists\, n \in \mathbb{N}$ så att $f^{(n)}(z_0) \neq 0$

Har $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n =$

$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ z nära z_0

Skriv $f(z) = \underset{(a_m \neq 0)}{a_m} \underset{m \geq 1}{(z-z_0)^m} + a_{m+1}(z-z_0)^{m+1} + \dots$ z nära z_0

Här kallas m nollställets ordning

Låt $g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & z \neq z_0 \\ a_m & z = z_0 \end{cases}$

Då är $g(z)$ analytisk i D och $g(z_0) = a_m \neq 0$

Har $f(z) = (z-z_0)^m g(z)$ OBS: $g(z) \neq 0$ nära z_0

$f(z) \neq 0$ i $\rho < |z - z_0| < r$
om $r > 0$ tillräckligt liten

Nollställena uppträder isolerat.

Ex Antag $h: D \to \mathbb{C}$ analytisk och $= 0$ på en liten kurva (ej en punkt)

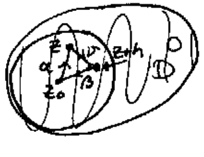
Då är $h \equiv 0$

Speciellt om $f, g: D \to \mathbb{C}$ analytiska och lika på γ så är de lika överallt i D.

Morera: Antag $f: D \to \mathbb{C}$ kont. och

$\int_\gamma f(z)\,dz = 0$ för varje triangel γ i D sådan att $\text{skiv}(\gamma) \subseteq D$. Då är f analytisk.

Bevis:



Def: $F(z) = \int_\alpha f(\xi)\,d\xi$

$F(z+h) = \int_\beta f(\xi)\,d\xi$

$$\int_{\alpha+\nu-\beta} f(\xi)\,d\xi = 0 \Rightarrow \int_\nu f(\xi)\,d\xi = \int_\beta f(\xi)\,d\xi - \int_\alpha f(\xi)\,d\xi \,;\quad F(z+h) - F(z) = \int_\nu f(\xi)\,d\xi =$$

$$= \begin{bmatrix} \xi = z + th \\ d\xi = h\,dt \end{bmatrix} = h \int_0^1 f(z+h)\,dt$$

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f(z+h)\,dt \to \int_0^1 f(z)\,dt =$$

$$= f(z) \int_0^1 dt = f(z) \quad \therefore F' = f \text{ i cirkelskivan}$$

$\Rightarrow$ F analytisk i cirkelskivan

$\Rightarrow$ f analytisk i cirkelskivan, därmed överallt i D.

Liouvilles sats

En begränsad, hel analytisk funktion är konstant.

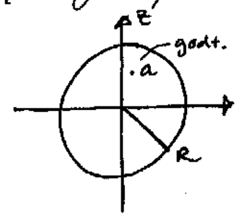
Bevis: Kalla funktionen F. Här är $|F| \leq M$ där $M \in \mathbb{R}$

Här: $\sum_{k=0}^{\infty} F^{(k)}(0) z^{(k)}, \quad \forall z \in \mathbb{C}$

sätt $g(z) = \begin{cases} \dfrac{F(z) - F(0)}{z} & z \neq 0 \\ F'(0) & z = 0 \end{cases}$

så är g analytisk. Vidare $|z| = R > 0 \Rightarrow$

$\Rightarrow |g(z)| \leq \dfrac{2M}{R}$



$$g(a) = \frac{1}{2\pi i} \int_\alpha \frac{g(z)}{z-a}\,dz$$

$$\text{M.L}: |g(a)| = \frac{1}{2\pi} \cdot \frac{2M}{R} \cdot \frac{1}{R-1}\, 2\pi R \to 0 \text{ då } R \to \infty$$

$\therefore g(a) = 0$ dvs $F(a) = F(0) \quad \forall a \in \mathbb{C}$

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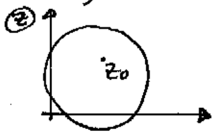
Isolerade singulariteter

Om $f(z)$ analytisk i $0<|z-z_0|<r$ säges $f(z)$ ha en isolerad singularitet i z_0



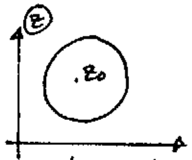
Ex 1) $f(z)=\dfrac{z^2-z_0^2}{z-z_0}$ (hävbar)

2) $f(z)=\dfrac{1}{(z-z_0)^7}$ (pol)

3) $f(z)=e^{\frac{1}{z-z_0}}$ (väsentlig singularitet)

<u>Fall (α)</u>: $|f(z)|$ begränsad då $z\to z_0$
<u>Fall (β)</u>: $|f(z)|\to+\infty$ då $z\to z_0$
<u>Fall (γ)</u>: varken α eller β gäller

Fall (a) disk:



$0<|z-z_0|<r$

Bilda $g(z)=\begin{cases}(z-z_0)^2f(z) & 0<|z-z_0|<r\\ 0 & ,\ z=z_0\end{cases}$

Funktionen $g(z)$ är analytisk i $0<|z-z_0|<r$

Vidare $\dfrac{g(z)-g(z_0)}{z-z_0}=(z-z_0)f(z)\to 0$ då $z\to z_0$

Får $g'(z_0)=0$

$\therefore$ $g(z)$ analytisk i $|z-z_0|<r$
och $g(z_0)=g'(z_0)=0$

Då $g(z)=\underset{g(z_0)=0}{b_0}+\underset{g'(z_0)=0}{b_1}(z-z_0)+b_2(z-z_0)^2+$
$+b_3(z-z_0)^3+\dots,\ |z-z_0|<r$

Får $g(z)=(z-z_0)^2(b_2+b_3(z-z_0)+\dots),\ |z-z_0|<r$
Slutsats: $f(z)=b_2+b_3(z-z_0)+\dots\quad 0<|z-z_0|<r$
Definiera $f(z_0)=b_2$. Då följer:
$$f(z)=b_2+b_3(z-z_0)+\dots,\ |z-z_0|<r$$

Vi har en <u>hävbar singularitet</u> i Fall α

Fall (β) disk.: Kan anta $|f(z)|>1$ då $0<|z-z_0|<r$
Sätt $g(z)=\dfrac{1}{f(z)}\quad 0<|z-z_0|<r$
Då är $|g(z)|$ begränsad då $z\to z_0$
$\therefore$ $g(z)$ har en hävbar singularitet.
Får $g(z_0)=\lim\limits_{\substack{z\to z_0\\ z\neq z_0}} g(z)=0$

Alltså $g(z)$, $|z-z_0|<r$, analytisk, $g(z_0)=0$
och $g(z)\neq 0$ då $0<|z-z_0|<r$
$\therefore$ $g\not\equiv 0$. Låt $g(z)=(z-z_0)^m h(z)$, där $m\in\mathbb{Z}$,
$h(z)$, $|z-z_0|<r$, är analytisk och $h(z_0)\neq 0$
Obs att $h(z)\neq 0$, $0<|z-z_0|<r$
Alltså $h(z)\neq 0$, $|z-z_0|<r$

Sätt $f(z) = \frac{1}{h(z)}$. Då $f(z) = \frac{H(z)}{(z-z_0)^m}$, $0 < |z - z_0| < r$,

där $H(z)$, $|z-z_0| < r$ och $H(z_0) \neq 0$.

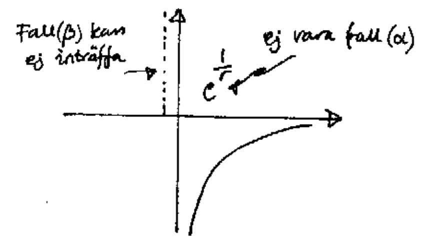
I fall (β) säger vi att f har en <u>pol av ordning</u> m.

<u>Ex</u> $e^{\frac{1}{z-z_0}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(z-z_0)^n} = 1 + \frac{1}{z-z_0} + \frac{1/2}{(z-z_0)^2} + \frac{1/6}{(z-z_0)^3} + \dots$

$z - z_0 = re^{i\theta}$

$e^{\frac{1}{z-z_0}} = e^{\frac{1}{r}e^{-i\theta}}$; $|e^{\xi}| = e^{\operatorname{Re}\xi}$

$\left| e^{\frac{1}{z-z_0}} \right| = e^{\frac{\cos\theta}{r}} = \begin{cases} 1, & \theta = \frac{\pi}{2} \\ e^{\frac{1}{r}}, & \theta = 0 \end{cases}$

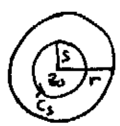


$e^{\frac{1}{z-z_0}} \notin$ Fall (α) $\cup$ Fall (β)

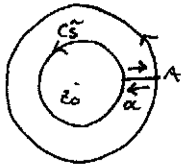
$e^{\frac{1}{z-z_0}} \in$ Fall γ

Antag $f(z)$, $0 < |z - z_0| < r$ analytisk

Då är $\int_{C_s} f(\xi)\,d\xi$ oberoende av s



$\therefore \int_{C_s + \alpha - \tilde{C_s} - \alpha} f(\xi)\,d\xi = 0$



$\therefore \int_{C_s} f(\xi)\,d\xi - \int_{\tilde{C_s}} f(\xi)\,d\xi = 0$

$\therefore \int_{C_s} f(\xi)\,d\xi = \int_{\tilde{C_s}} f(\xi)\,d\xi$

<u>Def</u> Integralen $\frac{1}{2\pi i}\int_C f(\xi)\,d\xi$ kallas residuen av $f(z)$ i punkten z_0 och betecknas $\operatorname{Res}(f; z_0)$ eller $\operatorname{Res}(f(z); z_0)$

<u>Problem</u> Sök $\operatorname{Res}((z-z_0)^n; z_0)$ där $n \in \mathbb{Z}$

<u>$\mathcal{L}$</u>: $\frac{1}{2\pi i}\int_{C_s} (z-z_0)^n\,dz = \begin{bmatrix} z = z_0 + se^{it} \\ dz = sie^{it}dt \end{bmatrix} =$

$= \frac{1}{2\pi}\int_0^{2\pi} s^{n+1} e^{i(n+1)t}\,dt = \begin{cases} 1, & n = -1 \\ 0, & n \neq -1 \end{cases}$

$\therefore \operatorname{Res}((z-z_0)^n; z_0) = \begin{cases} 1, & n = -1 \\ 0, & n \neq -1 \end{cases}$

<u>Problem</u> Antag $f(z)$ har en pol av ordning $m \geq 1$ i z_0. Sök $\operatorname{Res}(f(z); z_0)$

<u>$\mathcal{L}$</u>: $f(z) = \frac{C_0}{(z-z_0)^m} + \frac{C_1}{(z-z_0)^m} + \dots + \frac{C_{m-1}}{z-z_0} + C_m + C_{m+1}(z-z_0) + \dots$, $0 < |z-z_0| < r$

Låt $0 < s < r$: $\frac{1}{2\pi i}\int_{C_s} f(z)\,dz = \frac{1}{2\pi i}\int_{C_s} \frac{C_{m-1}}{z-z_0}\,dz =$

$= C_{m-1} = \frac{H^{(m-1)}(z_0)}{(m-1)!} = \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} \left\{ (z-z_0)^m f(z) \right\} \Big|_{z=z_0}$

Problem: Antag att $F(z), G(z)$ analytiska nära z_0 sådana att $G(z_0)=0$ och $G'(z_0) \neq 0$

Då $\operatorname{Res}\left(\frac{F(z)}{G(z)} ; z_0\right) = \frac{F(z_0)}{G'(z_0)}$

B: $G(z) = G'(z_0)(z-z_0) + b_2(z-z_0)^2 + \ldots)$

Då $\frac{F(z)}{G(z)} = \frac{1}{z-z_0} \cdot \frac{F(z)}{G'(z_0) + G(z-z_0) + \ldots} =$

$= \frac{1}{z-z_0}\left(\frac{F(z_0)}{G'(z_0)} + C(z-z_0) + C_2(z-z_0)^2 + \ldots\right)$

$\therefore \operatorname{Res}\left(\frac{F(z)}{G(z)} ; z_0\right) = \frac{F(z_0)}{G'(z_0)}$

Ex: $\operatorname{Res}\left(\frac{z^7}{z^2+1} ; i\right) = \frac{z^7}{2z}\Big|_{z=i} = \frac{i^6}{2} = -\frac{1}{2}$

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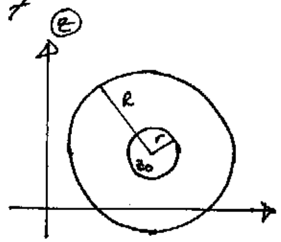
Laurentserie



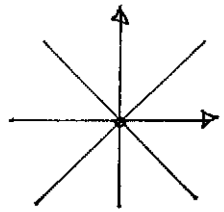
antag $f(z)$ analytisk i $r < |z-z_0| < R$

Då gäller

$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n, \quad r < |z-z_0| < R$

principaldelen av f i z_0 $\Big\}$ $\sum_{n=-\infty}^{-1} a_n (z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n$

Ex:



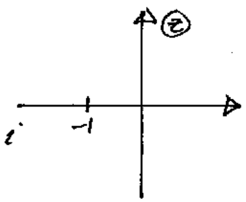
$f(z) = e^{\frac{1}{z}} \quad z \neq 0$

$r = 0, \quad R = +\infty$

$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} =$

$= \sum_{k=-\infty}^{0} \frac{z^k}{(-k)!}$

Ex: $f(z) = \frac{1}{z(z+1)}$



Utveckla $f(z)$ i Laurentserie i

a) $0 < |z| < 1$ b) $1 < |z|$

a) $f(z) = \frac{1}{z} \cdot \frac{1}{z+1} = \frac{1}{z} \sum_{n=0}^{\infty} (-z)^n = \frac{1}{z}(1 - z + z^2 - z^3 + \ldots) =$

$= \frac{1}{z} - 1 + z - z^2 + \ldots, \quad 0 < |z| < 1$

alt: $f(z) = \frac{1}{z} - \frac{1}{1+z} = \frac{1}{z} - \sum_{n=0}^{\infty}(-z)^n = \frac{1}{z} - 1 + z - z^2 + \ldots$

$0 < |z| < 1$

b) $f(z) = \frac{1}{z^2} \cdot \frac{1}{\frac{1}{z}+1} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{n+2}} =$

$= \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \ldots, \quad 1 < |z|$

alt: $f(z) = \frac{1}{z} - \frac{1}{1+z} = \frac{1}{z} - \frac{1}{z} \cdot \frac{1}{\frac{1}{z}+1} =$

$= \frac{1}{z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^4} - \ldots \quad 1 < |z|$

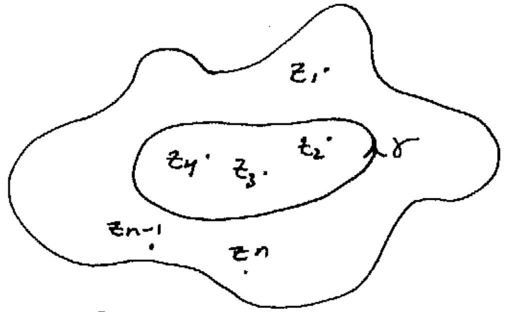
Ex Bestäm principaldelen av $f(z) = \frac{z^3}{(z-1)^2}$ i $z = 1$

α: $f(z) = \begin{bmatrix} z-1 = w \\ z = 1+w \end{bmatrix} = \frac{(w+1)^3}{w^2} = \frac{1}{w^2}(w^2 + 3w^3 + 3w + 1)$:

$= w + 3 + \frac{3}{w} + \frac{1}{w^2} = (z-1) + 3 + \underbrace{\left(\frac{3}{z-1} + \frac{1}{(z-1)^2}\right)}_{\text{principaldelen i } z=1}$

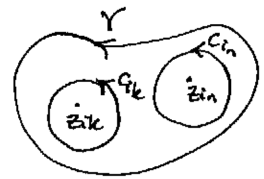
Residusatsen

Antag D är ES, $z_1, \ldots, z_n \in D$
och $f: D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ analytisk
Låt γ vara en enkel sluten kurva i D
som ej går genom någon av punkterna
$z_1, \ldots, z_n$ och som är positivt orienterad.



Då $\int_\gamma f(z)\,dz = 2\pi i \sum_{z_j \text{ innanför } \gamma} \text{Res}\,(f(z); z_j)$

Bevis: Antag $z_1, \ldots, z_m$ innanför γ

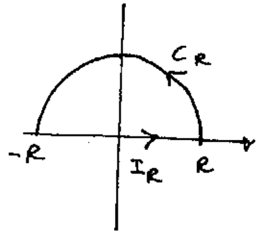


Nu $\int_{\gamma - C_{i_1} - C_{in}} f(z)\,dz = 0$

För $\int_\gamma f(z)\,dz = \int_{C_1} f(z)\,dz + \ldots + \int_{C_{in}} f(z)\,dz =$

$= 2\pi i\left(\text{Res}(f; z_1) + \ldots + \text{Res}(f; z_{in})\right)$

Ex



$$\int_\gamma \frac{dz}{1+z^2} = 2\pi i \, Res\left(\frac{1}{1+z^2}\,;\, i\right)$$

$\gamma = I_R + C_R$

$$= 2\pi i\left(\frac{1}{2z}\right)\Bigg|_{z=1} = 2\pi i\,\frac{1}{2i} = \pi$$

$$\int_\gamma = \int_{I_R} + \int_{C_R} \; ; \; \left|\int_{C_R}\frac{dz}{1+z^2}\right| \le \frac{1}{(R^2-1)}\,\pi R \to 0 \text{ då } R\to+\infty$$

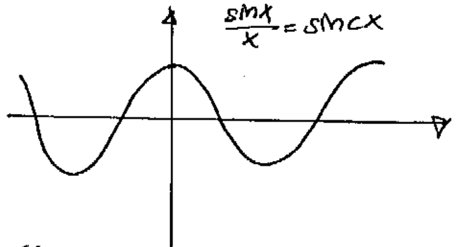
$$\pi = \int_{-R}^{R}\frac{dx}{(1+x^2)} + \int \frac{dz}{1+z^2}$$

Låt $R\to+\infty$

Då $\pi = \int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

Koll: $$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \Big[\arctan x\Big]_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

Ex: $$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx =$$



Går att beräkna med
residukalkyl + många fler integraler

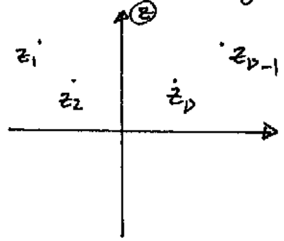
(43)

Sats: Antag P, Q polynom sådana att

(α) $m = \text{grad } Q \ge 2 + \text{grad } P = 2 + n$

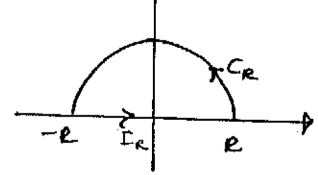
(β) $Q(x) \neq 0, \; \forall x \in \mathbb{R}$

Då: $$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{j=1}^{\nu} Res\left(\frac{P(z)}{Q(z)}\,;\, z_j\right),$$ där $z_1, \dots, z_\nu$ är polerna för $\frac{P(z)}{Q(z)}$ i $Im z > 0$



Bevis $P(z) = a_n z^n + \dots + a_1 z + a_0, \; a_n \neq 0$

$Q(z) = b_m z^m + \dots + b_1 z + b_0 \quad b_m \neq 0$



$$\int_{I_R + C_R}\frac{P(z)}{Q(z)}\,dz = 2\pi i \sum_{j=1}^{\nu} Res\left(\frac{P(z)}{Q(z)}\,;\, z_j\right)$$

$$= \int_{I_R}\frac{P(x)}{Q(x)}\,dx + \int_{C_R}\frac{P(z)}{Q(z)}\,dz$$

$\frac{P(z)}{z^n} \to a_n$ då $|z| \to \infty$. Alltså $\left|\frac{P(z)}{z^n}\right| \le 2|a_n|$

om $|z| \ge R_1$

(44)

Får $|P(z)| \le 2|a_n||z|^n$ om $|z| \ge R_1$

Vidare $\dfrac{Q(z)}{z^m} \to b_m$ då $|z| \to \infty$

Alltså $\left|\dfrac{Q(z)}{z^m}\right| \ge \frac{1}{2}|b_m|$ om $|z| \ge R_2$

Får $|Q(z)| \ge \frac{1}{2}|b_m||z|^m$, $|z| \ge R_2$

Vidare $$\left|\int_{C_R} \frac{P(z)}{Q(z)}\,dz\right| \underset{R \ge \max(R_1, R_2)}{\le} \frac{2|a_n|R^n}{\frac{1}{2}|b_m|R^m}\pi R =$$

$$= \frac{4\pi|a_m|}{|b_m|} \cdot \frac{1}{R^{m-1-n}} \to 0 \text{ då } R \to +\infty$$

$$\therefore R \to \infty: \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{j=1}^{\nu} \operatorname{Res}\left(\frac{P(z)}{Q(z)}; z_j\right)$$

<u>Sats</u>: Antag $a \ge 0$ samma förutsättningar som i föregående sats

Då $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iax}\,dx = 2\pi i \sum_{j=1}^{\nu} \operatorname{Res}\left(\frac{P(z)}{Q(z)} e^{iaz}; z_j\right)$$

<u>Bevis</u>: $\displaystyle\int \frac{P(z)}{Q(z)} e^{iaz}\,dz = \xi_R$

Men om $z \in C_R$ så $|e^{iaz}| = |e^{a(ix-y)}| = e^{-ay} \le 1$

som förut $\xi_R \to 0$ då $R \to +\infty$

<u>Ex</u> Antag $a > 0$

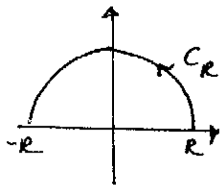
Beräkna $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = J$

<u>Lösn</u>: $$J = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = 2\pi i \operatorname{Res}\left(\frac{e^{iaz}}{z^2+1}; i\right) =$$

$$= 2\pi i \left(\frac{e^{iaz}}{2z}\right)\Bigg|_{z=i} = \pi e^{-a}$$

$$\left[J = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx + i \underbrace{\int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\,dx}_{=0} = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx \right]$$

<u>Jordans lemma</u>



$a > 0$

Antag $|f(z)| \le M(R)$ då $z \in C_R$

Då $$\left|\int_{C_R} f(z) e^{iaz}\,dz\right| \le \frac{\pi}{a} M(R)$$

Bevis: $C_R: \ z = Re^{it}, \ 0 \le t \le \pi$

Då $dz = iRe^{it}dt$ så $\left|\int_{C_R} f(z)e^{iaz}dz\right| =$

$= \left|\int_0^{\pi} f(Re^{it})e^{iaRe^{it}} \, iRe^{it}dt\right| \le$

$\le \int_0^{\pi} |f(Re^{it})| e^{\mathrm{Re}(iaRe^{it})} R\,dt \le$

$\le RM(R)\int_0^{\pi} e^{-aR\sin t}dt = 2RM(R)\int_0^{\frac{\pi}{2}} e^{-aR\sin t}dt \le$

$\le 2RM(R)\int_0^{\frac{\pi}{2}} e^{-\frac{aR^2}{\pi}t}dt \le 2RM(R)\int_0^{\infty} e^{-\frac{aR^2}{\pi}t}dt =$

$= \frac{2RM(R)}{aR^2}\pi = \frac{M(R)\pi}{a}$

Sats Antag P, Q polynom sådana att

(α) $m = \mathrm{grad}\, Q \ge 1 + \mathrm{grad}\, P = 1 + n$

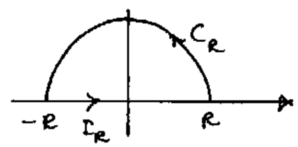
(β) $Q(x) \ne 0, \ \forall x \in \mathbb{R}$

Låt $a > 0$. Då $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iax}dx =$

$= 2\pi i \sum_{j=1}^{\nu} \mathrm{Res}\left(\frac{P(z)}{Q(z)}e^{iaz}; z_j\right)$

där $z_1, \dots, z_\nu$ är $\frac{P(z)}{Q(z)}$:s polen i $\mathrm{Im}\, z > 0$

Bevis:



$\int \frac{P(x)}{Q(x)} e^{iax}dx + \underbrace{\int \frac{P(z)}{Q(z)} e^{iaz}dz}_{= \xi_R} =$

$= 2\pi i \sum \mathrm{Res}\left(\frac{P(z)}{Q(z)} e^{iaz}; z_j\right)$

Påst: $\xi_R \to 0$ då $R \to +\infty$ (sedan klart)

Bevis: Jordan $\Rightarrow |\xi_R| \le \frac{2|a_n|R^n}{\frac{1}{2}|b_m|R^m} \cdot \frac{\pi}{a} =$

$= \frac{4|a_n|\pi}{a|b_m|} \frac{1}{R^{m-n}} \to 0$ då $r \to +\infty$ $\square$

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Residusatsen

D är ES, $z_1, \dots, z_n \in D$

$f: D \setminus \{z_1, \dots, z_n\} \to \mathbb{C}$ analytisk

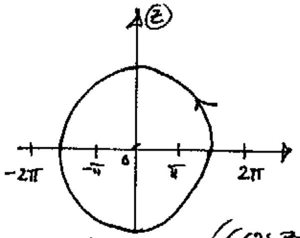
γ enkel sluten positivt orienterad kurva i $D \setminus \{z_1, \dots, z_n\}$

Då $\displaystyle\int_\gamma f(z)\,dz = 2\pi i \sum_{z_j \text{ innanför } \gamma} \operatorname{Res}(f(z); z_j)$

Ex $\displaystyle\oint \cot z \, dz \qquad |z| = \frac{3\pi}{2}$

$\cot z = \dfrac{\cos z}{\sin z}$

$\sin z = 0 \Leftrightarrow z = n\pi$

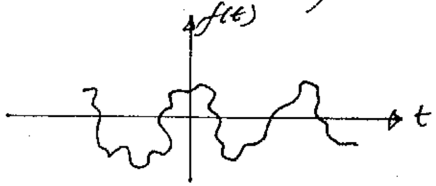


$$\oint \cot z \, dz =$$

$$= 2\pi i \left(\operatorname{Res}\left(\frac{\cos z}{\sin z}; 0\right) + \operatorname{Res}\left(\frac{\cos z}{\sin z}; \pi\right) + \operatorname{Res}\left(\frac{\cos z}{\sin z}; -\pi\right) \right) =$$

$$= 2\pi i \left(\frac{\cos z}{\cos z}\Big|_{z=0} + \frac{\cos z}{\cos z}\Big|_{z=\pi} + \frac{\cos z}{\cos z}\Big|_{z=-\pi} \right) = 6\pi i$$

Fourier transform



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \quad \omega \in \mathbb{R}$$

viktigt: $\displaystyle f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \frac{d\omega}{2\pi}$ (F.A)

(49)

(rep)

Sats Antag P, Q polynom och

(α) $m = \text{grad } Q \geq 1 + \text{grad } P = 1 + n$

(β) $Q(x) \neq 0$, $x \in \mathbb{R}$

Då gäller för $\forall a > 0$ att

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iax}\, dx = 2\pi i \sum_{j=1}^{\nu} \operatorname{Res}\left(\frac{P(z)}{Q(z)} e^{iaz}; z_j\right)$$

där $z_1, \dots, z_\nu$ är polerna för $\dfrac{P(z)}{Q(z)}$ i $\operatorname{Im} z > 0$

Problem Bestäm Fouriertransformen till

$$f(t) = \frac{t}{1+t^2}, \quad t \in \mathbb{R}$$

L: $\displaystyle \hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{t}{1+t^2} e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} \frac{x}{1+x^2} e^{iax}\, dx$, där $a = -\omega$

Fall $\omega < 0$: $\displaystyle \hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{x}{1+x^2} e^{iax}\, dx =$

$$= 2\pi i \operatorname{Res}\left(\frac{z}{1+z^2} e^{aiz}; i\right) = 2\pi i \left(\frac{z e^{aiz}}{2z}\Big|_{z=i}\right) = \pi i e^{-a} =$$

$= \pi i e^{\omega}$

Fall $\omega > 0$: $\displaystyle \hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{x}{1+x^2} e^{iax}\, dx = [x = -\xi] =$

$$= -\int_{-\infty}^{\infty} \frac{\xi}{1+\xi^2} e^{i(-a)\xi}\, d\xi = -2\pi i \left(\frac{z e^{i(-a)z}}{2z}\right)_{z=i} =$$

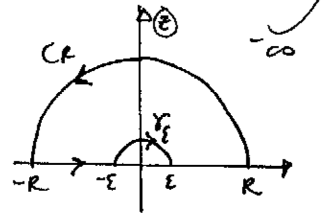
$= -i\pi e^{a} = i\pi e^{-\omega}$

Svar: $\hat{f}(\omega) = -i\pi (\operatorname{sgn} \omega) e^{-|\omega|}$

(50)

Problem: Beräkna $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

$\measuredangle$:



$f(z) = \dfrac{e^{iz}}{z}$

$$\int_{C_R} f(z)\,dz + \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) f(x)\,dx + \int_{\gamma_\varepsilon} f(z)\,dz = 0 \quad (*)$$

$$\left|\int_{C_R} \frac{1}{z} e^{iz}\,dz\right| \underset{\text{Jordan}}{\le} \max_{C_R}\left|\frac{1}{z}\right| \cdot \frac{\pi}{1} = \frac{\pi}{R}$$

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,dz = \begin{bmatrix} z = \varepsilon e^{it} & \pi \ge t \ge 0 \\ dz = \varepsilon i e^{it}dt & \end{bmatrix} = \int_{\pi}^{0} \frac{e^{i\varepsilon e^{it}}}{\varepsilon e^{it}} \varepsilon i e^{it}\,dt =$$

$$= -i\int_0^{\pi} e^{i\varepsilon e^{it}}\,dt \to -i\int_0^{\pi} 1\,dt = -i\pi \quad \text{då } \varepsilon \to 0$$

Vidare: $\displaystyle\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \frac{e^{ix}}{x}\,dx = \int_{\varepsilon<|x|<R} \frac{\cos x + i\sin x}{x}\,dx =$

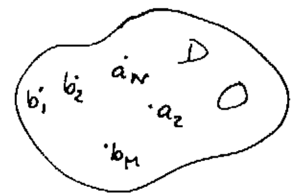
$$= \int_{\varepsilon<|x|<R} \frac{\cos x}{x}\,dx + i\int_{\varepsilon<|x|<R} \frac{\sin x}{x}\,dx = 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx$$

Låt $R \to \infty$ och $\varepsilon \to 0+$ i $(*)$:

$$0 + 2i\int_0^{\infty} \frac{\sin x}{x}\,dx - i\pi = 0$$

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Motargumentprincipen



D domän

$h: D \setminus \{b_1, \dots, b_M\} \to \mathbb{C}$ analytisk
b_k pol av ordning m_k $k = 1, \dots, M$
h:s nollställen $a_1, \dots, a_N$
a_k nollst. av ordning n_k, $k = 1, \dots, N$

γ är en enkel sluten kurva, positivt orienterad, som ligger i D och ej går genom någon av punkterna $a_1, \dots, a_N, b_1, \dots, b_M$

<u>Sats</u>: $\displaystyle\frac{1}{2\pi i}\int_{\gamma} \frac{h'(z)}{h(z)}\,dz = \sum_{\{k;\, a_k \in \Omega\}} n_k - \sum_{\{k;\, b_k \in \Omega\}} m_k \quad (\in \mathbb{Z}) =$

$= N_\gamma - P_\gamma$

Där Ω är det område som ligger innanför γ, och det förutsätts att $\Omega \subseteq D$

Bevis:

a) $h(z) = (z-a_k)^{n_k} \cdot H_k(z)$, där $H_k(z)$ analytisk nära a_k och $H_k(a_k) \neq 0$

$$h'(z) = n_k (z-a_k)^{n_k-1} H_k(z) + (z-a_k)^{n_k} H_k'(z) \rightarrow$$

$$\rightarrow \frac{h'(z)}{h(z)} = \frac{n_k}{z-a_k} + \frac{H_k'(z)}{H_k(z)}; \quad \operatorname{Res}\left(\frac{h'(z)}{h(z)}; a_k\right) = n_k$$

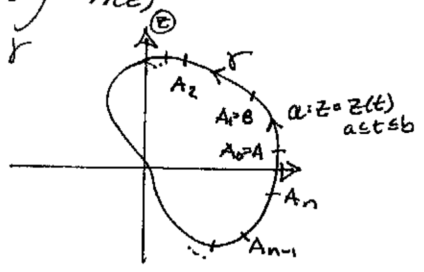
b) $h(z) = (z-b_k)^{-m_k} G_k(z)$, där $G_k(z)$ analytisk nära b_k och $G_k(b_k) \neq 0$

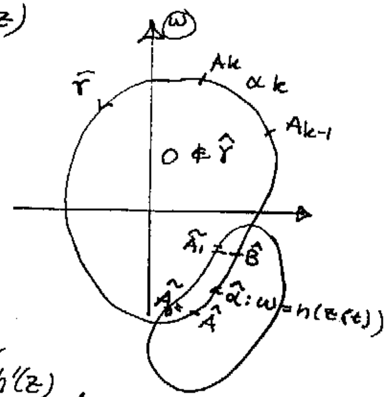
Som ovan $\operatorname{Res}\left(\frac{h'(z)}{h(z)}; b_k\right) = -m_k$

Residuesatsen $\Rightarrow$ satsen

Tolkning av

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz \qquad w = h(z)$$





$$\int_{\tilde{\alpha}} \frac{dw}{w} = \int_a^b \frac{h'(z(t))}{h(z(t))} z'(t)\,dt = \int_\alpha \frac{h'(z)}{h(z)}\,dz$$

$$\int_{\tilde{\alpha}} \frac{dw}{w} = \Big[\ln|w| + i\theta\Big]_{\hat{A}}^{\hat{B}} = \ln\frac{|\hat{B}|}{|\hat{A}|} + i\,\Delta_{\tilde{\alpha}} \arg w$$

$$\int_{\hat{\gamma}} \frac{dw}{w} = \sum_{k=1}^{n} \int \frac{dw}{w} = \sum_{k=1}^{n}\left(\ln\frac{|\hat{A}_k|}{|\hat{A}_{k-1}|} + i\,\Delta_{\hat{\alpha}_k} \arg w\right) =$$

$$= i\sum_{k=1}^{n} \Delta_{\hat{\alpha}_k} \arg w = i\,\Delta_{\hat{\alpha}} \arg w$$

$$\int_\gamma \frac{h'(z)\,dz}{h(z)} = \sum_{k=1}^{n}\int_{\alpha_k} \frac{h'(z)}{h(z)}\,dz = \sum_{k=1}^{n}\int_{\tilde{\alpha}_k} \frac{dw}{w} =$$

$$= \int_{\hat{\gamma}} \frac{dw}{w} = i\,\Delta_{\hat{\gamma}} \arg w$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \frac{1}{2\pi}\Delta_{\hat{\gamma}} \arg w = \frac{1}{2\pi}\Delta_\gamma \arg f(z)$$

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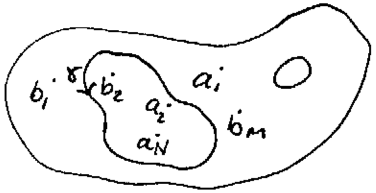
Argumentprincipen



$h: D \setminus \{b_1, \dots, b_M\} \to \mathbb{C}$ analytisk

b_k pol av ordning m_k, $k = 1, \dots, M$

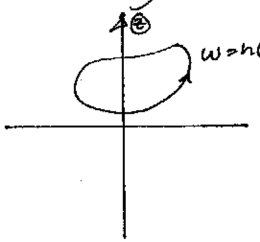
$a_1, \dots, a_N$ h:s nollställen

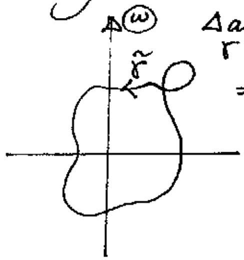
a_k nollställe av ordning n_k, $k = 1, \dots, N$

γ är en enkel sluten positivt orienterad kurva i $D \setminus \{a_1, \dots, a_N, b_1, \dots, b_M\}$ som innesluter ett område $\Omega \subseteq D$

Sats: $\displaystyle \frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \sum_{\{k;\, a_k \in \Omega\}} n_k - \sum_{\{k;\, b_k \in \Omega\}} m_k$

Sats: $\displaystyle \frac{1}{2\pi i}\int \frac{h'(z)}{h(z)}\,dz = \frac{1}{2\pi}\Delta \arg h(z)$





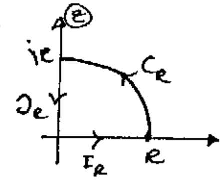
$\Delta \arg_\gamma h(z) = \Delta \arg_{\tilde{\gamma}} w$

Argumentprincipen: $\displaystyle \frac{1}{2\pi}\Delta \arg_\gamma h(z) = \sum_{a_k} n_k - \sum_{b_k \in \Omega} m_k$

Exempel: Antag $f(z) = z^3 - 2z^2 + 4$

Hur många nollställen har funktionen i första kvadranten?

α:



På I_R: $f(x) = x^3 - 2x^2 + 4$

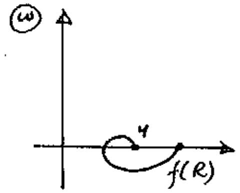
$0 \le x \le R$

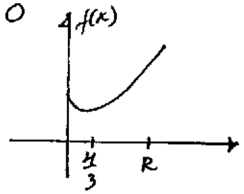
$f'(x) = 3x^2 - 4x = 3x(x - \frac{4}{3})$

$x = \frac{4}{3}$ ger min.

$\gamma: I_R + C_R + J_R$ $\quad f(\frac{4}{3}) = \frac{64}{27} - \frac{32}{9} + 4 = \frac{64 - 96 + 4 \cdot 27}{27} > 0$

$\Delta \arg_{I_R} f(z) = 0$

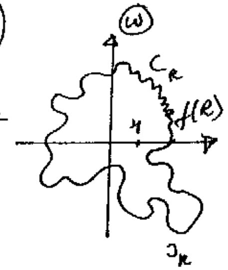




På C_R: $\quad C_R: z = Re^{it} \quad 0 \le t \le \frac{\pi}{2}$

$f(Re^{it}) = R^3 e^{3it}\underbrace{\left(1 - \frac{2}{Re^{it}} + \frac{4}{R^3 e^{3it}}\right)}_{\approx 1,\ R \text{ stort}}$

$\Delta \arg_{C_R} f(z) \approx \frac{3\pi}{2}$, R stort



På J_R: $f(iy) = 2y^2 + 4 - iy^3 \quad R \ge y \ge 0$

$\mathrm{Re}\, f(iy) > 0$, $\quad R \ge y \ge 0$

$\mathrm{Im}\, f(iy) < 0$, $\quad R \ge y > 0$

$\Delta \arg_{J_R} f(z) \approx \frac{\pi}{2}$, R stort

$\Delta_\gamma \arg h(z) \approx 0 + \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi$, R stort

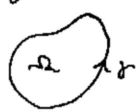
Svar: 1 enl. a.p

Sats (Rouché) Låt γ vara en enkel sluten kurva som innesluter ett område Ω. Antag f & g analytiska i en omgivning av $\gamma \cup \Omega$ och sådana att:

$$|f(z)+g(z)| < |f(z)|, \quad \forall z \in \gamma$$

Då har f, g lika många nollställen i Ω räknade med multiplicitet.

Bevis OBS: $f(z) \neq 0$ och $g(z) \neq 0$ på γ



Låt $0 \le t \le 1$ och sätt $h_t(z) =$

$= f(z) - t(f(z) + g(z))$

Då är h_t analytisk, $h_0 = f$ och $h_1 = -g$

Vidare så är

$|h_t(z)| \geq |f(z)| - t|f(z)+g(z)|$ så

$|h_t(z)| \geq |f(z)| - |f(z)+g(z)| \geq \varepsilon \geq 0, \ \forall z \in \gamma$

Vidare är $\frac{1}{2\pi i}\int \frac{h'(z)}{h(z)}\,dz$ antalet nollställen för h_t i Ω

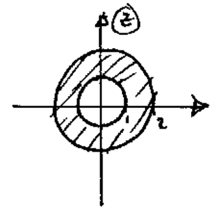
och $t \to \frac{1}{2\pi i}\int \frac{h'(z)}{h(z)}\,dz$ konstant & heltalsvärd.

Antalet nollställen för $f = h_0$ i Ω =

= Antalet nollställen för $g = -h_1$ i Ω

Ex: Hur många nollställen har

$$g(z) = 3z^3 - 2z^2 + 2iz - 8 \quad i \quad 1 < |z| < 2$$



L: på $|z| = 1$:

$$|\underbrace{3z^3 - 2z^2 + 2iz - 8}_{g(z)} \underbrace{+8}_{f(z)}| \le$$

$\le 3 + 2 + 2 = 7 < 8 = |f(z)|$

Utvidgad Rouché: $g(z)$ och $f(z)$ har lika många nollställen i $|z| \le 1$, dvs 0

På $|z| = 2$:

$$|\underbrace{3z^3 - 2z^2 + 2iz - 8}_{g(z)} + \underbrace{(-3z^3)}_{f(z)}| = 20 < 24 = |-3z^3| =$$
$$= |f(z)|$$

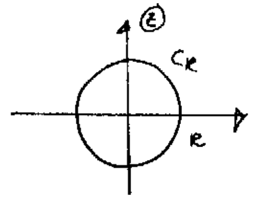
$f(z)$ och $g(z)$ har lika många nollställen i $|z| < 2$ dvs 3   Svar: $3 - 0 = 3$

<u>Sats</u> (Algebrans fundamentalsats)

Låt $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$ där $n \geq 1$

Då har $p(z) = 0$ exakt n rötter r.m.m

<u>Bevis</u>



På C_R: $|p(z) + (-z^n)| = |a_{n-1} z^{n-1} + \dots + a_0| \leq$

$\leq |a_{n-1}| R^{n-1} + \dots + |a_1| R + |a_0| =$

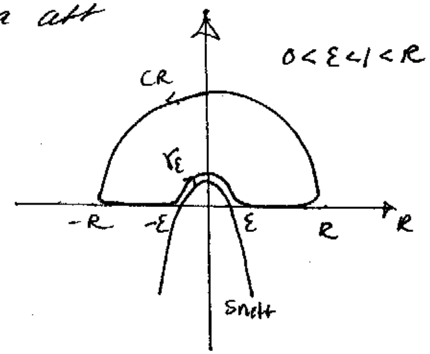
$= R^n \left(\frac{|a_{n-1}|}{R} + \dots + \frac{|a_1|}{R^{n-1}} + \frac{|a_0|}{R^n} \right) < R^n = |-z^n|$ då R stort

Rouché $\Rightarrow$ $p(z)$ och $f(z)$ har lika många nollställen i $|z| < R$ r.m.m. Men $-z^n$ har n sådana nollställen, vilket visar satsen.

<u>Ex sid 163</u>

antag $-1 < p < 1$ Visa att

$$\int_{-\infty}^{\infty} \frac{x^p}{1+x^2}\,dx = \frac{\pi}{2} \frac{1}{\cos \frac{p\pi}{2}}$$



α: $f(z) = \frac{z^p}{1+z^2}$ $\qquad \widetilde{\log} z = \ln|r| + i\theta$

$z = re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$$\left\{ \begin{array}{l} z^p = e^{p \widetilde{\log} z} = e^{p(\ln r + i\theta)} = \\ = r^p e^{ip\theta} \\ z = re^{i\theta}, \ -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{array} \right.$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{-\varepsilon} f(x)\,dx + \int_{\gamma_\varepsilon} f(z)\,dz + \int_{\varepsilon}^{R} f(x)\,dx = 2\pi i \operatorname{Res}(f(z); i)$$

$$\left| \int_{C_R} f(z)\,dz \right| \underset{\text{M.L}}{\leq} \frac{R^p}{R^2-1} \overset{\text{mult m. kurvans längd}}{\downarrow} \pi \cdot R \to \infty \text{ då } R \to +\infty, \text{ ty } p+1 < 2$$

$$\left| \int_{\gamma(\varepsilon)} f(z)\,dz \right| \underset{\text{M.L}}{\leq} \frac{\varepsilon^p}{1-\varepsilon^2} \pi \varepsilon \to 0 \text{ då } \varepsilon \to 0^+, \text{ ty } \xi + 1 > 0$$

$$2\pi i \operatorname{Res}(f(z); i) = 2\pi i \left(\frac{z^p}{2z} \right)_{z=i} = \pi i^p = \pi e^{i\frac{p\pi}{2}}$$

$\varepsilon \to 0$, $R \to +\infty$ i (*)

$$0 + \int_{-\infty}^{0} \frac{|x|^p e^{ip\pi}}{1+x^2}\,dx + \int_0^{\infty} \frac{x^p}{1+x^2}\,dx = \pi e^{\frac{ip\pi}{2}}$$

$$\int_0^{\infty} \frac{x^p}{1+x^2}\,dx \left(e^{ic\pi} + 1 \right) = \pi e^{\frac{ip\pi}{2}}$$

$$\int_0^{\infty} \frac{x^p}{1+x^2}\,dx \cdot 2\cos \frac{p\pi}{2} = \pi$$

$$\int_0^{\infty} \frac{x^p}{1+x^2}\,dx = \frac{\pi}{2} \frac{1}{\cos \frac{p\pi}{2}} \Rightarrow \int_0^{\infty} \frac{dx}{1+x^\beta} = \frac{\pi/\beta}{\sin \pi/\beta}$$

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Möbiusavbildningar

Antag $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A \neq 0$

Sätt $w = T_A(z) = T(z) = \dfrac{az+b}{cz+d}$

T möbiusavbildning

$D_{T_A} = \mathbb{C}$ om $c=0$

$D_{T_A} = \mathbb{C} \setminus \{-\frac{d}{c}\}$ om $c \neq 0$

Problem: Sök V_{T_A}

α: $c=0$: $w = \frac{a}{d}z + \frac{b}{d}$

$dw = az + b$

$z = \dfrac{dw-b}{a}$ $\quad V_{T_A} = \mathbb{C}$

$c \neq 0$: $w = \dfrac{az+b}{cz+d}$

$\Leftrightarrow czw + dw = az + b$

$\Leftrightarrow z(cw-a) = -dw + b$

Fall $w = \frac{a}{c}$: $0 = -d\frac{a}{c} + b = -\frac{1}{c}\det A$

$\det A = 0$ Motsägelse!

Fall $w \neq \frac{a}{c}$: $z = \dfrac{dw-b}{-cw+a} = T_{\underline{\underline{B}}}(w)$ $\quad V_{T_A} = \mathbb{C} \setminus \{\frac{a}{c}\}$

$\underline{\underline{B}} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

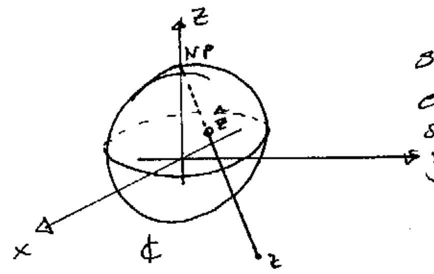
Def $c=0$: $T(\infty) = \infty$

$c \neq 0$: $T(\infty) = \frac{a}{c}$

$T(-\frac{d}{c}) = \infty$

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$T = T_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $\qquad T_A^{-1} = T_B$



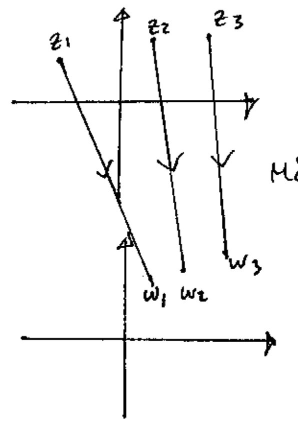
står vid NP och drar
ett rakt streck mot z.
skär sfär-skalet i $\hat{z}$

$z \to \infty$

$\hat{z} \to NP$

$\hat{\mathbb{C}} =$ 

Problem:



$$\frac{w-w_1}{w-w_2} \; \frac{w_3-w_2}{w_3-w_1} = \frac{z-z_1}{z-z_2} \; \frac{z_3-z_2}{z_3-z_1}$$

$w = f(z)$ blir Möbius

Speialfall

w	z
0	0
∞	i
i	$2i$

$$\frac{w-0}{w-\infty}\,\frac{i-\infty}{i-0} = \frac{z-0}{z-1}\,\frac{2i-1}{2i-0}$$

$$\frac{w}{i} = \frac{z}{z-1}\,\frac{2i-1}{2i}\,;\quad w = \left(i-\frac{1}{2}\right)\frac{z}{z-1}$$

Avbildningsegenskaper

$$w = \frac{1}{z}$$

Problem: Hur avbildas linjer/cirklar?

Svar: På linje/cirkel.

Antag $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ och

$$4\alpha\delta + \beta^2 + \gamma^2 > 0$$

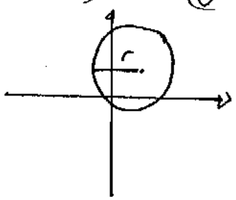
Tolka $\alpha(x^2+y^2) + \beta x - \gamma y = \delta$

Fall $\alpha = 0$: $\begin{cases} \beta x + \gamma y = \delta \\ (\beta,\gamma) \neq (0,0) \end{cases}$ linje

Fall $\alpha \neq 0$: $x^2 + y^2 + \frac{\beta}{\alpha}x + \frac{\gamma}{\alpha}y = \frac{\delta}{\alpha}$

$$\left(x+\frac{\beta}{2\alpha}\right)^2 + \left(y+\frac{\gamma}{2\alpha}\right)^2 = \frac{\delta}{\alpha} + \frac{\beta^2}{4\alpha^2} + \frac{\gamma^2}{4\alpha^2}$$

$$\left(x+\frac{\beta}{2\alpha}\right)^2 + \left(y+\frac{\gamma}{2\alpha}\right)^2 = \frac{4\alpha\delta+\beta^2+\gamma^2}{4\alpha^2}$$



cirkel m $r = \frac{\sqrt{4\alpha\delta+\beta^2+\gamma^2}}{2|\alpha|}$

centrum i $\left(-\frac{\beta}{2\alpha}, \frac{-\gamma}{2\alpha}\right)$

Tillbaka till $w = \frac{1}{z}$

Låt $z = x+iy$, $w = u+iv$

Får: $u+iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$$\begin{cases} u = \frac{x}{x^2+y^2} & x = \frac{u}{u^2+v^2} \\ v = \frac{-y}{x^2+y^2} & y = \frac{-v}{u^2+v^2} \end{cases}$$

Antag (ε):

$$\alpha\left(\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right) + \beta\frac{u}{u^2+v^2} + \gamma\frac{(-v)}{u^2+v^2} = \delta$$

$$\frac{\alpha}{u^2+v^2} + \beta\frac{u}{u^2+v^2} - \gamma\frac{v}{u^2+v^2} = \delta$$

$$\alpha + \beta u - \gamma v = \delta(u^2+v^2)$$

$$\delta(u^2+v^2) + (-\beta)u + \gamma v = \alpha$$

$$4\delta\alpha + (-\beta)^2 + \gamma^2 = 4\alpha\delta + \beta^2 + \gamma^2 > 0$$

$\therefore$ linje/cirkel i uv-planet.

Allmänna fallet $(c \neq 0)$

$$w = T(z) = \frac{az+b}{cz+d} = \overset{\text{konst.}}{A} + \frac{\overset{\text{konst}}{B}}{cz+d}$$

$ad - bc \neq 0$

$z \to cz = z_1 \to z_1 + d = z_1 \to \frac{1}{z_2} = z_3 \to \beta z_3 =$

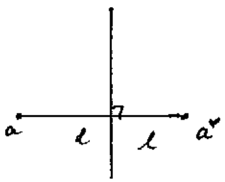
$= \ell/c \to \ell/c \to \ell/c \longrightarrow \ell/c \longrightarrow \ell/c \to$

$= z_4 \to w = A + z_4$

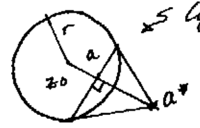
$\longrightarrow \ell/c$

Inversa Punkter

relativt linje:



relativt cirkel:



a behöver ej ligga i cirkeln

a, a^* inversa punkter

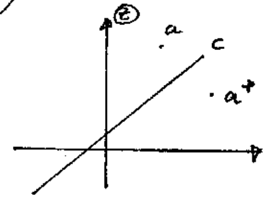
$a = z_0\,,\quad a^* = \infty$

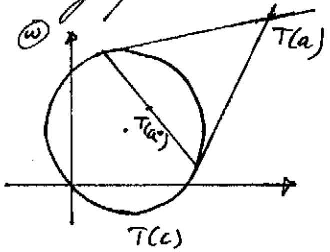
$a = \infty\,,\quad a^* = z_0$

enl. def → om $a \neq z_0$, $a \neq \infty$

$$(a - z_0)\,\overline{(a^* - z_0)} = r^2$$

Antag T, Möbius och C är linje/cirkel.





a, a^* inversa rel. c

Sats: $T(a), T(a^*)$ inversa relativt $T(c)$

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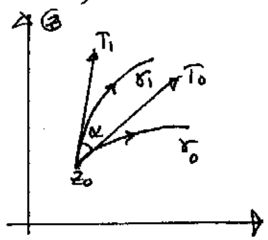
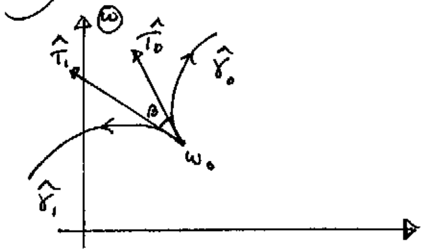
Konform avbildning





$w = f(z)$ analytisk

$w_0 = z_0$

$\gamma_0 \to \hat{\gamma}_0$

$\gamma_1 \to \hat{\gamma}_1$

Sats: Om $f'(z) \neq 0$ så är $\alpha = \beta$

Bevis: $\gamma_0 = \begin{cases} \gamma_0(t), & 0 \le t \le 1 \\ \gamma_0(0) = z_0 \end{cases}$

$$\hat{\gamma}_0 = \begin{cases} f(\gamma_0(t)), & 0 \le t \le 1 \\ f(\gamma_0(0)) = w_0 \end{cases}$$

$T_0 = \gamma_0'(0)$ (förutsätts vara $\neq 0$)

$\hat{T}_0 = f'(\gamma_0(0))\,\gamma_0'(0) = f'(z_0)\,T_0$

$f'(z_0) = A_0 e^{i\theta_0}$ $\quad (A_0 > 0,\ \theta_0 \in [0, 2\pi[\,)$

$\hat{T}_0 = e^{i\theta_0} A_0 T_0$

$T_0 \to \hat{T}_0$

sträck i skalan A_0

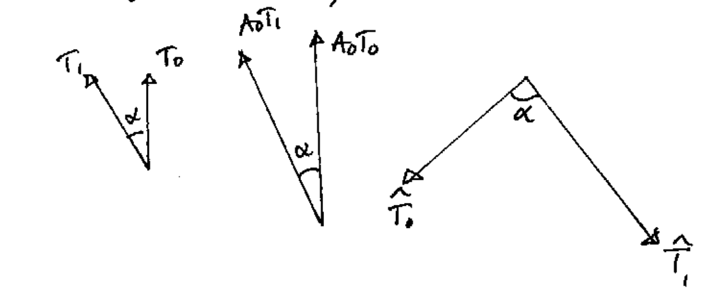
vrid θ_0 radianer i positiv led

(66)

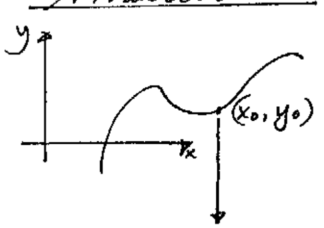
$T_1 \to \hat{T}_1$

Sträck i skalan A_0

Vrid θ_0 rad. i positiv led



Nivåkurva



$\{(x,y);\ u(x,y) = c\}$

$n = \nabla u(x_0, y_0)$

om $\neq 0$

Sats: Antag $f(z)$ analytisk och $f'(z_0) \neq 0$

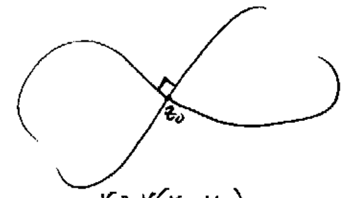
Skriv $f = u + iv$ och $z_0 = x_0 + iy_0$

Nivåkurvorna $u(x,y) = u(x_0, y_0)$

$v(x,y) = v(x_0, y_0)$

skär varandra under rät vinkel i z_0.

Bevis



$$\nabla u(x_0, y_0) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \Big/ (x_0, y_0)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$if'(z) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Om $\nabla u(x_0, y_0) = 0$ så $\frac{\partial u}{\partial x}(x_0, y_0) = 0$

och $\frac{\partial v}{\partial x}(x_0, y_0) \underset{CR}{=} -\frac{\partial u}{\partial y}(x_0, y_0) = 0$ dvs $f'(z_0) = 0$

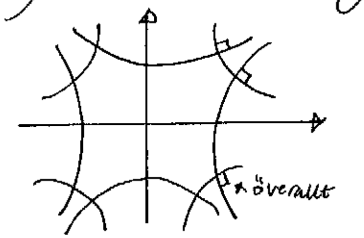
Om $\nabla v(x_0, y_0) = 0$ pss $f'(z_0) = 0$ går ej

$\therefore \underbrace{\nabla u(x_0, y_0)}_{n_1} \neq 0$ & $\underbrace{\nabla v(x_0, y_0)}_{n_2} \neq 0$

$$n_1 \cdot n_2 = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right) \Big/ (x_0, y_0) \overset{CR}{=}$$

$$= \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial x} + \left(-\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial y}\right) \Big/_{(x_0, y_0)} = 0$$

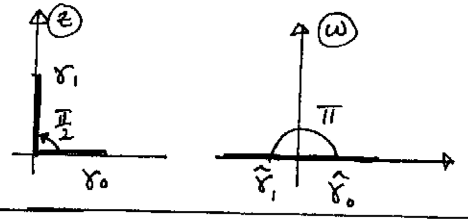
Ex: $f(z) = z^2 = x^2 - y^2 + 2ixy \qquad x^2 - y^2 = c$



ligger olika, beroende på om c är pos. el. neg.

Ex $f(z) = z^2$

$f'(z) = 2z = 0$ om $z = 0$



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Harmoniskt Konjugat

Antag $u: D \to \mathbb{R}$ harmonisk.

Vi säger att $v: D \to \mathbb{R}$ är ett harmoniskt konjugat till u om

$$f(z) = u(x,y) + iv(x,y), \quad z = x+iy \in D$$

är analytisk

Sats: Om v_1, v_2 harmoniskt konjugat till u så är $v_1 - v_2 = $ konst.

Bevis: $f_1 = u + iv_1$ analytisk

$f_2 = u + iv_2$ analytisk

$\Rightarrow f_1 - f_2 = i(v_1 - v_2)$ analytisk

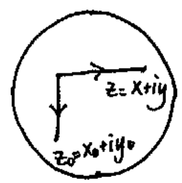
$$CR \to \left(\frac{\partial(v_1-v_2)}{\partial x}, \frac{\partial(v_1-v_2)}{\partial y}\right) = (0,0) \Rightarrow v_1 - v_2 = \text{konst.}$$

ty D är en domän

Sats: Antag $D = \{z \in \mathbb{C};\ |z - z_0| < r\}$ och $u: D \to \mathbb{R}$ harmonisk. Då har u ett harmoniskt konjugat

Bevis:

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0,s)\,ds - \int_{x_0}^{x} \frac{\partial u}{\partial y}(t,y)\,dt$$



$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}(x,y)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}(x_0,y) - \int_{x_0}^{x} \frac{\partial^2 u}{\partial y^2}(t,y)\,dt =$$

$$= \frac{\partial u}{\partial x}(x_0,y) + \int_{x_0}^{x} \frac{\partial^2 u}{\partial x^2}(t,y)\,dt =$$

$$= \frac{\partial u}{\partial x}(x_0,y) + \left[\frac{\partial u}{\partial x}(t,y)\right]_{t=x_0}^{x} = \frac{\partial u}{\partial x}(x_0,y) + \frac{\partial u}{\partial x}(x,y) -$$

$$- \frac{\partial u}{\partial x}(x_0,y) = \frac{\partial u}{\partial x}(x,y)$$

CR gäller, u och v är $C^2 \Rightarrow v$ harmoniskt konjugat till u.

$\boxed{\text{Om } u: D \to \mathbb{R} \text{ harmonisk och } D \text{ är ES så har } u \text{ ett harmoniskt konjugat.}}$

Ex: Låt $u(x,y) = e^x(x\cos y - y\sin y)$. Bestäm en analytisk funktion $f(z)$ sådan att

$$\mathrm{Re}\,f(z) = u(x,y), \quad z = x + iy$$

Li: $(\Delta x = 0)$ Låt $f = u + iv$

Har $f'(z) = u_x' + iv_x' = u_x' - iu_y' =$

$= e^x(x\cos y - y\sin y) + e^x\cos y - ie^x(-\sin y - y\cos y - x\sin y)$

För: $f'(x + i\cdot 0) = xe^x + e^x$

$f'(z)$ och $ze^z + e^z$ hela funktioner som är lika på x-axeln.

$\therefore f'(z) = ze^z + e^z, \ \forall z \in \mathbb{C} \Rightarrow f(z) = ze^z + iA \quad A \in \mathbb{R}$

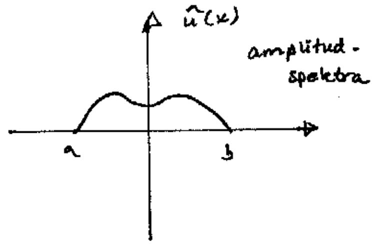
ty $ze^z\big|_{z=0} = u(0,0) = 0$

Fouriertransform

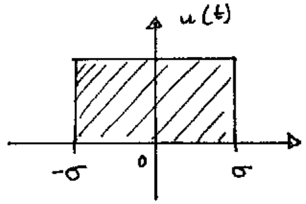
Givet $u: \mathbb{R} \to \mathbb{C}$ och $\int_{-\infty}^{\infty} |u(t)|\,dt < +\infty$

Vi definierar Fouriertransformen $\hat{u}: \mathbb{R} \to \mathbb{C}$ genom att $\hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-ixt}\,dt \quad , x \in \mathbb{R}$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(x)e^{ixt}\frac{dx}{dt} =$$

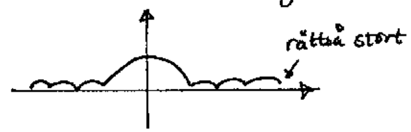


Ex:

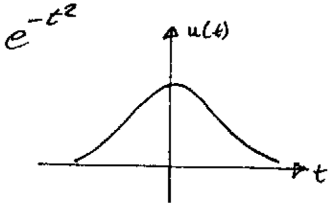


$$\hat{u}(x) = \int_{-\infty}^{\infty} u(t)e^{-ixt}\,dt = \int_{-\sigma}^{\sigma} 1\cdot e^{-ixt}\,dt =$$

$$= \int_{-\sigma}^{\sigma} \cos xt\,dt = 2\int_0^{\sigma} \cos xt\,dt = 2\left[\frac{\sin xt}{x}\right]_0^{\sigma} = \frac{2\sin\sigma x}{x}$$



Ex: $u(t) = e^{-t^2}$



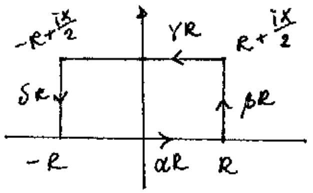
$$\int_{-\infty}^{\infty} u(t)\,dt = \sqrt{\pi}$$

Visar sig att $\hat{u}(x) = \sqrt{\pi}\,e^{-\frac{x^2}{4}}$

$$\boxed{\int_{-\infty}^{\infty} e^{-t^2}e^{-ixt}\,dt = \sqrt{\pi}\,e^{-\frac{x^2}{4}}}$$

Fall $x>0$: $\hat{u}(x) = \int_{-\infty}^{\infty} e^{-t^2-ixt}\,dt = e^{-\frac{x^2}{4}} \int_{-\infty}^{\infty} e^{-(t+\frac{ix}{2})^2}\,dt =$

$$= \int_{-\infty}^{\infty} e^{-(t+\frac{ix}{2})^2}\,dt = \int_{-\infty}^{\infty} e^{-(\xi+\frac{ix}{2})^2}\,d\xi \;;\quad e^{-\zeta^2}$$



(✿)

$$\int_{\alpha_R+\beta_R+\gamma_R+\delta_R} e^{-\zeta^2}\,d\zeta = 0 \quad (*)$$

På β_R: $\left|\int_{\beta_R} e^{-\zeta^2}\,d\zeta\right| = \max_{0\le\eta<\frac{x}{2}} \left|e^{-(R+i\eta)^2}\right| \cdot \frac{x}{2} \le \max_{0\le\eta\le\frac{x}{2}} e^{\eta^2-R^2}\frac{x}{2} \le$

$\le e^{\frac{x^2}{4}-R^2}\frac{x}{2} \to 0$ då $R\to+\infty$

på δ_R: $\left|\int_{\delta_R} e^{-\zeta^2}\,d\zeta\right| \overset{\text{M.L}}{\le} \max_{0\le\eta\le\frac{x}{2}} \left|e^{-(-R+i\eta)^2}\right| \frac{x}{2} \le$

$\le \max_{0\le\eta\le\frac{x}{2}} e^{\eta^2-R^2}\frac{x}{2} \le e^{\frac{x^2}{4}-R^2}\frac{x}{2} \to 0$ då $R\to+\infty$

$$\int_{\alpha_R} e^{-\zeta^2}\,d\zeta + \int_{\beta_R} e^{-\zeta^2}\,d\zeta + \int_{\gamma_R} e^{-\zeta^2}\,d\zeta + \int_{\delta_R} e^{-\zeta^2}\,d\zeta = 0$$

$\left[\begin{array}{l}\zeta = \xi + i\eta \\ z = x+iy\end{array}\right]$ Låt $R\to\infty$:

$$\int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi + 0 + \int_{\infty}^{-\infty} e^{-(\xi+i\frac{x}{2})^2}\,d\xi + 0 = 0$$

$$\int_{-\infty}^{\infty} e^{-(\xi+i\frac{x}{2})^2}\,d\xi = \int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi = \sqrt{\pi}$$

(✿) Klar då $x>0$

Fall $x<0$: $\hat{u}(x) = \int_{-\infty}^{\infty} e^{-t^2}\cos xt\,dt = \int_{-\infty}^{\infty} e^{-t^2}\cos(-x)t\,dt =$

$= \sqrt{\pi}\, e^{-\frac{(-x)^2}{4}}$

(✿) klar.

$\hat{u} = \mathcal{F}u$; $u(t) \supset \hat{u}(x)$ $\quad u(t) \underset{\mathcal{F}}{\supset} \hat{u}(x)$

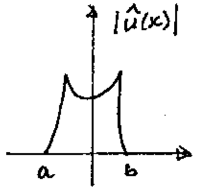
Egenskaper: 1) $\mathcal{F}$ linjär

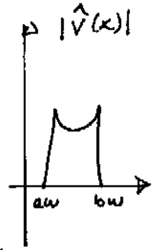
2) Multiplikation med karaktär:

$u(t) \supset \hat{u}(x)$

$e^{i\omega t}u(t) \supset \hat{u}(x-\omega)$

B $\displaystyle\int_{-\infty}^{\infty} e^{iwt}u(t)e^{-ixt}\,dt = \int_{-\infty}^{\infty} u(t)e^{-i(x-w)t}\,dt = \hat{u}(x-w)$





3. Skalning $\quad u(bt) \supset \frac{1}{|b|}\hat{u}\left(\frac{x}{b}\right),\ b \neq 0$

B: $\displaystyle\int_{-\infty}^{\infty} u(bt)e^{-ixt}\,dt = \left[\begin{matrix}\text{Fall } b<0\\ s = bt\end{matrix}\right] = \frac{1}{b}\int_{\infty}^{-\infty} u(s)e^{-i\frac{x}{b}s}\,ds =$

$\displaystyle = \frac{1}{-b}\int_{-\infty}^{\infty} u(s)e^{-i\frac{x}{b}s}\,ds = \frac{1}{|b|}\hat{u}\left(\frac{x}{b}\right)$

Koroll: u jämn $\Rightarrow$ $u(t) \supset \hat{u}(x) \Rightarrow \hat{u}$ jämn

$\qquad\qquad u(t) = u(-t) \supset \hat{u}(-x)$

u udda $\Rightarrow$ -övning- $\Rightarrow \hat{u}$ udda

Faltning $\quad$ Antag $u, v : \mathbb{R} \to \mathbb{C}$ och

$\displaystyle\int_{-\infty}^{\infty} |u(t)|\,dt < +\infty, \quad \int_{-\infty}^{\infty} |u(t)|\,dt < \infty$

Man definierar faltningen $u * v$ av u och v genom att:

$$(u*v)(t) = \int_{-\infty}^{\infty} u(t-s)v(s)\,ds, \quad t \in \mathbb{R}$$

Sats: $(u*v)(t) \underset{\mathcal{F}}{\supset} \hat{u}(x)\hat{v}(x)$

Beviset bygger på $e^{z} \cdot e^{w} = e^{z+w}$

Bevis: $\displaystyle\int_{-\infty}^{\infty} (u*v)(t)e^{-itx}\,dt =$

$\displaystyle = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(t-s)v(s)\,ds\right)e^{-itx}\,dt =$

$\displaystyle = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(t-s)v(s)e^{-itx}\,dt\right)ds =$

$\displaystyle = \int_{-\infty}^{\infty} v(s)\left(\int_{-\infty}^{\infty} u(t-s)e^{-itx}\,dt\right)ds =$ $\quad t = s+r$

$\displaystyle = \int_{-\infty}^{\infty} v(s)\left(\int_{-\infty}^{\infty} u(r)\underbrace{e^{-i(s+r)x}}_{e^{-isx}\cdot e^{-irx}}\,dr\right)ds =$

$\displaystyle = \int_{-\infty}^{\infty} v(s)e^{-isx}\underbrace{\left(\int_{-\infty}^{\infty} u(r)e^{-irx}\,dr\right)}_{\hat{u}(x)}ds =$

$\displaystyle = \hat{u}(x)\int_{-\infty}^{\infty} v(s)e^{-isx}\,ds = \hat{u}(x)\hat{v}(x)$

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Möbiusavbildningar (Linear... s. 1.9.6)

Antag $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A \neq 0$

Sätt $w = T_A(z) = T(z) = \dfrac{az+b}{cz+d}$

T: möbiusavbildning

$\mathcal{D}_{T_A} = \mathbb{C}$ om $c=0$

$\mathcal{D}_{T_A} = \mathbb{C} \setminus \{-\frac{d}{c}\}$ om $c \neq 0$

Ex sök V_{T_A}

Lösn. $\underline{c=0}$: $w = \frac{a}{d}z + \frac{b}{d}$ (d kan ej vara noll, ty $\det A \neq 0$)

$dw = az+b$

$z = \dfrac{dw-b}{a}$ $\qquad V_{T_A} = \mathbb{C}$

$\underline{c \neq 0}$: $w = \dfrac{az+b}{cz+d} \Rightarrow z(cw-a) = -dw+b$

fall $w = \frac{a}{c}$ $\qquad 0 = -d\frac{a}{c} + b = -\frac{1}{c}\det A$

motsägelse att $\det A = 0$

fall $w \neq \frac{a}{c}$ $\qquad z = \dfrac{dw-b}{-cw+a} = T_B(w)$

$\therefore V_{T_A} = \mathbb{C} \setminus \{\frac{a}{c}\}$, $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Def $\quad c=0: \ T(\infty) = \infty$

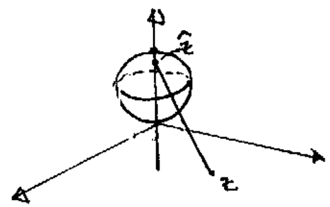
$c \neq 0: \ T(\infty) = \frac{a}{c}$

$T(-\frac{d}{c}) = \infty$

Ger: $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$T = T_A : \hat{\mathbb{C}} \xrightarrow[1\text{-}1]{} \hat{\mathbb{C}}$ omvändbart entydig

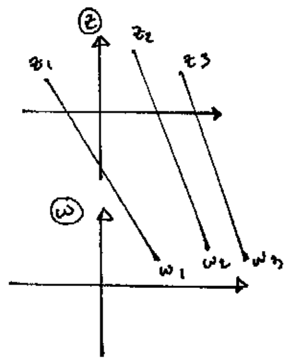
$T_A^{-1} = T_B$



$z \to \infty \Rightarrow \hat{z} \to NP$

$\hat{\mathbb{C}} =$ ⊖ riemanssfären

Ex



$$\frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1} = \frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1}$$

$w = f(z)$ blir möbius

specialfall

w	z
0	0
∞	1
i	$2i$

$$\frac{w-0}{\cancel{w-\infty}_{1}} \cdot \frac{\overset{1}{\cancel{i-\infty}}}{i-0} = \frac{z-0}{z-1} \cdot \frac{2i-1}{2i-0}$$

$$\Rightarrow \frac{w}{i} = \frac{z(2i-1)}{(z-1)2i} \Rightarrow w = \left(i - \frac{1}{2}\right)\frac{z}{z-1}$$

Avbildningsegenskaper

$$w = \frac{1}{z}$$

Ex: Hur avbildas linjer och cirklar?

Svar: På linjer och cirklar. (en linje på en cirkel el. en linje)

Antag $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ och $4\alpha\delta + \beta^2 + \gamma^2 > 0$

Tolka $\alpha(x^2+y^2) + \beta x + \gamma y = \delta$

Fall $\alpha = 0$

$$\begin{cases} \delta = \beta x + \gamma y \\ (\beta, \gamma) \neq (0,0) \end{cases}$$

Detta är en linje

Fall $\alpha \neq 0$

$$x^2 + y^2 + \frac{\beta}{\alpha} x + \frac{\gamma}{\alpha} y = \frac{\delta}{\alpha}$$

kvadr. kompl $\left(x + \frac{\beta}{2\alpha}\right)^2 + \left(y + \frac{\gamma}{2\alpha}\right)^2 = \frac{\delta}{\alpha} + \frac{\beta^2}{4\alpha^2} + \frac{\gamma^2}{4\alpha^2}$

$$\left(x + \frac{\beta}{2\alpha}\right)^2 + \left(y + \frac{\gamma}{2\alpha}\right)^2 = \frac{4\alpha\delta + \beta^2 + \gamma^2}{4\alpha^2}$$

$$r = \frac{\sqrt{4\alpha\delta + \beta^2 + \gamma^2}}{2|\alpha|} \quad \text{cirkel!}$$

Tillbaka till $w = \frac{1}{z}$

Låt $x + iy = z$, $u + iv = w$

Får $u + iv = \frac{1}{x+iy} = \frac{x - iy}{x^2+y^2}$

$$u = \frac{x}{x^2+y^2}\ ; \quad v = \frac{-y}{x^2+y^2}$$

$$\left.\begin{aligned} x &= \frac{u}{u^2+v^2} \\ y &= -\frac{v}{u^2+v^2} \end{aligned}\right\} \text{pga } w = \frac{1}{z} \Leftrightarrow z = \frac{1}{w} \text{ kan det bara bytas ut.}$$

Antag

$$\alpha\left(\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right) + \beta\frac{u}{u^2+v^2} + \gamma\frac{-v}{u^2+v^2} = \delta$$

$$\frac{\alpha}{(u^2+v^2)} + \frac{\beta u}{u^2+v^2} - \frac{\gamma v}{u^2+v^2} = \delta$$

$$\Leftrightarrow \alpha + \beta u - \gamma v = \delta(u^2+v^2)$$

$$\delta(u^2+v^2) + (-\beta)u + \gamma v = \alpha$$

$$4\delta\alpha + (-\beta)^2 + \gamma^2 = 4\alpha\delta + \beta^2 + \gamma^2 > 0$$

$\therefore$ linje/cirkel i uv-planet

Allmänna fallet $(c \neq 0)$

$$w = T(z) = \frac{az+b}{cz+d} = A + \frac{B}{cz+d}$$

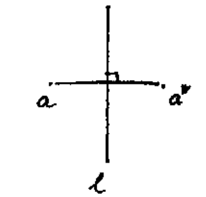
$$ad - bc \neq 0$$

$$z \longrightarrow cz = z_1 \longrightarrow z_1 + d = z_2 \longrightarrow \frac{1}{z_2} = z_3 \longrightarrow Bz_3 = z_4 \longrightarrow w = A + z$$

$$l/c \xrightarrow{\text{avb.}} l/c \longrightarrow l/c \xrightarrow[\text{enl. ovan}]{} l/c \longrightarrow l/c \longrightarrow l/c$$

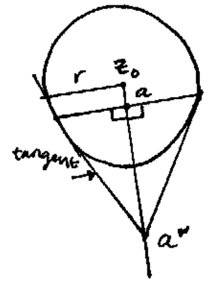
Inversa punkter

relativt linje



a, a^* inv. punkter

relativt cirkel



a, a^* inv. pkter

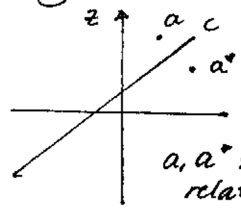
$a = z_0,\ a^* = \infty$

$a = \infty,\ a^* = z_0$

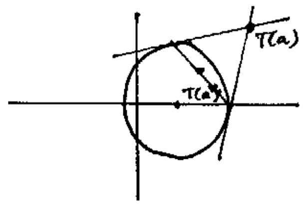
om $a \neq z_0,\ a \neq \infty$

$(a - z_0)\overline{(a^* - z_0)} = r^2$

Antag att T är en Möbius och C är en linje/cirkel

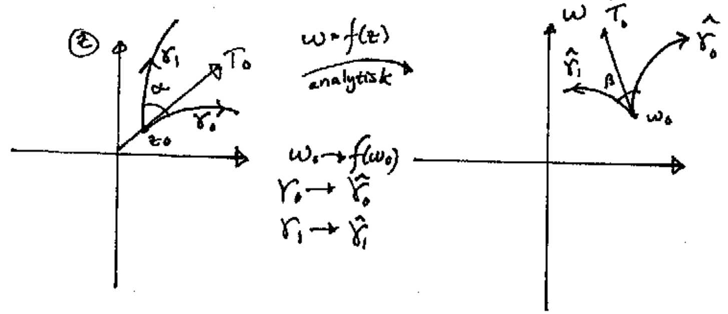


a, a^* inv. relativt C



Sats: $T(a), T(a^*)$ inversa relativt $T(C)$

Konform avbildning



Sats: Om $f'(z_0) \neq 0$ så är $\alpha = \beta$

Bevis $\gamma_0 = \begin{cases} \gamma_0(t), & 0 \le t \le 1 \\ \gamma_0(0) = z_0 \end{cases}$

$\hat{\gamma}_0 = \begin{cases} f(\gamma_0(t)), & 0 \le t \le 1 \\ f(\gamma_0(0)) = w_0 \end{cases}$

$T_0 = \gamma_0'(0)$ (förutsätt $\gamma_0'(0) \neq 0$)

$\hat{T}_0 = f'(\gamma_0(0))\gamma_0'(0) = f'(z_0)T_0$

$f'(z_0) = A_0 e^{i\theta_0}$ $(A_0 > 0,\ \theta_0 \in [0, 2\pi[\,)$

$\hat{T}_0 = e^{i\theta_0} A_0 T_0$

$T_0 \to \hat{T}_0$

sträck i skalan A_0

vrid θ_0 radianer i

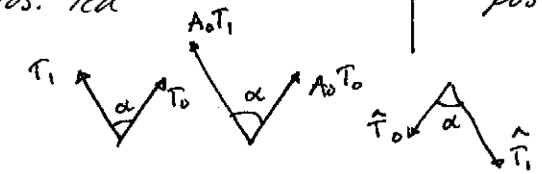
pos. led

$T_1 \to \hat{T}_1$

sträck i skalan A_0

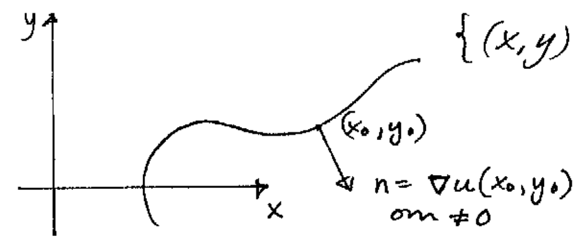
vrid θ_0 radianer i

pos. led



Nivålinjer (Nivåkurva)

$\{(x,y) \,;\, u(x,y) = c\}$



Sats Antag $f(z)$ analytisk och $f'(z_0) \neq 0$

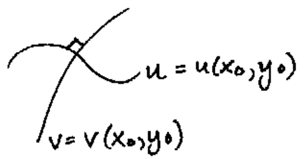
Skriv $f = u + iv$ och $z_0 = x_0 + iy_0$

Nivåkurvorna $u(x,y) = u(x_0, y_0)$ och

$v(x,y) = v(x_0, y_0)$

skär varandra under rät vinkel i z_0

Bevis



$$\nabla u(x_0,y_0) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\Big|_{(x_0,y_0)}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$if'(z) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

Om $\nabla u(x_0,y_0) = 0$ så $\frac{\partial u}{\partial x}(x_0,y_0) = 0$ och

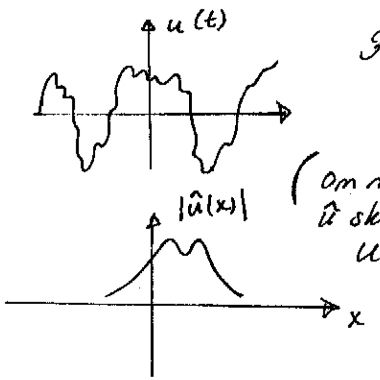
$\frac{\partial v}{\partial x}(x_0,y_0) \underset{CR}{=} -\frac{\partial u}{\partial y}(x_0,y_0) = 0$ dvs $f'(z_0) = 0$ 〉 Går ej!

Om $\nabla v(x_0,y_0) = 0$ pss $f'(z_0) = 0$

$\therefore \underbrace{\nabla u(x_0,y_0)}_{\hat{n}_1} \neq 0$ & $\underbrace{\nabla v(x_0,y_0)}_{\hat{n}_2} \neq 0$

$\hat{n}_1 \cdot \hat{n}_2 = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right) = 0$ enl. CR.ekv $\forall z$ där f analytisk

Bok lite väl omfattande



$$\mathcal{F}u(x) = (x) = \int_{-\infty}^{\infty} u(t)e^{-ixt}\,dt, \quad x \in \mathbb{R}$$

(Om man har fouriertransf. $\hat{u}$ skriver man $u(t) \supset \hat{u}(x)$) $\int_{-\infty}^{\infty} |u(t)|\,dt < +\infty$

x kan tolkas som vinkelfrekv. ω lite dumt använda x

Ex $e^{-t^2} \supset \sqrt{\pi}\, e^{-x^2/4}$

$$\begin{cases} 1, & |t| \leq \sigma \\ 0, & \text{f.ö.} \end{cases} \supset 2\frac{\sin \sigma x}{x}$$

Skalning $u(bt) \supset \frac{1}{|b|}\hat{u}\left(\frac{x}{b}\right)$ $b \in \mathbb{R}\setminus\{0\}$

Ex $e^{-b^2t^2} \supset \frac{\sqrt{\pi}}{|b|} e^{-\frac{x^2}{4b^2}}$, $b \neq 0$ (obs: b reell!)

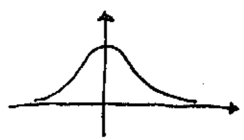
$b = \frac{1}{\sqrt{2}\,\sigma}$ där $\sigma > 0$

$$e^{\frac{-t^2}{2\sigma^2}} \supset \sqrt{2\pi\sigma^2}\, e^{-\frac{\sigma^2 x^2}{2}}$$

$$\underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}}_{\text{normal förd.}} \supset e^{-\frac{\sigma^2 x^2}{2}}$$

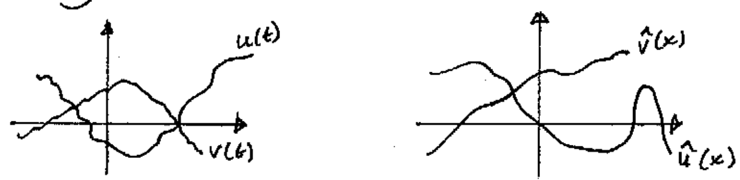
Om $T \in N(0,\sigma^2)$

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$$



$$\hat{f}(t) \supset e^{\frac{-\sigma^2 x^2}{2}}$$

Faltning



$u(t) + v(t)$	$\supset$	$\hat{u}(x) + \hat{v}(x)$
$a\,u(t)$	$\supset$	$a\hat{u}(t)$
$(u * v)(t) = \int_{-\infty}^{\infty} u(t-s)v(s)\,ds$	$\supset$	$\hat{u}(x)\hat{v}(x)$

Ex $\sigma_1, \sigma_2 > 0$

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{t^2}{2\sigma_1^2}} \supset e^{-\sigma_1^2 \frac{x^2}{2}}$$

$$\frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{t^2}{2\sigma_2^2}} \supset e^{-\sigma_2^2 \frac{x^2}{2}}$$

$$\frac{1}{2\pi\sigma_1^2} e^{-\frac{t^2}{2\sigma_1^2}} * \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{t^2}{2\sigma_2^2}} \supset e^{-\sigma_1^2 \frac{x^2}{2}} \cdot e^{-\sigma_2^2 \frac{x^2}{2}} =$$

$$= e^{-(\sigma_1^2+\sigma_2^2)\frac{x^2}{2}} = e^{-(\sqrt{\sigma_1^2+\sigma_2^2})^2\frac{x^2}{2}} \subset \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \cdot e^{\frac{-t^2}{2(\sigma_1^2+\sigma_2^2)}}$$

Problem!

Om $\hat{u} = \hat{v}$, är då $u = v$??

sid 340 i boken har Borell aldrig förstått.

Sats: Om u är begränsad, kontinuerlig och

$\int_{-\infty}^{\infty} |u(t)|\,dt < +\infty$, $\int_{-\infty}^{\infty} |\hat{u}(x)|\,dx < +\infty$

så är $u(t) = \int_{-\infty}^{\infty} \hat{u}(x)\, e^{ixt} \frac{dx}{2\pi}$, $x \in \mathbb{R}$

Bevis $\varphi_b(t) = e^{-b^2t^2} \supset \frac{\sqrt{\pi}}{b} e^{\frac{-x^2}{4b^2}} = \hat{\varphi}_b(x)$; $b > 0$

fås ur Beta

$$\hat{u}(x) = \int_{-\infty}^{\infty} u(s) e^{-ixs}\, ds$$

Betrakta $\int_{-\infty}^{\infty} \underbrace{\hat{u}(x) e^{ixt}}_{\text{divergent}} \cdot \underbrace{e^{\frac{-\sigma^2x^2}{2}}}_{\text{gaussfunkt.}}\, dx =$

slår ihjäl i stort sett allt ⇒ konvergerar totalt

= ej divergent så vi kan kasta om integrationsordningen =

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(s) e^{-ixs}\, ds\right) e^{ixt} e^{-\sigma^2\frac{x^2}{2}}\, dx =$$

$$= \int_{-\infty}^{\infty} u(s) \Big(\int_{-\infty}^{\infty} \underbrace{e^{\frac{-\sigma^2x^2}{2}}}_{\varphi_b(x)} e^{-ix(s-t)}\, dx \Big)\, ds = \quad b = \frac{\sigma}{\sqrt{2}}$$

$$= \int_{-\infty}^{\infty} u(s)\, \hat{\varphi}_b(s-t)\, ds = \int_{-\infty}^{\infty} u(s) \frac{\sqrt{\pi}}{b} e^{-\frac{(s-t)^2}{2\sigma^2}} \frac{ds}{\sigma} =$$

känd enl. Beta

konstanter-tok

$$= \begin{bmatrix} s = t + \sigma y \\ ds = \sigma\, dy \end{bmatrix} = \sqrt{2\pi} \int_{-\infty}^{\infty} u(t+\sigma y) e^{-\frac{y^2}{2}}\, dy$$

Låt $\sigma \to 0^+$: $\int_{-\infty}^{\infty} \hat{u}(x) e^{ixt} \cdot 1\, dx =$

$$= \sqrt{2\pi} \int_{-\infty}^{\infty} u(t) e^{-\frac{y^2}{2}}\, dy = u(t)\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy =$$

$$= u(t)\sqrt{2\pi}\sqrt{2\pi} = 2\pi\, u(t)$$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(x) e^{ixt} \frac{dx}{2\pi}$$

minns x = vinkelfr.
kan byta df och
$e^{i2\pi ft}$ men det
ser dumt ut!

<u>Koroll:</u>

Om $\hat{u} = 0$ så $u = 0$

<u>Bevis</u> $u(t) = \int_{-\infty}^{\infty} \underbrace{\hat{u}(x)}_{0} e^{ixt} \frac{dx}{2\pi} = 0$ (kan visas gälla alltid)

<u>Koroll:</u> Om $\hat{u} = \hat{v}$ så är $u = v$

$$\widehat{u - v} \underset{\substack{\uparrow \\ \text{f transf.} \\ \text{linjär}}}{=} \hat{u} - \hat{v} = 0 \Rightarrow u - v = 0 \Rightarrow u = v$$

<u>Derivera</u>

$$u'(t) \underset{\text{u snäll}}{\supset} ix\hat{u}(x)$$

<u>Bevis</u> (antydan till bevis):

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(x) e^{ixt} \frac{dx}{2\pi} \qquad \hat{u}(x) \text{ unik}$$

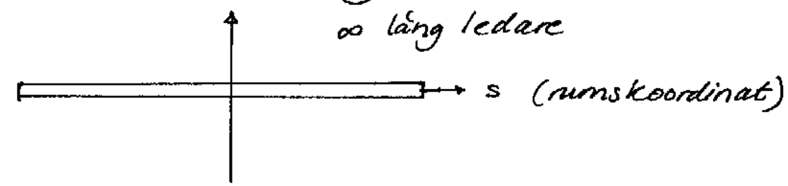
$$u'(t) = \int_{-\infty}^{\infty} \frac{d}{dt} \hat{u}(x) e^{ixt} \frac{dx}{2\pi} = \int_{-\infty}^{\infty} ix\hat{u}(x) e^{ixt} \frac{dx}{2\pi}$$

Men obs att $\frac{d}{dt}$ ej bara kan flyttas in (gen. integral)

[se Schwartz distributionsteori. Fungerar bara i linjära funktioner, ej i $\frac{\partial}{\partial t} u = \nabla u + u^3$]

$$u'(t) \supset ix\hat{u}(x)$$
$$u''(t) \supset ix\,ix\hat{u}(x)$$
$$u''(t) \supset -x^2\hat{u}(x)$$

tillämpn. <u>Värmeledning</u>



Låt $u(t,s)$ vara temperaturen i punkten s vid tiden $t > 0$.

$u(0,s) = f(s)$ given

$u(t,s)$?

Detta funderade Fourier på...

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial s^2} \cdot \underset{\substack{\uparrow \\ \text{ignorera idag}}}{\text{konst}}, \quad t > 0, \ s \in \mathbb{R}$$

<u>Problem</u> Best $u(t,s)$

<u>Lösning</u> $\frac{\partial u}{\partial t}(t,s) = \frac{\partial^2 u}{\partial s^2}(t,s)$, $t > 0$

Sätt $\hat{u}(t,x) = \int_{-\infty}^{\infty} u(t,s) e^{-ixs} ds$

$$u(t,s) \underset{\mathcal{F}}{\supset} \hat{u}(t,x)$$

$$\frac{\partial u}{\partial s}(t,s) \supset ix\hat{u}(t,x)$$

$$\frac{\partial^2 u}{\partial s^2}(t,s) \supset -x^2\hat{u}(t,x)$$

$\frac{\partial u}{\partial s}(t,s) = ?$

$$\frac{\partial \hat{u}}{\partial t}(t,x) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(t,s)e^{-ixs}\,ds$$

$$\frac{\partial u}{\partial t}(t,s) \underset{\mathcal{F}_s}{\supset} \frac{\partial \hat{u}}{\partial t}(t,x)$$

Kör $\mathcal{F}_s$ på PDE:

$\frac{\partial \hat{u}}{\partial t}(t,x) = -x^2\hat{u}(t,x)$

$\hat{u}(t,x) = Ce^{-tx^2}$

$t=0 \Leftrightarrow \hat{u}(0,x) = C$

$\hat{u}(0,x) = \hat{f}(x)$

$\Rightarrow \hat{u}(t,x) = \hat{f}(x)e^{-tx^2}$

$e^{-b^2s^2} \underset{\mathcal{F}_s}{\supset} \frac{\sqrt{\pi}}{b}e^{-x^2/4b^2}$

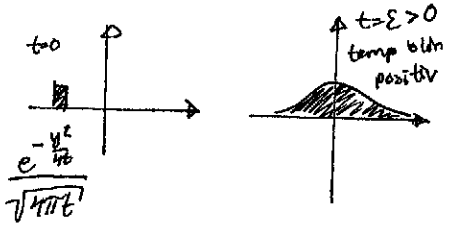
$\frac{1}{4b^2} = t;\quad b = \frac{1}{2\sqrt{t}} \Rightarrow e^{-\frac{s^2}{4t}} \supset \sqrt{4\pi t}\,e^{-tx^2}$

$\frac{1}{\sqrt{4\pi t}}e^{\frac{-s^2}{4t}} \underset{\mathcal{F}_s}{\supset} e^{-tx^2}$

faltning

$$u(t,s) = \int_{-\infty}^{\infty} f(y)e^{-\frac{(s-y)^2}{4t}}\frac{dy}{\sqrt{4\pi t}}$$

$$u(t,s) = \int_{-\infty}^{\infty} f(s-y)e^{-\frac{y^2}{4t}}\frac{dy}{\sqrt{4\pi t}}$$



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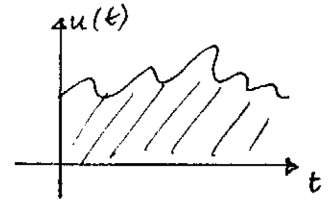
lv 7
föreл. 13

Laplacetransform



$\exists\, M, a \in \mathbb{R}$ så att

$|u(t)| \le \overset{\text{pos}}{M}e^{at}$

Transform

$u(t) \supset \hat{u}(s)$

lägg till bokstav under för Laplace el. Fourier

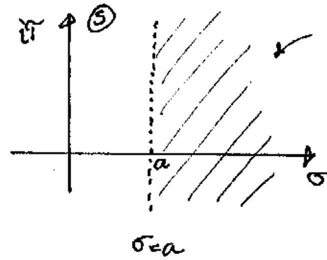
u styckvis kont.

Laplacetransformen

$$\begin{cases} \hat{u}(s) = \int_0^{\infty} u(t)e^{-st}\,dt \\ s = \sigma + i\tau;\quad \sigma,\tau \in \mathbb{R} \end{cases}$$

$|u(t)e^{-st}| = |u(t)|e^{-(\mathrm{Re}(s))t} \le Me^{at}e^{-\sigma t} = Me^{(a-\sigma)t}$

Om σ stort (större än a) har vi konvergens då $t \to \infty$



$\hat{u}(s)$ ∃ här är kontinuerlig

$$\int_\gamma \hat{u}(s)\,ds = \int_\gamma\left(\int_0^{\infty} u(t)e^{-st}dt\right)ds$$

$$= \int_0^{\infty}\underbrace{\left(\int_\gamma \underbrace{u(t)e^{-st}}_{\text{analytisk}}ds\right)}_{\substack{=0 \text{ enl.} \\ \text{Cauchys integralsats}}}dt = \int_0^{\infty} 0\,dt = 0$$

Moreras sats $\Rightarrow$ $\tilde{u}(s)$ analytisk i $\mathrm{Re}\, s > 0$

Skriver ofta

$\mathcal{L} u = \tilde{u}$ (Laplace på u)

$$\boxed{\begin{aligned} &\mathcal{L}(\alpha u) = \alpha \mathcal{L}(u), \ \alpha \in \mathbb{C} \\ &\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v \end{aligned}}$$

$\therefore$ linearitet

Sätt $u(t) = 0$, $t < 0$

Då är $\tilde{u}(s) = \int_0^\infty u(t) e^{-st} dt = \int_{-\infty}^\infty u(t) e^{-st} dt =$

$= \int_{-\infty}^\infty (u(t) e^{-\sigma t}) e^{-i\tau t} dt = \widehat{u(t) e^{-\sigma t}}(\tau)$

"Sats" $\tilde{u} = 0 \Rightarrow u = 0$

AB: $\tilde{u} = 0 \Rightarrow \widehat{u(t)e^{-\sigma t}} = 0 \Rightarrow u(t) e^{-\sigma t} = 0 \Rightarrow u = 0$

"Sats" $\tilde{u} = \tilde{v} \Rightarrow u = v$

AB: $\widetilde{u-v} = \tilde{u} - \tilde{v} = 0 \Rightarrow u - v = 0 \Rightarrow u = v$

Ex $1 \supset \frac{1}{s}$, då $\mathrm{Re}\, s > 0$ ty $\int_0^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_{0=t}^{\infty = t} =$

$= 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$

Ex $t \supset \frac{1}{s^2}$, då $\mathrm{Re}\, s > 0$ ty $\int_0^\infty t e^{-st} dt = \left[\frac{t e^{-st}}{-s}\right]_0^\infty +$

$+ \frac{1}{s} \int_0^\infty 1 e^{-st} dt = \frac{1}{s^2}$

(91)

Antag $u(t) \supset \tilde{u}(s)$ dvs $\tilde{u}(s) = \int_0^\infty u(t) e^{-st} dt$ $\mathrm{Re}\, s > a$

Får $\frac{d}{ds} \tilde{u}(s) = \int_0^\infty \left(\frac{d}{ds}(u(t) e^{-st})\right) dt = \int_0^\infty (-t u(t) e^{-ts}) dt$

och $\left(\frac{d}{ds}\right)^n \tilde{u}(s) = \int_0^\infty (-t)^n u(t) e^{-st} dt$

så $(-t)^n u(t) \supset \left(\frac{\partial}{\partial s}\right)^n \tilde{u}(s)$

dvs $t^n(u(t)) \supset \left(-\frac{d}{ds}\right)^n \tilde{u}(s)$ [viktig]

Ex $1 \supset \frac{1}{s} \Rightarrow t \supset -\frac{d}{ds} \frac{1}{s} = \frac{1}{s^2} \Rightarrow t^2 \supset \frac{2}{s^3} \Rightarrow \ldots \Rightarrow$

$\Rightarrow t^n \supset \frac{n!}{s^{n+1}}$

Ex $e^{at} u(t) \supset \tilde{u}(s-a)$ ty $\int_0^\infty e^{at} u(t) e^{-st} dt =$

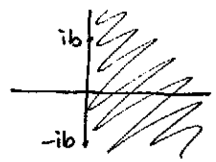
$= \int_0^\infty u(t) e^{(a-s)t} dt = \int_0^\infty u(t) e^{-(s-a)t} dt = \tilde{u}(s-a)$

Ex $1 \supset \frac{1}{s} \Rightarrow e^{at} \supset \frac{1}{s-a}$ a kan vara komplex

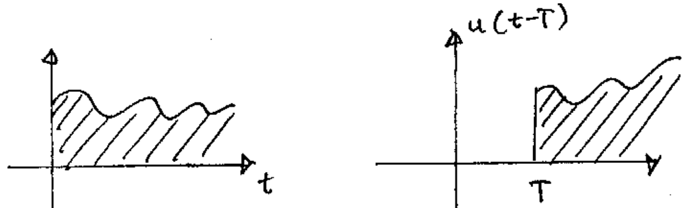
Ex $\left.\begin{aligned} e^{ibt} &\supset \frac{1}{s-ib} \\ e^{-ibt} &\supset \frac{1}{s+ibt} \end{aligned}\right\} + \text{mult } \frac{1}{2} \Big\}$ $\cos(bt) \supset \frac{1}{2}\left(\frac{1}{s-ib} + \frac{1}{s+ib}\right) = \frac{s}{s^2+b^2}$

Ex $\sin bt \supset \frac{1}{2i}\left(\frac{1}{s-ib} - \frac{1}{s+ib}\right) = \frac{b}{s^2+b^2}$

Vilkor på s: $\mathrm{Re}(s) > 0$



(92)



$u(t)=0$
$t<0$

$$u(t-T) \supset \int_0^\infty u(t-T)e^{-st}dt = \int_T^\infty u(t-T)e^{-st}dt =$$

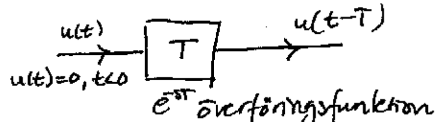
$$= \begin{bmatrix} t = x+T \\ dt = dx \end{bmatrix} = \int_0^\infty u(x)e^{-s(x+T)}dx =$$

$$= e^{-sT}\int_0^\infty u(x)e^{-sx}dx = e^{-sT}\tilde{u}(s)$$

$u(t-T) \supset e^{-sT}\,\tilde{u}(s)$ om $u(t)=0,\ t<0$

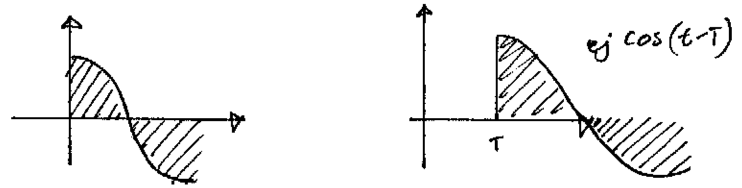
(obs $T>0$)

se det som ett filter

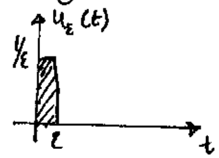


$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases}$$ Heavysides funktion

$T>0 \quad H(t-T)\cos(t-T) \supset \dfrac{s}{s^2+1}e^{-sT}$



Vilken funktion har Laplacetransformen 1 ?
svar: Ingen! (Ingen vanlig funktion)



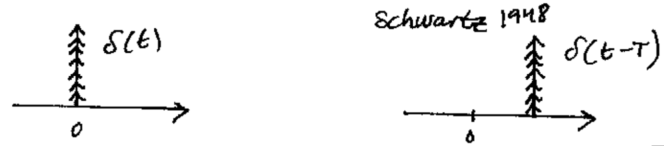
$$u_\varepsilon(t) = \frac{1}{\varepsilon}\left(H(t)-H(t-\varepsilon)\right) \supset \frac{1}{\varepsilon}\left(\frac{1}{s} - \frac{e^{-s\varepsilon}}{s}\right) =$$

$$= \frac{1-e^{-\varepsilon s}}{\varepsilon s} \to 1, \quad \varepsilon \to 0$$

$\tilde{u}_\varepsilon(s) \to 1, \ \varepsilon \to 0$

$u_\varepsilon(t) \to \delta(t)$ Diracfunktionen i 0

Schwartz 1948



$\delta(t-T) \supset e^{-sT}$

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)dt = f(T)$$

Faltning

Faltning $(u*v)(t) = \int_0^t u(t-x)v(x)dx, \quad t \geq 0$
Sätt $u(t) = v(t),\ t<0$

Då $$\int_{-\infty}^{\infty} u(t-x)v(x)dx = \int_{0 \leq x \leq t} u(t-x)v(x)dx = \begin{cases} 0, & t<0 \\ \int_0^t u(t-x)v(x)dx, & t \geq 0 \end{cases}$$

$\underline{Ex}$ $\left(e^{at}u(t)\right) * \left(e^{at}v(t)\right) = e^{at}(u*v)(t)$

$\mathcal{L}: \int_0^t e^{a(t-x)} u(t-x)\, e^{ax} v(x)\,dx =$

$= e^{at}\int_0^t u(t-x)v(x)\,dx$

$\underline{Sats}$ $v = \widetilde{u*v}(s) = \tilde{u}(s)\tilde{v}(s) = H.L$

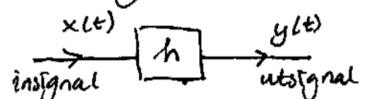
$\underline{Bevis}$: Sätt $u(t) = v(t) = 0$, $t<0$ Alltid i Laplace.

Har $\tilde{u}(s) = \widehat{e^{-\sigma t}u(t)}(\tau)$, $s = \sigma + i\tau$

$VL = \widehat{e^{-\sigma t}(u*v)(t)}(\tau) \underset{Ex}{=} \left(e^{-\sigma t}u(t)\right)*\left(e^{-\sigma t}v(t)\right)(\tau) =$

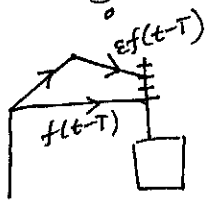
$= \widehat{e^{-\sigma t}u(t)}(\tau)\,\widehat{e^{-\sigma t}u(t)}(\tau) = \tilde{u}(s)\tilde{v}(s)$

Faltningsfilter



om x, y, h kausala

$$y(t) = \int_0^t h(t-\tau)x(\tau)\,d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$$



$x(t) = f(t) - \varepsilon f(t-T)$

$f(t) = h * x(t)$ Laplacetransf.

och räkna fram $h(t)$

$x(t) - \varepsilon x(t-T) = f(t) - \varepsilon^2 f(t-T)$

Derivering: $\int_0^\infty u'(t)e^{-st}\,dt \overset{PI}{=} \left[u(t)e^{-st}\right]_0^\infty +$

$+ s\int_0^\infty u(t)e^{-st}\,dt = s\tilde{u}(s) - u(0)$

$\therefore u'(t) \supset s\tilde{u}(s) - u(0) \Leftrightarrow (\mathcal{L}u')(s) = s(\mathcal{L}u)(s) - u(0)$

$(\mathcal{L}u'') = s(\mathcal{L}u')(s) - u'(0) = s^2(\mathcal{L}u)(s) - su(0) - u'(0)$

$\underline{Ex}$ Lös $\begin{cases} u''(t) + 4u(t) = 1 \\ u(0) = 0,\ u'(0) = 1 \end{cases}$ $\tilde{u} = U$

$\underline{Lösn.}$ $s^2U(s) - su(0) - u'(0) + 4U(s) = \frac{1}{s}$

$s^2U(s) - 1 + 4U(s) = \frac{1}{s}$

$(s^2+4)U(s) - 1 = \frac{1}{s} \Rightarrow (s^2+4)U(s) = \frac{s+1}{s}$

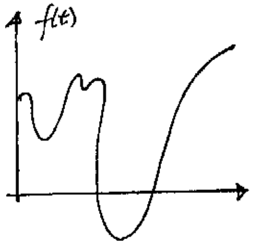
$U(s) = \frac{s+1}{s(s^2+4)} = \frac{1/4}{s} - \frac{\frac{1}{4}s+1}{s^2+4} =$

$= \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{s^2+4}$

$u(t) = \frac{1}{4} - \frac{1}{4}\cos(2t) - \frac{1}{2}\sin 2t$

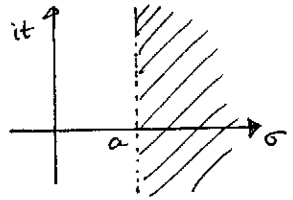
01.10.19
Lv 7
sista föreställningen.

Laplacetransform



$|f(t)| \le Me^{at}, \quad t \ge 0$

$(\mathcal{L}f)(s) = \tilde{f}(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad s = \sigma + i\tau \quad (\sigma, \tau \in \mathbb{R})$



När är en analytisk funktion en Laplacetransf.

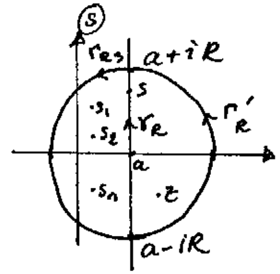
Sats (Kjellfil) Låt $g: \mathbb{C} \setminus \{s_1, \dots, s_n\} \to \mathbb{C}$ vara analytisk och antag $g(s) \to 0$ då $|s| \to +\infty$

Då är $g(s) = \tilde{f}(s)$ där $f(t) = \sum_{k=1}^{n} \operatorname{Res}(g(s)e^{st}; s_k) \quad t \ge 0$

Ex $g(s) = \frac{s}{s^2+1}$; $f(t) = \operatorname{Res}\left(\frac{se^{st}}{s^2+1}; i\right) + \operatorname{Res}\left(\frac{se^{st}}{s^2+1}; -i\right)$

$= \frac{e^{it}}{2} + \frac{e^{-it}}{2} = \cos t$

Bevis



$C = \gamma_R + \Gamma_R$

$|e^{st}| = e^{(\operatorname{Re} s)t}$

$\operatorname{Re} s_k < a, \quad k = 1, \dots, n$

$|s_k - a| < R, \quad k = 1, \dots, n$

$\frac{1}{2\pi i}\int_C g(s)e^{st}\,ds = \underbrace{\sum_{k=1}^{n} \operatorname{Res}(g(s)e^{st}; s_k)}_{\text{Residusatsen}} = f(t)$

$|f(t)| = \frac{1}{2\pi}\left|\int_C g(s)e^{st}\,ds\right| \le \frac{1}{2\pi}\max|g(s)e^{st}|\,l(C) \le$

$\le \frac{1}{2\pi}\max_C |g(s)|e^{at}(2R + \pi R) = Me^{at} \quad (t \ge 0)$

$\tilde{f}(z) = \int_0^\infty f(t)e^{-zt}\,dt = \int_0^\infty \frac{1}{2\pi i}\left(\int_C g(s)e^{st}\,ds\right)e^{-zt}\,dt =$

$= \frac{1}{2\pi i}\int_C g(s)\left(\int_0^\infty e^{-(z-s)t}\,dt\right)ds = \frac{1}{2\pi i}\int \frac{g(s)}{z-s}\,ds$, ty $\operatorname{Re} z > a$

$\left[\frac{e^{-(z-s)t}}{-(z-s)}\right]_{0=t}^{\infty}$

$\tilde{f}(z) = -\frac{1}{2\pi i}\int \frac{g(s)}{s-z}\,ds = -\frac{1}{2\pi i}\int_\gamma \frac{g(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(s)}{s-z}\,ds$

$g(z) = \frac{1}{2\pi i}\int_{\Gamma'_R - \gamma_R} \frac{g(s)}{s-z}\,ds = -\frac{1}{2\pi i}\int_{\gamma_R} \frac{g(s)}{s-z}\,ds + \frac{1}{2\pi i}\int_{\Gamma'_R} \frac{g(s)}{s-z}\,ds$

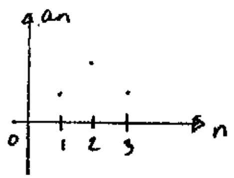
$$\tilde{f}(z) - g(z) = \frac{1}{2\pi i} \int_{\Gamma_R + \Gamma_R'} \frac{g(s)}{s-z}\, ds$$

$$|\tilde{f}(z) - g(z)| = \frac{1}{2\pi i} \left| \int_{\Gamma_R + \Gamma_R'} \frac{g(s)}{(s-a)-(z-a)}\, ds \right| \underset{ML}{\leq}$$

$$\leq \frac{1}{2\pi} \frac{\max_{s \in \Gamma_R + \Gamma_R'} |g(s)|}{R - |z-a|} \cdot 2\pi R = \frac{1}{1 - \frac{(z-a)}{R}} \max_{\Gamma_R + \Gamma_R'} |g(s)| \to 0 \quad \text{då } R \to +\infty$$

$$\therefore \tilde{f}(z) - g(z) = 0 \Rightarrow \tilde{f}(z) = g(z)$$

Pragos' sats



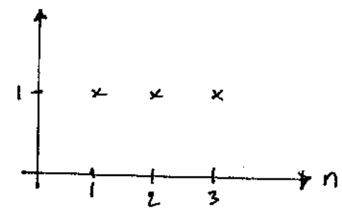
$$A(s) = \sum_{n=0}^{\infty} a_n s^n \quad \text{genererande funktion}$$

Införs av Laplace

$$s = z^{-1} \quad z\text{-transf.} \Rightarrow \sum_0^{\infty} a_n z^{-n} \text{ betecknas } Z(\{a_n\})$$

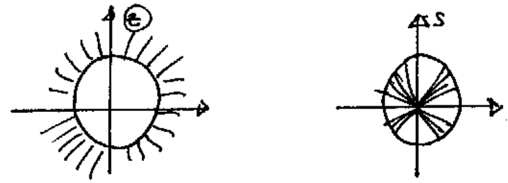
[z transformen av $\{a_n\}_0^{\infty}$]

Ex : $H(n) > ?$



Gretzky

$$\sum_0^{\infty} H(n) z^{-n} = \sum_0^{\infty} z^{-n} = \sum_0^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \quad \text{om } |z| > 1$$



$$s = \frac{1}{z}$$

$$H(n) > \frac{z}{z-1} \qquad \text{tabell: } 1 > \frac{z}{z-1} \qquad \theta(n) > \frac{z}{z-1} \quad \theta = H$$

Ex $H(n) c^n > ?$

$$\sum_0^{\infty} H(n) c^n z^{-n} = \sum_0^{\infty} (cz^{-1})^n = \frac{1}{1-cz^{-1}} = \frac{z}{z-c}$$

om $|z| > |c|$

z-transform

$$a_n > \sum_0^{\infty} a_n z^{-n} = A(z)$$

$$x_n > \sum_0^{\infty} x_n z^{-n} = X(z)$$

Ex $1 > \frac{z}{z-1}$

$$c^n > \frac{z}{z-c}$$

$$nc^n > ?$$

$$\sum_0^{\infty} c^n z^{-n} = \frac{1}{1-cz^{-1}}$$

$$s = z^{-1}$$

$$\sum_0^\infty c^n s^n = \frac{1}{1-cs}$$

$$\sum_0^\infty n c^n s^n = \frac{c}{(1-cs)^2}$$

$$\sum_0^\infty n c^n s^n = \frac{cs}{(1-cs)^2}$$

$$\sum_0^\infty n c^n z^{-n} = \frac{cz^{-1}}{(1-cz^{-1})^2} = \frac{cz}{(z-c)^2}$$

$$nc^n \supset \frac{cz}{(z-c)^2}$$

$$\{a_n\} * \{b_n\} = \{c_n\}$$

$$c_n = \sum_{k=0}^{n} a_{n-k}\, b_k = \sum_{k=0}^{n} a_k\, b_{n-k}$$

<u>Sats</u> $\{a_n\} * \{b_n\} \supset A(z)B(z)$

$B:$ $\sum_0^\infty c_n z^{-n} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k}\, b_k\right) z^{-n} =$

$$= \sum\sum_{0\le k\le n<+\infty} a_{n-k} b_k z^{-n} = \sum_{k=0}^{\infty}\left(\sum_{n=k} a_{n-k} z^{-n}\right) b_k =$$

$$= \sum_{k=0}^{\infty}\left(\sum_{\nu=0}^{\infty} a_\nu z^{-k-\nu}\right) b_k = \sum_{k=0}^{\infty}\underbrace{\left(\sum_{\nu=0}^{\infty} a_\nu z^{-\nu}\right)}_{A(z)} b_k z^{-k} =$$

$$= A(z)\sum_{k=0}^{\infty} b_k z^{-k} = A(z)B(z)$$

<u>Sats</u> Antag $N \in \mathbb{N}_+$ Då $a_{n+N} \supset z^N A(z) - z^N a_0 - \dots -$
$- z a_{N-1}$

<u>Bevis:</u> $\sum_{n=0}^{\infty} a_{n+N} z^{-n} = z^N \sum_{n=0}^{\infty} a_{n+N} z^{-(n+N)} =$

$$= z^N\left(a_N z^{-N} + a_{N+1} z^{-N-1} + \dots\right) =$$

$$= z^N\left(A(z) - a_0 - \dots - a_{N-1} z^{-(N-1)}\right) =$$

$$= z^N A(z) - z^N a_0 - \dots - z a_{N}$$

<u>Sats</u> Antag $N \in \mathbb{N}_+$

$H(n-N)a_{n-N} \supset z^{-N}A(z)$ (formel i tabell)

är ej definierad

<u>Bevis</u>: $\sum_{n=0}^{\infty} H(n-N) a_{n-N} z^{-n} = \sum_{n=N}^{\infty} a_{n-N} z^{-n}$

$$\sum_{n=N}^{\infty} a_{n-N} z^{-n} = \sum_{\nu=0}^{\infty} a_\nu z^{-N-\nu}$$

$$\begin{pmatrix} n-N=\nu \\ n = N+\nu \end{pmatrix}$$

$$z^{-N}\sum_{\nu=0}^{\infty} a_\nu z^{-\nu} = z^{-N} A(z)$$

Ex Lös $\begin{cases} X_{n+2} - 5X_{n+1} + 6X_n = 1 \\ X_0 = 0 \quad x_1 = 1 \end{cases}$

L: z-transformera ekv.:

$$\left(z^2X(z)\right) - z^2 \cdot 0 - z) - 5\left(zX(z) - z \cdot 0\right) +$$

$$+ 6X(z) = \frac{z}{z-1}$$

$$\left(z^2 - 5z + 6\right)X(z) = z + \frac{z}{z-1} = \frac{z^2}{z-1}$$

$$X(z) = \frac{z^2}{(z-1)(z-2)(z-3)} = z \frac{z}{(z-1)(z-2)(z-3)} =$$

$$= z\left(\frac{\frac{1}{2}}{z-1} + \frac{(-2)}{z-2} - \frac{\frac{3}{2}}{z-3}\right)$$

$$X_n = \frac{1}{2} - 2 \cdot 2^n + \frac{3}{2} \cdot 3^n = \frac{1}{2} - 2^{n+1} + \frac{1}{2}3^{n+1}$$