FFM232− Vektorfält och klassisk fysik F

Övningar
2010

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Demostrationsövningar

Vektorfält och klassisk fysik

Med Martin Cederwall och Per Salomonsson

F2 LP1 2010 FFM 232

1.13 $T(x,y,z) = (x^2 + 2yz - z^2)\,T_0\,, \qquad T_0 = 1^\circ\ C/m^2$

Fluga i $(x,y,z) = (1,1,2)$

- Vilket håll ska den flyga för att bli varmare fortast?
- Hast. $v = 0{,}3\ m/s$ i riktn. $(-2,2,1)$, vad blir $\dot{T}$

Temp. ökar snabbast i riktn. som ges av ∇T

$$\nabla T(x,y,z) = \ldots = 2T_0(x, z, y-z)$$
$$\nabla T(1,1,2) = 2T_0(1,2,-1)$$

Flyg i riktn. $\frac{1}{\sqrt{6}}(1,2,-1)$. Längs $\mathbf{n} = \frac{1}{3}(-2,2,1)$:

Temp. ändring / längdenhet:

$$\mathbf{n} \cdot \nabla T = \frac{1}{3}\, 2T_0 \cdot 1 = \frac{2}{3}\, T_0$$

Temp. ändring / tidsenhet: $\frac{2}{3}\, T_0 \cdot v = 0{,}2\ K/s \qquad (\mathbf{v} \cdot \nabla T)$

2.11

a) Nivåytor till $\phi(r,\theta,\varphi) = r^2\cos 2\theta - 2ar\sin\theta\cos\varphi$

b) Riktn. deriv. av ϕ i $r = a$, $\theta = \frac{\pi}{4}$, $\varphi = \pi$ i riktn. $\hat{\theta}$

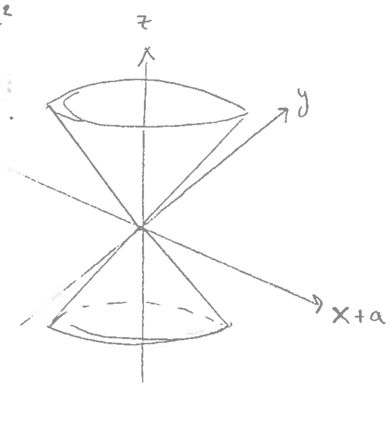
c) Fältlinjer till $\nabla\phi$

$$\phi = r^2(\cos^2\theta - \sin^2\theta) - 2ar\sin\theta\cos\varphi = z^2 - (x^2+y^2) - 2ax = -(x+a)^2 - y^2 + z^2 + a^2$$

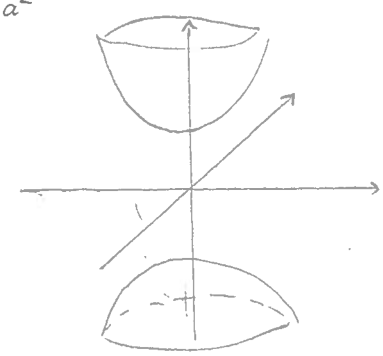
a) Nivåytorna ges av $\underbrace{-(x+a)^2}_{-x^2} - y^2 + z^2 = C - a^2$

$C = a^2$

kon

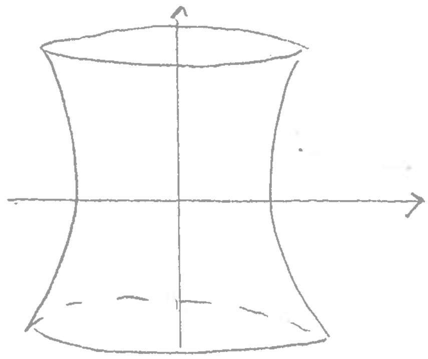


$C > a^2$



2-mantlad hyperboloid

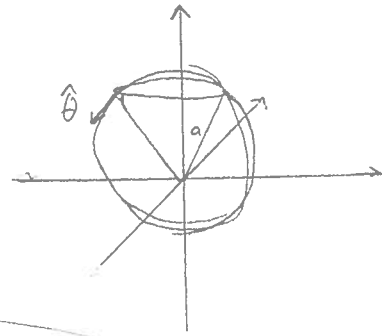
$C < a^2$



d) $\nabla\phi = -2(x+a)\hat{x} - 2y\hat{y} + 2z\hat{z}$

$r = a,\ \theta = \frac{\pi}{4},\ \varphi = \pi$

$(x,y,z) = \left(-\frac{a}{\sqrt{2}}, 0, \frac{a}{\sqrt{2}}\right),\quad \hat{\theta} = -\frac{1}{\sqrt{2}}(1,0,1)$

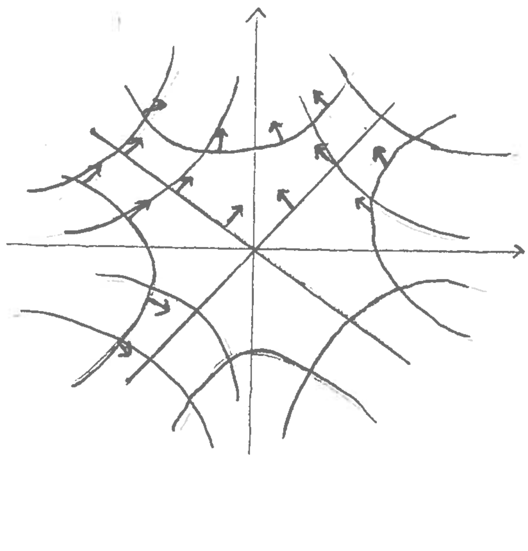


$\hat{\theta}\cdot\nabla\phi = -\frac{1}{\sqrt{2}}(1,0,1)\cdot\left(-2\left(-\frac{a}{\sqrt{2}}+a\right), 0, a\sqrt{2}\right) =$

$= -\frac{1}{\sqrt{2}}\left(a\sqrt{2} - 2a + \sqrt{2}\,a\right) = -(2-\sqrt{2})\,a$

$\hat{\theta} = \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta}$

e)



$\nabla\phi$

röd: nivåkurvor
blå: fältlinjer

Ekv. för fältlinjerna: $\frac{d\mathbb{r}}{dt} = \underbrace{f(t)}_{=\frac{1}{2}} \nabla\phi(\mathbb{r}(t))$

$$\begin{cases} \frac{dx}{dt} = -(x+a) & \overbrace{(x+a)}^{x'} = \overbrace{(x_0+a)}^{x_0'} e^{-t} \\ \frac{dy}{dt} = -y & (y(t) = y_0 e^{-t}) \\ \frac{dz}{dt} = z & z(t) = z_0 e^{-t} \end{cases}$$

$x'z$ = konst.
rot. kring z-axeln

$\boxed{\text{Nästan } 2.22}$

Koord. u, v
$$\begin{cases} x = uv \\ y = u^2 + \lambda v^2 \end{cases}$$

Bestäm λ s.a. uv-syst. är ortogonalt. Beräkna skalfaktorer!

Koord. ytor/-linjer

Basvektorer $\frac{1}{h_u}\frac{\partial \mathbb{r}}{\partial u} = \hat{u}$, $\frac{1}{h_v}\frac{\partial \mathbb{r}}{\partial v} = \hat{v}$

$\mathbb{r} = uv\hat{x} + (u^2 + \lambda v^2)\hat{y}$

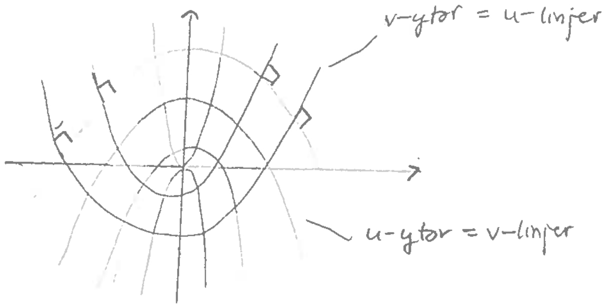
$$\frac{\partial \mathbb{r}}{\partial u} = v\hat{x} + 2u\hat{y}$$
$$\frac{\partial \mathbb{r}}{\partial v} = u\hat{x} + 2\lambda v\hat{y}$$

$$\frac{\partial \mathbb{r}}{\partial u} \cdot \frac{\partial \mathbb{r}}{\partial v} = uv + 4\lambda uv = 0 \Rightarrow \lambda = -\frac{1}{4}$$

$h_u = \sqrt{4u^2 + v^2}$, $h_v = \sqrt{u^2 + \frac{1}{4}v^2} = \frac{1}{2}\sqrt{4u^2+v^2}$

$ds^2 = (h_u du)^2 + (h_v dv)^2 = (4u^2+v^2)(du^2 + \frac{1}{4}dv^2)$

koord.yta $u = u_0$: $\begin{cases} x = u_0 v \\ y = u_0^2 - \frac{1}{4}v^2 \end{cases}$ $\hookleftarrow v = \frac{x}{u_0}$ $\Rightarrow y = u_0^2 - \frac{x^2}{4u_0^2}$



$v = v_0$: $\begin{cases} x = u v_0 \\ y = u^2 - \frac{1}{u} v_0^2 \end{cases} \longleftarrow u = \frac{x}{v_0}$

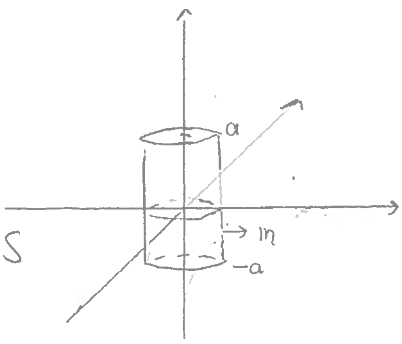
$\Rightarrow y = -\frac{1}{4} v_0^2 + \frac{x^2}{v_0^2} = -\left(\frac{v_0}{2}\right)^2 + \frac{1}{4}\frac{x^2}{\left(\frac{v_0}{2}\right)^2}$

Lv 2

Ex. 3.16

$S = x^2 + y^2 = a$
$|z| < a$

Beräkna $\mathbb{I} = \int_S (\mathbf{r} + \rho\hat{\varphi}) \times d\mathbf{S}$



$d\mathbf{S} = \mathbf{n}\, dS$

Parametrisera ytan med φ, z

$S: \begin{cases} \rho = a \\ 0 \leq \varphi < 2\pi \\ -a < z < a \end{cases}$ $\quad d\mathbf{S} = \underbrace{a\, dz\, d\varphi}_{dS}\, \underbrace{\hat{\rho}}_{\mathbf{n}}$ $\quad \mathbf{r} + \rho\hat{\varphi} = \rho\hat{\rho} + z\hat{z} + \rho\hat{\varphi}$

$(\mathbf{r} + \rho\hat{\varphi}) \times d\mathbf{S} = (\rho\hat{\rho} + z\hat{z} + \rho\hat{\varphi}) \times a\hat{\rho}\, dz\, d\varphi = a\, dz\, d\varphi\, (z\hat{\varphi} - \underset{a}{\rho}\hat{z})$

$\mathbb{I} = \int_{-a}^{a} dz \int_0^{2\pi} d\varphi\, a\,(z\hat{\varphi} - a\hat{z}) = \hat{z}(-a^2)\, 2\pi\, 2a = -4\pi a^3 \hat{z}$

konst. vektor

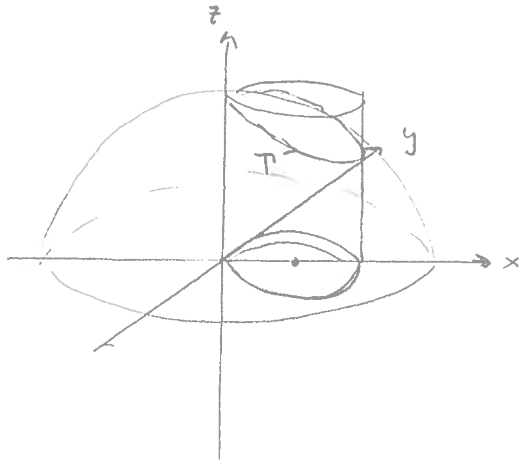
$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$

Ex) 4.4. Beräkna $\oint_{\Gamma} \mathbb{F} \cdot d\mathbf{r}$

$$\mathbb{F} = [x^2 - a(y+z)]\hat{x} + (y^2 - az)\hat{y} + [z^2 - a(x+y)]\hat{z}$$

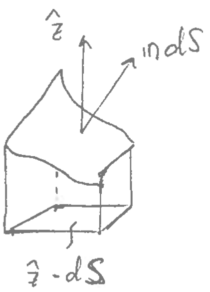
Γ: skärningen mellan cylindern $(x-a)^2 + y^2 = a^2$, $z > 0$ och sfären $x^2+y^2+z^2 = R^2$, $R > 2a$

Stokes? $\displaystyle\int_{\Gamma = \partial S} \mathbb{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbb{F}) \cdot d\mathbf{S}$



$$\nabla \times \mathbb{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - a(y+z & y^2 - a+z^2 & z^2 - a(x+y) \end{vmatrix}$$

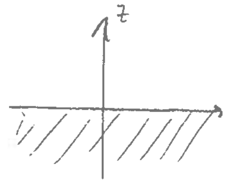
$$= \ldots = a\hat{z}$$



$$\int_{\Gamma} \mathbb{F} \cdot d\mathbf{r} \overset{\text{stokes}}{=} \int_S a\hat{z} \cdot d\mathbf{S} = a(\text{arean av ytans proj. i xy-planet}) = \pi a^3$$

4.21

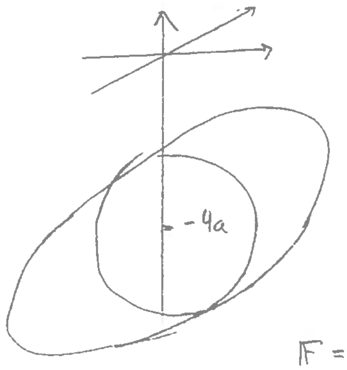
$$p(z) = p_0 - \rho g z + k z^2, \quad z<0$$

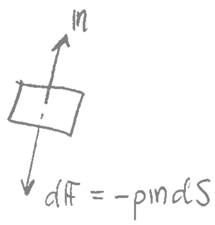


Beräkna kraften från vätskan på kroppen med begr. ytan

$$S: \; 4\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \underbrace{4\left(\frac{z}{a}\right)^2 + 32\frac{z}{a}}_{4\left(\left(\frac{z}{a}\right)^2 + 8\frac{z}{a}\right) = 4\left(\frac{z}{a}+4\right)^2 - 64} + 48 = 0$$

$$S: \; 4\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + 4\left(\frac{z}{a}+4\right)^2 = 16, \qquad \frac{1}{4a^2}x^2 + \frac{1}{16a^2}y^2 + \frac{1}{4a^2}(z+4a)^2 = 1$$





$$\mathbb{F} = -\int_S p \, d\mathbb{S}$$

fortsätter sen...

4.2. a) Bevisa att $\int_V \nabla f \, dV = \int_{\partial V} f \, d\mathbb{S}$

Tag Gauss med $\mathbb{F} = a f$ (a konst) $\Rightarrow$ $\nabla \cdot \mathbb{F} = a \cdot \nabla f$

Gauss: $\cancel{a\cdot} \int_V \nabla f \, dV = \cancel{a\cdot} \int_{\partial V} f \, d\mathbb{S}$

4.21 (forts.)

Gaussanalog sats från 4.2:

$$\Rightarrow \quad \mathbf{F} = -\int_S p \, d\mathbf{S} = -\int_V \nabla p \, dV$$

$\int \underbrace{(z+4a)}\, dz = 0$

udda kring $z=-4a$
men kroppen är sym. kring ~~axl~~ z

sätt $z=-4a$
flytta origo

$$\nabla p = (-\rho g + 2kz)\,\hat{z}$$

$$\mathbf{F} = \hat{z}\int_V (\rho g - 2kz)\, dV = \hat{z}\rho g \int \Big(1 + \frac{8k}{\rho g}\underset{z?}{a}\Big)\, dV = \hat{z}\rho g\Big(1+\frac{8ka}{\rho g}\Big)\cdot \underbrace{\frac{4\pi}{3}\cdot(2a)^2\cdot 4a}_{V} =$$

$$= \frac{64\pi}{3}\rho g a^3\,\hat{z}\Big(1+\frac{8ka}{\rho g}\Big), \qquad k=0:\ \mathbf{F} = \hat{z}\rho g V$$

Lv.3

2.8

$$\mathbf{E} = \frac{m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Fältlinje genom $(r_0, \theta_0, \varphi_0) = (2, \frac{\pi}{4}, \frac{\pi}{6})$, $\mathbf{r}(\tau) = \mathbf{r}(r(\tau), \theta(\tau), \varphi(\tau))$

$$\frac{\partial \mathbf{r}}{\partial r} = h_r\hat{r} = \hat{r}, \quad \frac{\partial \mathbf{r}}{\partial \theta} = h_\theta\hat{\theta} = r\hat{\theta}, \quad \frac{\partial \mathbf{r}}{\partial \varphi} = h_\varphi\hat{\varphi} = r\sin\theta\,\hat{\varphi}$$

$$\frac{\partial \mathbf{r}}{\partial \tau} = \frac{\partial \mathbf{r}}{\partial r}\frac{\partial r}{\partial \tau} + \frac{\partial \mathbf{r}}{\partial \theta}\frac{\partial \theta}{\partial \tau} + \frac{\partial \mathbf{r}}{\partial \varphi}\frac{\partial \varphi}{\partial \tau} = \hat{r}\frac{\partial r}{\partial \tau} + r\hat{\theta}\frac{\partial \theta}{\partial \tau} + r\sin\theta\,\hat{\varphi}\cdot\frac{\partial \varphi}{\partial \tau} = C\,\mathbf{E} =$$

skalfaktor av läget

$$= \left[\text{sätt } C = 4\pi r^3/m\right] = 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} + 0\cdot\hat{\varphi}$$

$$\left\{\begin{array}{l} \dfrac{dr}{d\tau} = 2\cos\theta \\[2mm] r\dfrac{d\theta}{d\tau} = \sin\theta \\[2mm] \dfrac{d\varphi}{d\tau} = 0 \quad \Rightarrow \varphi = \varphi_0 \end{array}\right. \quad \Longrightarrow \quad \frac{dr}{r\,d\theta} = \frac{2\cos\theta}{\sin\theta} \quad \Longleftrightarrow \quad \int_{r_0}^{r}\frac{dr}{r} = \int_{\theta_0}^{\theta}\frac{2\cos\theta}{\sin\theta}\,d\theta$$

$$\ln\frac{r}{r_0} = 2\ln\frac{\sin\theta}{\sin\theta_0} = \ln\left(\frac{\sin\theta}{\sin\theta_0}\right)^2$$

$$r = r_0\left(\frac{\sin\theta}{\sin\theta_0}\right)^2 = \frac{2\sin^2\theta}{(1/\sqrt{2})^2} = 4\sin^2\theta$$

$$\varphi = \pi/6$$

5.9 Förenkla $\mathbf{B} = \mathbf{r} \times \{\nabla \times [\mathbf{r} \times (\mathbf{A} \times \mathbf{r})]\}$

$\mathbf{r}$ = ortsvektorn $= x\hat{x} + y\hat{y} + z\hat{z} \quad (= \rho\hat{\rho} + z\hat{z} = r\hat{r})$

$\mathbf{A}$ = konst. vektor

$$\mathbf{r} \times (\mathbf{A} \times \mathbf{r}) = (\mathbf{r} \cdot \mathbf{r})\mathbf{A} - (\mathbf{r} \cdot \mathbf{A})\mathbf{r} = r^2\mathbf{A} - (\mathbf{r} \cdot \mathbf{A})\mathbf{r}$$

$$\varepsilon_{ijk}\,\varepsilon_{klm}\,A_l r_m = \varepsilon_{ijk}\varepsilon_{klm}\, r_j A_l r_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, r_j A_l r_m =$$

$$= r_j A_i r_j - r_j A_j r_i \implies (\mathbf{r} \cdot \mathbf{r})\mathbf{A} - (\mathbf{r} \cdot \mathbf{A})\mathbf{r}$$

$$\nabla \times (r^2\mathbf{A} - (\mathbf{r} \cdot \mathbf{A})\mathbf{r}) = \nabla \times r^2\mathbf{A} - \nabla \times (\mathbf{r} \cdot \mathbf{A})\mathbf{r} - \nabla \times (\mathbf{r} \cdot \mathbf{A})\mathbf{r} =$$

$$= \underbrace{(\nabla r^2)}_{2\mathbf{r}} \times \mathbf{A} + \mathbf{r} \times \underbrace{\nabla(\mathbf{r} \cdot \mathbf{A})}_{\mathbf{A}} - (\mathbf{r} \cdot \mathbf{A})\underbrace{(\nabla \times \mathbf{r})}_{=0} = 3\mathbf{r} \times \mathbf{A}$$

∇r^2 : $\quad \partial_i r_j r_j = 2r_j \partial_i r_j = 2r_j \delta_{ij} = 2r_i \implies 2\mathbf{r}$

$\nabla(\mathbf{r} \cdot \mathbf{A})$: $\quad \partial_i r_j a_j = a_j \partial_i r_j = a_j \delta_{ij} = a_i \implies \mathbf{A}$

$\nabla \times \mathbf{r}$: $\quad \varepsilon_{ijk}\partial_j r_k = \varepsilon_{ijk}\delta_{jk} = \varepsilon_{ijj} = 0 \implies 0$

$$\mathbf{B} = \mathbf{r} \times (3\mathbf{r} \times \mathbf{A}) = 3[(\mathbf{r} \cdot \mathbf{A})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{A}] = 3(\mathbf{r} \cdot \mathbf{A})\mathbf{r} - 3r^2\mathbf{A}$$

6.13

$$\mathbf{F}(r,\theta,\varphi) = \frac{F_0}{ar\sin\theta}\Big[(a^2 + ar\sin\theta\cos\varphi\,(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}) - (a^2 + ar\sin\theta\sin\varphi - r^2\sin^2\theta)\,\hat{\varphi}\Big]$$

Linjeintegral av $\mathbf{F}$ längs kurva C

C: skärningskurvan mellan S_1 och S_2

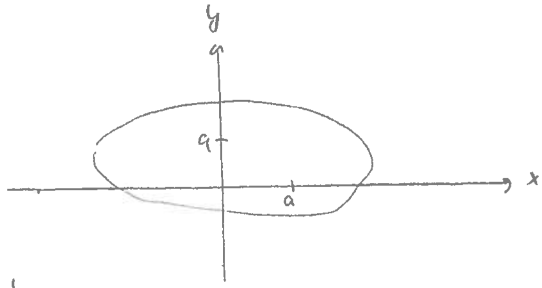
S_1: $x^2 + 4y^2 = 12a^2 + 8ay$

S_2: $x^2 + y^2 = 4az - 2ay - a^2$

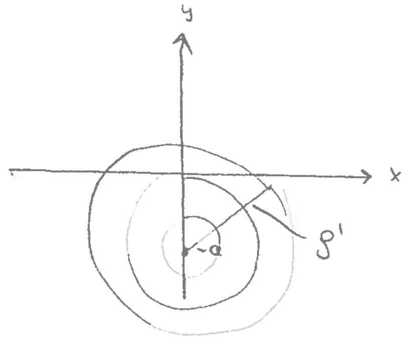
$S_1: \; x^2+4y^2-8ay+4a^2 = 12a^2+4a^2$

$$x^2+4(y-a)^2 = 16a^2$$

$$\left(\frac{x}{4a}\right)^2 + \left(\frac{y-a}{2a}\right)^2 = 1$$



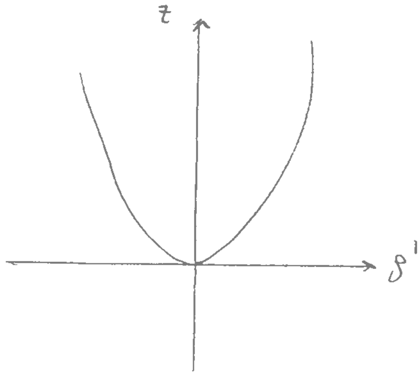
Elliptisk cylinder med axel parallell med z-axeln genom $x=0$, $y=a$



$S_2: \; x^2+y^2+2ay+a^2 = 4az$

$$x^2+(y+a)^2 = 4az = \rho'^2 \iff z = \frac{(\rho')^2}{4a}$$

ρ' radie

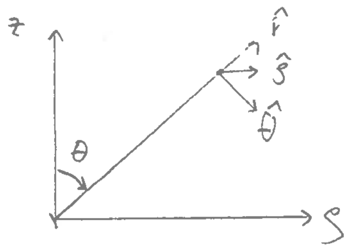


paraboloid med öppning mot pos. z-axeln. Spets i $(0,-a,0)$

Titta på $\mathbb{F}$

Utnyttja $\rho = r\sin\theta$

$\hat{\rho} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$



$$\mathbb{F} = \frac{F_0}{a\rho}\left[(a^2+a\rho\cos\varphi)\hat{\rho} - (a^2+a\rho\sin\varphi-\rho)\hat{\varphi}\right]$$

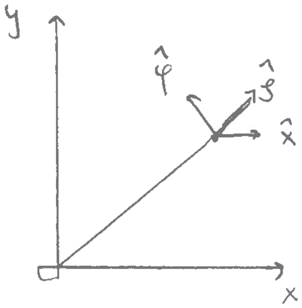
Dela upp fältet singulär del $\mathbb{F}_s$ och reguljär del $\mathbb{F}_r$

$\mathbb{F} = \mathbb{F}_s + \mathbb{F}_r$

$$\mathbb{F}_s = F_0 a\left(\frac{\hat{\rho}}{\rho} - \frac{\hat{\varphi}}{\rho}\right) \qquad \nabla\times\mathbb{F}_s = 0 \text{ om } \rho>0$$

linjekälla   virveltråd

$$\mathbb{F}_r = F_0\left[\cos\varphi\,\hat{\rho} - \sin\varphi\,\hat{\varphi} + \frac{\rho}{a}\hat{\varphi}\right] = F_0\left(\hat{x} + \frac{\rho}{a}\hat{\varphi}\right)$$



$$\nabla \times \mathbb{F}_\varrho = F_0\, a \frac{1}{\varrho} \begin{vmatrix} \hat{\varrho} & \varrho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \varrho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ \frac{1}{\varrho} & \varrho\left(-\frac{1}{\varrho}\right) & 0 \end{vmatrix} = 0$$

skalfaktor

$$\nabla \times \mathbb{F}_r = \nabla \times \frac{F_0}{a}\varrho\hat{\varphi} = \frac{F_0}{a}\frac{1}{\varrho} \begin{vmatrix} \hat{\varrho} & \hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial \varrho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & \varrho\cdot\varrho & 0 \end{vmatrix} =$$

$$= \frac{F_0}{a\varrho} 2\varrho\hat{z} = \frac{2F_0}{a}\hat{z}$$ (Stokes sats)

5.2 Visa $\nabla(\mathbb{A}\cdot\mathbb{B}) = \overbrace{(\mathbb{A}\cdot\nabla)}^{A_i\partial_i}\mathbb{B} + (\mathbb{B}\cdot\nabla)\mathbb{A} + \mathbb{A}\times(\nabla\times\mathbb{B}) + \mathbb{B}\times(\nabla\times\mathbb{A})$

$[\mathbb{a}\times(\mathbb{b}\times\mathbb{c}) = (\mathbb{a}\cdot\mathbb{c})\,\mathbb{b} - (\mathbb{a}\cdot\mathbb{b})\,\mathbb{c}]$

$\Rightarrow$ $\mathbb{A}\times(\nabla\times\mathbb{B}) = (\mathbb{A}\cdot\mathbb{B})\nabla - (\mathbb{A}\cdot\nabla)\mathbb{B}$ $\qquad \nabla(\mathbb{A}\cdot\mathbb{B})$

$\mathbb{B}\times(\nabla\times\mathbb{A}) = \nabla(\mathbb{A}\cdot\mathbb{B}) - (\mathbb{B}\cdot\nabla)\mathbb{A}$ $\qquad$ tensor matrisen

tillsammans: $\nabla(\mathbb{A}\cdot\mathbb{B})$ klart $\qquad A_j\partial_i B_j$

Samma med indexnotation

$$(\mathbb{A}\times(\nabla\times\mathbb{B}))_i = \varepsilon_{ijk}A_j(\nabla\times\mathbb{B})_k = \varepsilon_{ijk}A_j\varepsilon_{klm}\partial_l B_m =$$

$$= \left[\begin{array}{l} \varepsilon_{ijm}\varepsilon_{klm} = \delta_{il}\delta_{jl} - \delta_{il}\delta_{jk} \\ \varepsilon_{ikl}\varepsilon_{jkl} = 2\delta_{ij} \\ \varepsilon_{ijk}\varepsilon_{ijk} = 6 \end{array}\right] = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\, A_i\,\partial_l B_m$$

$$\delta_{il}\partial_l = \partial_i$$

$$= A_j\partial_i B_j - A_j\partial_j B_i$$

$$-((\mathbb{A}\cdot\nabla)\mathbb{B})_i \qquad -((\mathbb{B}\cdot\nabla)\mathbb{A})_i$$

$$\mathbb{A}\times(\nabla\times\mathbb{B}) + \mathbb{B}\times(\nabla\times\mathbb{A}) = A_j\partial_i B_j - A_j\partial_j B_i + B_j\partial_i A_j - B_j\partial_j A_i$$

$$\partial_i(A_j B_j) = \nabla(\mathbb{A}\cdot\mathbb{B})$$

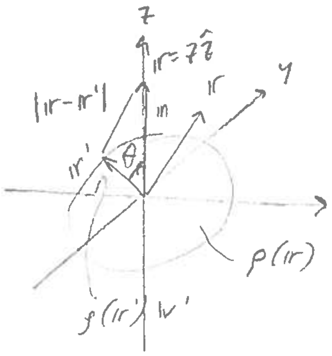
$\nabla \times (\nabla \phi) = 0$ $\qquad \varepsilon_{ijk} \underbrace{\partial_j \partial_k}_{= \partial_k \partial_j} \phi$

komponent 1: $\varepsilon_{ijk} \partial_j \partial_k \phi = + \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \phi - \frac{\partial}{\partial x^3} \frac{\partial}{\partial x^2} \phi = 0$

$$\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{array}$$

Ex 6.5

(ingen inskränkning)



$$\phi(\mathbb{r}) = \int_{\mathbb{R}^3} dV' \frac{\rho(\mathbb{r}')}{4\pi |\mathbb{r} - \mathbb{r}'|}$$

$$|\mathbb{r} - \mathbb{r}'|^2 = r^2 + r'^2 - 2rr'\cos\theta$$

$$\Rightarrow \quad = \int_0^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\varphi'\, r'^2 \sin\theta' \frac{2\pi\, \rho(r')}{4\pi\sqrt{r^2 + r'^2 - 2r'r\cos\theta'}} =$$

$$= \int_{-\infty}^{\infty} dr' \frac{1}{2\rho}(r') \frac{r'}{r} \underbrace{\left[\sqrt{r'^2 + r^2 - 2rr'\cos\theta'}\right]_{\theta=0}^{\theta=\pi}}_{|r+r'| - |r-r'|}$$

$$|r + r'| - |r - r'| = \begin{cases} 2r', & r > r' \\ 2r, & r < r' \end{cases} \quad \Rightarrow \text{integralen} = \int_{-\infty}^{\infty} dr' r' \rho(r') + \frac{1}{r} \int_0^r dr' r'^2 \rho(r')$$

Utanför en laddningsfördelning som ligger i $0 < r' < R$

$r > R \qquad \phi(r) = \frac{1}{r} \int_0^R dr' r'^2 \rho(r') \qquad$ Gör $\phi = \frac{Q}{4\pi r}$

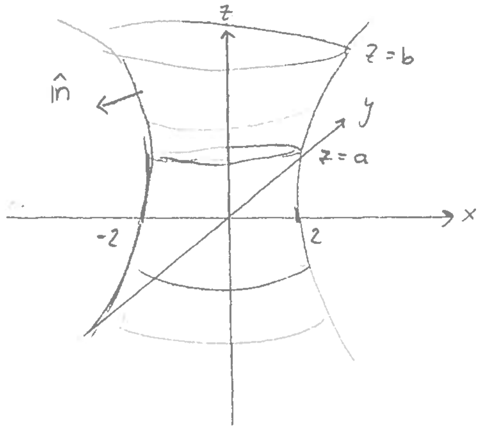
$\frac{Q}{4\pi}$

6.8

$$x^2+y^2-z^2=4 \quad \Rightarrow \quad \left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2-\left(\frac{z}{2}\right)^2=1, \quad u = \frac{(xz,\, yz,\, xy)}{x^2+y^2}$$

$a \le z \le b$

Beräkna $\int_S u \cdot dS$



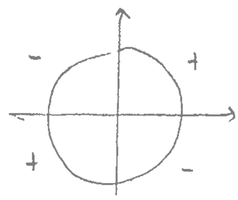
Testa Gauss!

(Reservstrategi: parametrisering i cyl koord.)

$$u = \underbrace{\frac{z}{\rho^2}\underbrace{(x\hat{x}+y\hat{y})}_{\rho\hat{\rho}}}_{\frac{z}{\rho}\hat{\rho}} + \cancel{\frac{xy}{x^2+y^2}}\,\hat{z} \to 0$$

inget bidrag från lock och botten

symmetrisk: uppifrån



Linjekälla på z-axeln med konstant täthet:

$$u = \frac{k}{2\pi\rho}\hat{\rho} \qquad \phi = \frac{-k}{2\pi}\log\rho$$

$k = 2\pi z$? $\qquad \phi \overset{?}{=} -z_1$

$\nabla \cdot u_1 = 0$ utanför z-axeln. Det finns en linjekälla med styrkan $2\pi z$ + divergensfritt

Gauss sats $\Longrightarrow$ Inget bidrag från locken.

$$= \int u \cdot dS = \text{innesluten källa}$$

$$= \int_a^b 2\pi z\, dz = \pi(b^2-a^2)$$

(Sten Salomonsson)

2010-09-29
Onsdag

6.13

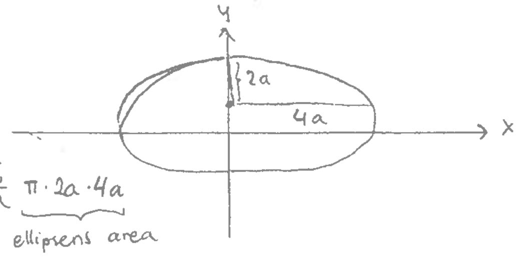
$\mathbb{F} = \mathbb{F}_s + \mathbb{F}_r$

$\mathbb{F}_s = F_0 a \left(\frac{\hat{\rho}}{\rho} - \frac{\hat{\varphi}}{\rho} \right)$ = linjekälla och virveltråd

$\mathbb{F}_r = F_0 \left(\hat{x} + \frac{\rho}{a} \hat{\varphi} \right)$, $\quad \nabla \times \mathbb{F}_r = \ldots = \frac{2F_0}{a} \hat{z}$

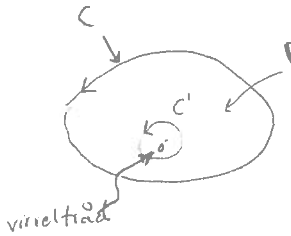
Beräkna $\int_C \mathbb{F} \cdot d\mathbb{r}$ C är en ~~kurva~~ kurva runt en elliptisk cylinder

Stokes sats: $\oint_C \mathbb{F}_r \cdot d\mathbb{r} = \int_S \overbrace{\nabla \times \mathbb{F}_r}^{\frac{2F_0}{a}\hat{z}} \cdot d\mathbb{S} =$



$= \underbrace{\int_{S_{\text{mantelyta}}}}_{\substack{=0 \\ \hat{z} \perp d\mathbb{S}}} + \int_{\substack{S_{\text{bottenyta}} \\ \text{i xy-planet}}} = \int_{\substack{S_{\text{bottenyta}} \\ \hat{z} \parallel d\mathbb{S}}} \frac{2F_0}{a}\, dS = \frac{2F_0}{a} \underbrace{\pi \cdot 2a \cdot 4a}_{\text{ellipsens area}}$

$= 16\pi a F_0$

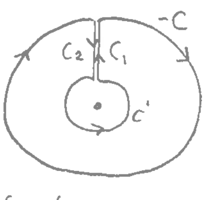


$\nabla \times \mathbb{F}_s = 0$

$\int_C \mathbb{F}_s \cdot d\mathbb{r} = \int_{C'} \mathbb{F}_s \cdot d\mathbb{r}$

$\int_{C'} + \int_{C_1} - \int_{C} - \int_{C_1} = 0$

$\Longrightarrow \int_{C'} = \int_C$



("integrerar man runt en virveltråd får man 2π")

bara virveltråden bidrar

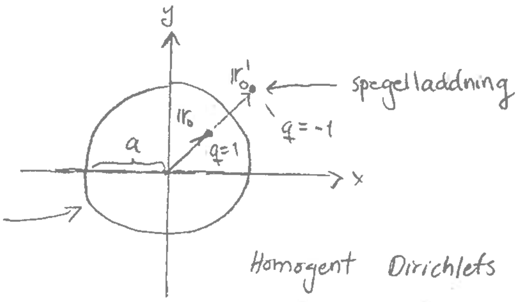
$\int_C \mathbb{F}_s \cdot d\mathbb{r} = -F_0 a \frac{1}{\rho} 2\pi \rho = -2\pi a F_0$

Svar: $\oint_C \mathbb{F} \cdot d\mathbb{r} = \pm 14\pi a F_0$

riktning runt kurvan var inte angiven

9.4

Två dimensioner



$G(\mathbb{r},\mathbb{r}_0)=0$ då $|\mathbb{r}|=a$

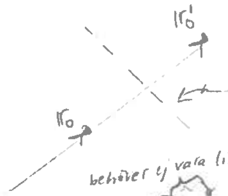
Homogent Dirichlets randvillkor

$\phi(\mathbb{r}) = G(\mathbb{r},\mathbb{r}_0)$ är potentialen från en punktladdning $q=1$ i $\mathbb{r}_0$ och har $\phi(\mathbb{r})\Big|_{|\mathbb{r}|=a} = 0$

På hela $\mathbb{R}^2$ fungerar $G(\mathbb{r},\mathbb{r}_0) = -\frac{1}{2\pi}\ln|\mathbb{r}-\mathbb{r}_0| + \text{konstant}$

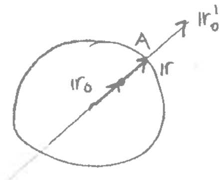
Prova om $G_{prov}(\mathbb{r},\mathbb{r}_0) = -\frac{1}{2\pi}\ln|\mathbb{r}-\mathbb{r}_0| + \frac{1}{2\pi}\ln|\mathbb{r}-\mathbb{r}_0'| = -\frac{1}{2\pi}\ln\frac{|\mathbb{r}-\mathbb{r}_0|}{|\mathbb{r}-\mathbb{r}_0'|}$

Var är $G_{prov}(\mathbb{r},\mathbb{r}_0) = 0 \Rightarrow |\mathbb{r}-\mathbb{r}_0| = |\mathbb{r}-\mathbb{r}_0'|$



Prova om vi kan få G_{prov} = konstant på randen.

$\frac{|\mathbb{r}-\mathbb{r}_0|}{|\mathbb{r}-\mathbb{r}_0'|}$ = konstant



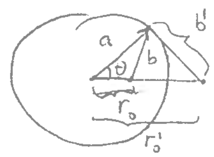
$$\frac{a-r_0}{r_0'-a} = \frac{a+r_0}{r_0'+a}$$

$$(a-r_0)(a+r_0') = (a+r_0)(r_0'-a)$$

$$a^2 + ar_0' - ar_0 - r_0r_0' = ar_0' - a^2 + r_0r_0' - r_0a$$

$$\boxed{a^2 = r_0r_0'}$$

är $\frac{b}{b'}$ konstant?



$$\left(\frac{b'}{b}\right)^2 = \frac{a^2+(r_0')^2-2ar_0'\cos\theta}{a^2+r_0^2-2ar_0\cos\theta} = \left[r_0' = \frac{a^2}{r_0}\right] = \frac{a^2+\frac{a^4}{r_0^2}-2\frac{a^3}{r_0}\cos\theta}{a^2+r_0^2-2ar_0\cos\theta}$$

Bryt ut $\frac{a^2}{r_0^2}$

$$\frac{\frac{a^2}{r_0^2}(r_0^2+a^2-2ar_0\cos\theta)}{a^2+r_0^2-2ar_0\cos\theta} = \frac{a^2}{r_0^2} = \text{konstant}$$

$$G_{prov}(\mathbb{r},\mathbb{r}_0)\Big|_{|\mathbb{r}|=a} = -\frac{1}{2\pi}\ln\frac{|\mathbb{r}-\mathbb{r}_0|}{|\mathbb{r}-\mathbb{r}_0'|}\Bigg|_{|\mathbb{r}|=a} = -\frac{1}{2\pi}\ln\frac{b}{b'} = -\frac{1}{2\pi}\ln\frac{r_0}{a} = \text{konstant}$$

Dra bort konstanten: $\boxed{G(\mathbb{r},\mathbb{r}_0) = -\frac{1}{2\pi}\ln\frac{|\mathbb{r}-\mathbb{r}_0|}{|\mathbb{r}-\mathbb{r}_0'|} + \frac{1}{2\pi}\ln\frac{r_0}{a} \quad \text{där } r_0r_0' = a^2 \text{ och } \mathbb{r}_0 \text{ parallell med } \mathbb{r}_0'}$

Om vi vill lösa $\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r})$ i cirkeln $|\mathbf{r}|<a$
$\phi(\mathbf{r})=0$ då $|\mathbf{r}|=a$

$$\phi(\mathbf{r}) = \int_{x'^2+y'^2\leq a^2} \underbrace{G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,dx'dy'}_{\text{tänk er detta som punktladdningar}}$$

$\rho(\mathbf{r})$ = ytladdningstäthet i xy-planet

2.14 Ett kroklinjigt koordinatsystem U, V, W ges av sambanden

(1) $U = r(1-\cos\theta)$
(2) $V = r(1+\cos\theta)$ där r, θ, φ är sfäriska koordinater
(3) $W=\varphi$

Visa att systemet är ortogonalt och beräkna skalfaktorerna.
Hur ser gradientoperatorn ∇ och ortsvektorn $\mathbf{r}$ ut i U,V,W-systemet?

$\mathbf{r}(U,V,W)$

$$\frac{d\mathbf{r}}{dU} = h_U\hat{e}_U \qquad h_U = \left|\frac{d\mathbf{r}}{dU}\right|$$

$$\frac{d\mathbf{r}}{dV} = h_V\hat{e}_V$$

$$\frac{d\mathbf{r}}{dW} = h_W\hat{e}_W$$

(1) och (2) ger $r = \frac{U+V}{2}$

$2r\cos\theta = V-U$, $\cos\theta = \frac{V-U}{2r} = \frac{V-U}{V+U}$

$$\begin{cases} r = \frac{U+V}{2} \\ \theta = \arccos\frac{V-U}{V+U} \\ \varphi = W \end{cases}$$

$r(U,V,W)$
$\downarrow$
$\mathbf{r}(r,\theta,\varphi)$
$\downarrow$

$$\frac{d\mathbf{r}}{d\theta} = r\hat{\theta}$$

$$\frac{d\mathbf{r}}{dU} = \frac{\partial\mathbf{r}}{\partial r}\frac{\partial r}{\partial U} + \frac{\partial\mathbf{r}}{\partial\theta}\frac{\partial\theta}{\partial U} + \frac{\partial\mathbf{r}}{\partial\varphi}\underbrace{\frac{\partial\varphi}{\partial U}}_{=0} = \hat{r}\frac{\partial r}{\partial U} + r\hat{\theta}\frac{\partial\theta}{\partial U}$$

$$\boxed{\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}}$$

$$\nabla U = \frac{1}{h_U}\hat{e}_U = \hat{r}\frac{\partial U}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial U}{\partial\theta} = \hat{r}(1-\cos\theta) + \hat{\theta}\frac{1}{r}r\sin\theta$$

$$= (1-\cos\theta)\hat{r} + \sin\theta\,\hat{\theta} = \frac{1}{h_U}\hat{U}$$

$$|\nabla U| = \sqrt{(1-\cos\theta)^2+\sin^2\theta} = \sqrt{1+\cos^2\theta-2\cos\theta+\sin^2\theta} = \sqrt{2(1-\cos\theta)} = \sqrt{\frac{2U}{r}} =$$

$$= \sqrt{\frac{2U}{\left(\frac{U+V}{2}\right)}} = \sqrt{\frac{4U}{U+V}} = \frac{1}{h_U}, \quad h_U = \frac{1}{2}\sqrt{\frac{U+V}{U}}$$

$$\nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial\theta} = (1+\cos\theta)\hat{r} - \sin\theta\,\hat{\theta}$$

$$|\nabla V| = \sqrt{(1+\cos\theta)^2+\sin^2\theta} = \sqrt{2(1+\cos\theta)} = \sqrt{\frac{2V}{r}} = \sqrt{\frac{4V}{U+V}} \qquad h_V = \frac{1}{2}\sqrt{\frac{U+V}{V}}$$

$$\nabla W = \hat{\varphi}\,\frac{1}{r\sin\theta}\,\frac{\partial W}{\partial\varphi} = \frac{1}{r\sin\theta}\,\hat{\varphi} = \frac{1}{h_\varphi}\,\hat{W}$$

$$|\nabla W| = \frac{1}{r\sin\theta} = \frac{1}{r\sqrt{1-\cos^2\theta}} = \frac{1}{r\sqrt{(1-\cos)(1+\cos)}} = \frac{1}{\sqrt{UV}} \qquad h_W = \sqrt{UV}$$

$$\hat{U} = h_u \nabla U = h_U\left((1-\cos\theta)\hat{r} + \sin\theta\,\hat{\theta}\right)$$
$$\hat{V} = h_V \nabla V = h_V\left((1+\cos\theta)\hat{r} - \sin\theta\,\hat{\theta}\right)$$

Kolla att $\hat{U}, \hat{V}$ och $\hat{W}$ verkligen är ortogonala mot varandra

Vi vill uttrycka ortsvektorn $\mathbb{r}$ i de nya koordinaterna.

$$\mathbb{r} = x\hat{x} + y\hat{y} + z\hat{z} = \rho\hat{\rho} + z\hat{z} = r\hat{r} = \;?\;(U,V,W)$$

ortsvektor

$$\frac{\hat{U}}{h_U} + \frac{\hat{V}}{h_V} = 2\hat{r} \qquad \mathbb{r} = r\hat{r} = \frac{r}{2}\left(\frac{\hat{U}}{h_U} + \frac{\hat{V}}{h_V}\right) = \underline{\underline{\frac{U+V}{4}\left(2\sqrt{\frac{U}{U+V}}\,\hat{U} + 2\sqrt{\frac{V}{U+V}}\,\hat{V}\right)}}$$

ortsvektorn

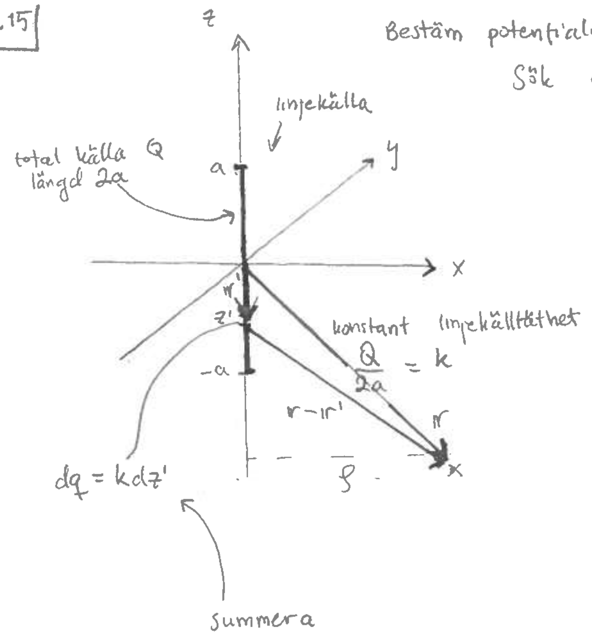
$$\nabla = \hat{U}\frac{1}{h_U}\frac{\partial}{\partial U} + \hat{V}\frac{1}{h_V}\frac{\partial}{\partial V} + \hat{W}\frac{1}{h_W}\frac{\partial}{\partial W} =$$
$$= \underline{\underline{\hat{U}\,2\sqrt{\frac{U}{U+V}}\,\frac{\partial}{\partial U} + \hat{V}\,2\sqrt{\frac{V}{U+V}}\,\frac{\partial}{\partial V} + \hat{W}\frac{1}{\sqrt{UV}}\frac{\partial}{\partial W}}}$$

$\boxed{\text{2010-10-06 Onsdag}}$

9.15

Bestäm potentialen!

Sök $\phi(\mathbb{r})$



dq ger bidrag $\quad d\phi(\mathbb{r}) = \dfrac{dq}{4\pi|\mathbb{r}-\mathbb{r}'|}$

kdz'

$\sqrt{\rho^2 + (z-z')^2}$

$$dq: \quad \text{linje}: \; k\,ds' \qquad \text{yta}: \; \sigma\, dS' \qquad \text{volym}: \; \rho\, dV'$$

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \int_{-a}^{a} \frac{k\,dz'}{\sqrt{(z'-z)^2+\rho^2}} =$$

$$= \left[\int \frac{dx}{\sqrt{x^2+c^2}} = -\log\left(\sqrt{x^2+c^2} - x\right) \right] = \frac{Q}{8\pi a} \left[-\log\left(\sqrt{(z'-z)^2+\rho^2} - (z'-z)\right) \right]_{z'=-a}^{a} =$$

$$= \frac{Q}{8\pi a} \log \frac{\sqrt{(z+a)^2+\rho^2} + z + a}{\sqrt{(z-a)^2+\rho^2} + z - a}$$

Kontrollera dim. analys

$\rho \to \infty$

$\rho \to 0$ (nära linjen)

Kontroll: $\frac{r}{a} \longrightarrow \infty$ (även $a \to 0$)

$(a \to 0)$

$$\phi = \frac{Q}{8\pi a} \log \frac{\sqrt{\underbrace{\rho^2 + z^2}_{r^2} + \underbrace{2az}_{r\cos\theta} + a^2} + z + a}{-\quad -}$$

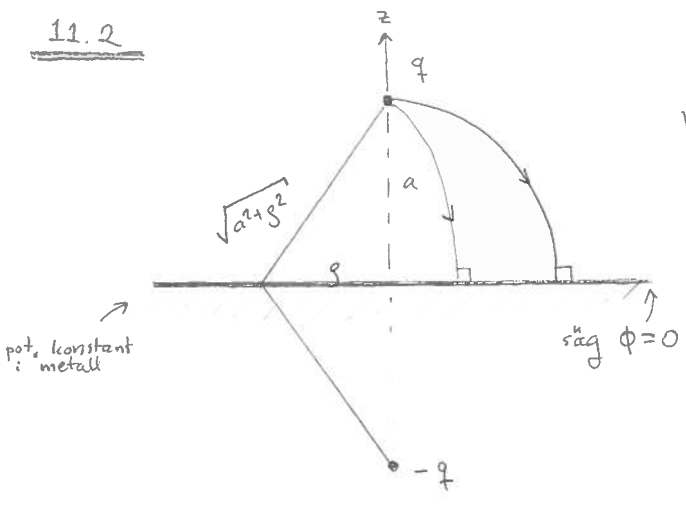
$$= \frac{Q}{8\pi a} \log \frac{\sqrt{r^2 + 2ar\cos\theta + a^2} + r\cos\theta + a}{-\quad -} = \frac{Q}{8\pi a} \log \frac{\sqrt{1 + \frac{2a\cos\theta}{r} + \frac{a^2}{r^2}} + \cos\theta + \frac{a}{r}}{-\quad -}$$

$$= \left[\sqrt{1+x} = 1 + \tfrac{1}{2}x + O(x^2) \right] = \frac{Q}{8\pi a} \log \frac{\overbrace{1 + \frac{a\cos\theta}{r} + O\left(\frac{1}{r^2}\right) + \cos\theta + \frac{a}{r}}^{(1+\cos\theta)(1+\frac{a}{r}) + O(\frac{1}{r^2})}}{\underbrace{-\quad -}_{(1+\cos\theta)(1-\frac{a}{r}) + O(\frac{1}{r^2})}} =$$

$$= \left[\frac{1}{1-\frac{a}{r}} = 1 + \frac{a}{r} + O\left(\frac{1}{r^2}\right) \right] = \frac{Q}{8\pi a} \underbrace{\log\left(1 + \frac{2a}{r} + O\left(\frac{1}{r^2}\right)\right)}_{\frac{2a}{r} + O(\frac{1}{r^2})} = \frac{Q}{4\pi r} + O\left(\frac{1}{r^2}\right) \quad \text{Ok!}$$

$\log(1+x) = x + O(x^2)$

11.2

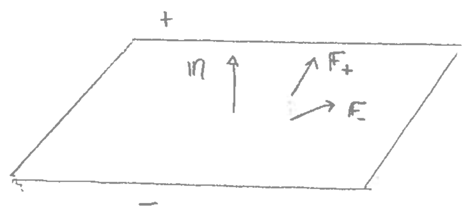


Vad är ytladdningen på metallen?

elektronerna kommer upp i ytan så att fältet under ytan blir 0

$$\Phi(z=0) = \frac{q}{4\pi\varepsilon_0\sqrt{a^2+\rho^2}} - \frac{q}{4\pi\varepsilon_0\sqrt{a^2+\rho^2}} = 0$$

$$\phi(\mathbf{r}) = \frac{q}{4\pi|\mathbf{r}-a\hat{z}|} - \frac{q}{4\pi|\mathbf{r}+a\hat{z}|}$$ = potentialen



elektriska fältet alldeles ovanför ytan

$$\sigma = \varepsilon_0\,\mathbf{n}\cdot(\mathbf{E}_+ - \mathbf{E}_-)$$

under, $\mathbf{E}_- = 0$

"precis ovanför", "precis under är det noll"

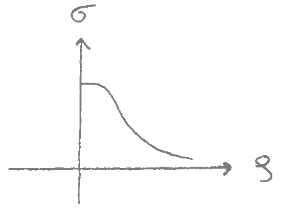
$$\mathbf{E} = -\nabla\phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{\mathbf{r}-a\hat{z}}{|\mathbf{r}-a\hat{z}|^3} - \frac{\mathbf{r}+a\hat{z}}{|\mathbf{r}+a\hat{z}|^3}\right)$$

På ytan: $|\mathbf{r}+a\hat{z}| = |\mathbf{r}-a\hat{z}| = \sqrt{\rho^2+a^2}$

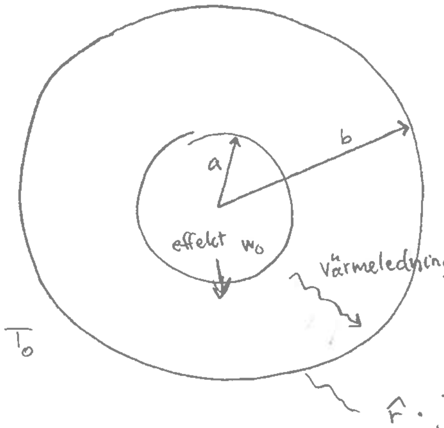
$$\mathbf{E}(z=0) = \frac{-qa\hat{z}}{2\pi\varepsilon_0(\rho^2+a^2)^{3/2}} \qquad \sigma = -\frac{qa}{2\pi(\rho^2+a^2)^{3/2}}$$

Total inducerad laddning: $-\frac{qa}{2\pi}2\pi\int_0^\infty d\rho\,\frac{\rho}{(\rho^2+a^2)^{3/2}} = -qa\left[-\frac{1}{\sqrt{\rho^2+a^2}}\right]_0^\infty$

$= \underline{-q}$



10.3



T: temp.

$\mathbf{J}$: värmeströmtäthet

$$\underline{\mathbf{J} = -\lambda \nabla T}$$

$\Delta T = 0$ (inget t-beroende innanför)

$\hat{r} \cdot \mathbf{J} = \alpha (T - T_0)$

$$\Delta T = 0 \quad \Rightarrow \quad \underbrace{T = A + \frac{B}{r}}_{\text{allmän lösn.}} \qquad \Delta T = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right)$$

$$\mathbf{J} = \lambda \frac{B \hat{r}}{r^2}$$

Effekt genom inre sfären : $-\oint \mathbf{J} \cdot d\mathbf{S} = \frac{\lambda B}{a^2} 4\pi a^2 = W_0$

$$B = \frac{W_0}{4\pi\lambda}$$

På yttre ytan : $\frac{\lambda B}{b^2} = \alpha \left(A + \frac{B}{b} - T_0\right) \Longrightarrow A = \ldots =$

$$T = T_0 + \frac{W_0}{4\pi\alpha b^2} + \frac{W_0}{4\pi b^2}\left(\frac{1}{r} - \frac{1}{b}\right)$$