Vektor

Röv 2001

62 sidor

30:—

LP1

010905

lv 1   * pappersuppgift 1.

övn 1

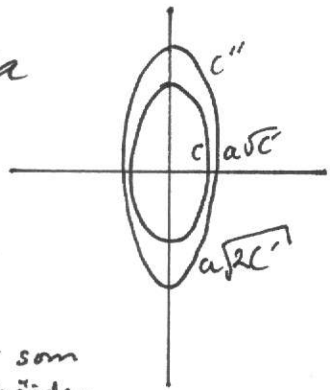
Inom ett bergsmassiv beskrivs den lokala nivån $h(x,y)$ över havsytan av funktionen

$$h(x,y) = \frac{k}{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{\sqrt{2}a}\right)^2 + 1}$$

a, k konstanter. Koordinatsystemet valt så att $\hat{x}$ - ($\hat{y}$ -) axeln ligger i väst-östlig (syd-nordlig) riktning.

a) Skissera nivå-linjerna

För att skissera nivå-linjerna: $h(x,y) = c$ ← konstant som anger höjden



$c'' < c$

ellips med ½axlarna $a\sqrt{c'}$ och $a\sqrt{2c'}$

$$\frac{k}{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{\sqrt{2}a}\right)^2 + 1} = c$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{\sqrt{2}a}\right)^2 = \frac{k}{c} - 1 = c'$$

$$\left(\frac{x}{a\sqrt{c'}}\right)^2 + \left(\frac{y}{a\sqrt{2c'}}\right)^2 = 1$$

* uppgift från utdelat papper

b) Bergsmassivet är brantast i området väster och öster om toppen. Bestäm stigningen och höjden över havet i den brantaste punkten. Givet: $a = 1$ m, $k = 1000$ m

För att derivera ett fält: $\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x_i}\hat{x}_i$

Stigningen fås ur $|\text{grad}\,h(x,y)| = |\nabla h(x,y)|$

$$\nabla h(x,y) = \frac{-k}{\left(x^2 + \frac{y^2}{2} + 1\right)^2}\left(2x\hat{x} + y\hat{y}\right)$$

$$|\nabla h(x,y)| = \frac{k}{\left(x^2 + \left(\frac{y}{2}\right)^2 + 1\right)^2}\sqrt{\left(4x^2 + y^2\right)}$$

$$\left.\frac{\partial |\nabla h(x,y)|}{\partial x}\right|_{y=0} = 0 \Rightarrow \left(-2 + 6x^2\right) = 0 \begin{cases} 0,0 \\ \pm\frac{1}{\sqrt{3}},0 \end{cases}$$

Stigningen i max:

$$\left|\nabla h\left(\pm\frac{1}{\sqrt{3}},0\right)\right| = 650 \text{ m}$$

Höjden vid maxstigningen:

$$h\left(\pm\frac{1}{\sqrt{3}},0\right) = 750 \text{ m}$$

pappersuppg. 2)

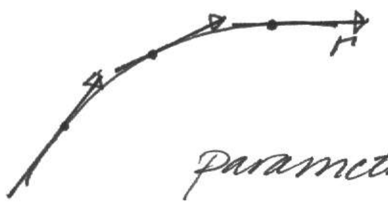
Ett vektorfält A är givet i cylinderkoordinater som: $\mathbf{A} = \rho\cos\alpha\,\hat{\rho} + \rho\,\hat{\alpha} + \rho\cos\alpha\,\hat{z}$

Härled ekvationerna för fältlinjerna till A

Betrakta fältlinjen som går genom punkten $\rho = 3$, $\alpha = \frac{\pi}{2}$, $z = 2$

I vilka punkter går denna fältlinje genom planet $y = 0$

Fältlinjens tangentvektor r är parallell med vektorfältet A i varje punkt p.



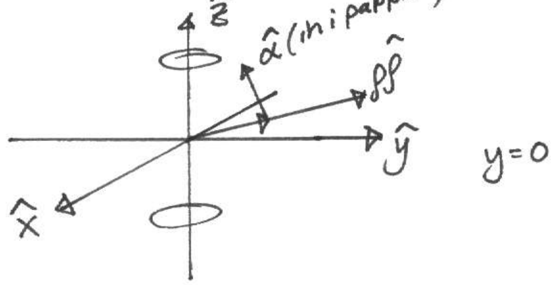
parametrisering: $\mathbf{r} = \mathbf{r}(u)$

$$\frac{d\mathbf{r}(u)}{du} \parallel \mathbf{A}$$

$$\frac{\frac{dx}{du}}{A_x} = \frac{\frac{dy}{du}}{A_y} = \frac{\frac{dz}{du}}{A_z}$$

$x(u\ c_1, c_2)$, $y(u\ c_1\ c_2)$, $z(u\ c_1\ c_2)$

Vektorfältet $\mathbb{A} = \rho\cos\alpha\,\hat{\rho} + \rho\hat{\alpha} + \rho\cos\alpha\,\hat{z}$



$$\frac{d\mathbf{r}}{du} = \frac{d\rho}{du}\hat{\rho} + \rho\frac{d\alpha}{du}\hat{\alpha} + \frac{dz}{du}\hat{z}$$

$$\frac{\frac{d\rho}{du}}{\rho\cos\alpha} = \frac{\rho\frac{d\alpha}{du}}{\rho} = \frac{\frac{dz}{du}}{\rho\cos\alpha}$$

$$\begin{cases} \dfrac{d\rho}{\rho} = \cos\alpha\, d\alpha \Rightarrow \ln\rho = \sin\alpha = c_1 \Rightarrow \\ \qquad\qquad \Rightarrow \rho = \exp(\sin\alpha + c_1) \\ d\rho = dz \Rightarrow \rho = z + c_2 \\ \rho\cos\alpha\, d\alpha = dz \end{cases}$$

Begynnelsevillkor $P(\rho = 3, \alpha = \frac{\pi}{2}, z = 2)$

$c_2 = 1$, $c_1 = \ln 3 - 1$

$\rho = 3\exp(\sin\alpha - 1)$; $z = 3\exp(\sin\alpha - 1) - 1$

$y = 0 \Rightarrow \alpha = \pi\cdot n$

$\rho = 3\exp(-1)$, $z = 3\exp(-1) - 1$

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lv 2 (*) studentuppgift 1:(□)PLK 1.3.1:6
övn. 2

6a) $T = T_0(x^2 + 2yz - z^2)$

$T_0 = 1°C/m^2$

Frusen fluga i $(1,1,2)$

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = T_0(2x, 2z, 2y - 2z)$$

$\Rightarrow (2,4,-2)$ riktningen som flugan skall flyga i.

b) $v = 0.3\ m/s$ $\quad \tau = (-2, 2, 1)$

Hur snabbt ökar temperaturen?

$$\hat{\tau} = \frac{\tau}{|\tau|} = \frac{1}{3}(-2,2,1)$$

$$\frac{dT}{d\tau} = \hat{\tau} \cdot \nabla T = \frac{1}{3}(-2,2,1) \cdot 1 \cdot (2,4,-2) = \frac{2}{3}\left[°C/m\right]$$

$$\frac{dT}{d\tau} \cdot v = 0.2\left[°C/s\right]$$

(*) uppgift som teknolog har redovisat vid tavlan
(□) ProblemLösningsKompendium

studentuppgift 2

PLK 1.2.1:2

$$\mathbb{E}(\mathbb{r}) = \frac{m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \qquad m \text{ konstant}$$

$$(r,\theta,\varphi) = \left(2, \frac{\pi}{4}, \frac{\pi}{6}\right)$$

$$\mathbb{r} = r\hat{r} + \theta\hat{\theta} + \varphi\hat{\varphi}$$

$$\frac{d\mathbb{r}}{du} = \frac{dr}{du}\hat{r} + r\frac{d\theta}{du}\hat{\theta} + r\sin\theta\frac{d\varphi}{du}\hat{\varphi}$$

$$\begin{cases} \dfrac{dr}{du} = \dfrac{m}{4\pi r^3} 2\cos\theta \\ r\dfrac{d\theta}{du} = \dfrac{m}{4\pi r^3}\sin\theta \\ r\sin\dfrac{d\varphi}{du} = 0 \Rightarrow \varphi = C \end{cases}$$

$$du = \cancel{\frac{4\pi r^3}{m}} \cdot \frac{dr}{2\cos\theta} = \cancel{\frac{4\pi r^3}{m}}\, r\frac{d\theta}{\sin\theta}$$

$$\int \frac{1}{r}\,dr = \int \frac{2\cos\theta}{\sin\theta}\,d\theta$$

$$\ln r = \ln \sin^2\theta + D$$

$$r = e^D \sin^2\theta$$

$$r = E\sin^2\theta$$

$$\varphi = \frac{\pi}{6} \qquad 2 = E\sin^2\frac{\pi}{4} \quad 2 = E\left(\frac{1}{\sqrt{2}}\right)^2 \quad E = 4$$

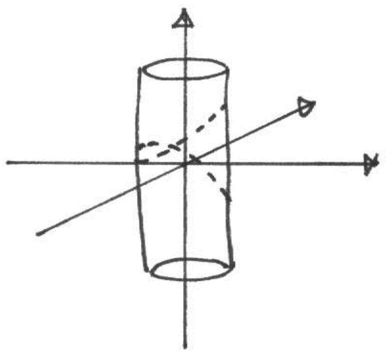
$$\begin{cases} \varphi = \dfrac{\pi}{6} \\ r = 4\sin^2\theta \end{cases}$$

Pappersuppg. 1

En partikel attraheras till origo m. kraft omvänt prop. mot avst. Vilket arbete uträttar fältet längs skruvlinjen $\mathbf{r} = a\cos t\,\hat{x} + a\sin t\,\hat{y} + bt\hat{z}$

från $t=0$ till $t=2\pi$

Kraft: $\mathbf{F} = -\frac{k\hat{r}}{r}$



cylindriska koordinater:

$$\mathbf{F} = -\frac{k}{r}\hat{r} = -\frac{k}{r^2}\mathbf{r}$$

$$\mathbf{r} = \rho\hat{\rho} + z\hat{z}$$

$$r^2 = \rho^2 + z^2$$

$$\mathbf{F} = \frac{-k}{\rho^2+z^2}(\rho\hat{\rho} + z\hat{z}) = \frac{-k}{a^2+b^2t^2}(a\hat{\rho} + bt\hat{z})$$

$$\mathbf{r}(t) = a\hat{z} + bt\hat{z} \quad ; \quad \frac{d\mathbf{r}}{dt} = b\hat{z}$$

Arbetet: $\int \mathbf{F}\cdot d\mathbf{r} = \int \mathbf{F}\frac{d\mathbf{r}}{dt} = \int_0^{2\pi} \frac{-kb^2t}{a^2+b^2t^2}\,dt =$

$$= -k\ln\sqrt{1+\left(\frac{2\pi b}{a}\right)^2}$$

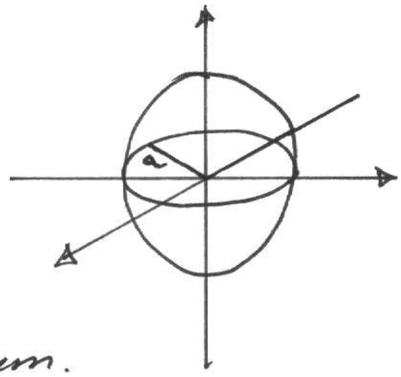
Alternativ: $\mathbf{F} = \frac{-k}{r}\hat{r}$

$$\phi = k\ln r = k\ln\sqrt{x^2+y^2+z^2}$$

PLK 2.3.1:1

Kraften $\mathbb{F} = \left(xz, yz, \frac{z^3}{a}\right)$

Beräkna $\int_S \mathbb{F} \cdot d\mathbb{S}$



S = sfär m. radie = a och origo i centrum.

$\mathbb{r} = (a\sin\theta\cos\varphi, a\sin\theta\sin\varphi, a\cos\theta)$

$\frac{d\mathbb{r}}{d\theta} = (a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, -a\sin\theta)$

$\frac{d\mathbb{r}}{d\varphi} = (-a\sin\theta\sin\varphi, a\sin\theta\cos\varphi, 0)$

$d\mathbb{S} = \frac{d\mathbb{r}}{du} \times \frac{d\mathbb{r}}{dv} \, du \, dv$

$$\frac{d\mathbb{r}}{d\theta} \times \frac{d\mathbb{r}}{d\varphi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a\cos\theta\cos\varphi & a\cos\theta\sin\varphi & -a\sin\theta \\ -a\sin\theta\cos\varphi & a\sin\theta\cos\varphi & 0 \end{vmatrix} =$$

$= (a^2\sin^2\theta\cos\varphi, a^2\sin^2\theta\sin\varphi, a^2\sin\theta\cos\theta)$

$\mathbb{F} = \left(xz, yz, \frac{z^3}{a}\right) =$

$= (a^2\sin\theta\cos\theta\cos\varphi, a^2\sin\theta\cos\theta\sin\varphi, a^2\cos^3\theta)$

$$\Rightarrow \iint \mathbf{F} \cdot d\mathbf{S} = \int_0^{\pi}\int_0^{2\pi} a^4 \sin^3\theta \cos\theta \cos^2\varphi + a^4 \sin^3\theta \cos\theta \sin^2\varphi +$$

$$+ a^4 \cos^4\theta \sin\theta \, d\theta \, d\varphi =$$

$$= 2\pi \int_0^{\pi} a^4 \sin^3\theta \cos\theta + a^4 \cos^4\theta \sin\theta \, d\theta =$$

$$2\pi a^4 \left[\frac{\sin^4\theta}{4} - \frac{\cos^5\theta}{5} \right] = \frac{4\pi}{5} a^4$$

[fel i facit]

010912
Lv2
övn3

stud.uppg. PLK 121:3

$$\mathbb{A} = \rho\cos\alpha\,\hat{\rho} + \rho^2\hat{\alpha} + \rho\sin\alpha\,\hat{z}$$

$$\frac{d\mathbf{r}}{du} = \frac{d\rho}{du}\hat{\rho} + \rho\frac{d\alpha}{du}\hat{\alpha} + \frac{dz}{du}\hat{z}$$

$$\frac{\frac{d\rho}{du}}{\rho\cos\alpha} = \frac{\rho\frac{d\alpha}{du}}{\rho^2} = \frac{\frac{dz}{du}}{\rho\sin\alpha}$$

$$\frac{\frac{d\rho}{du}}{\frac{d\alpha}{du}} = \frac{\rho\cos\alpha}{\rho} \Rightarrow \frac{d\rho}{d\alpha} = \cos\alpha \qquad d\rho = \cos\alpha\, d\alpha$$

$$\frac{\frac{d\alpha}{du}}{\frac{dz}{du}} = \frac{\rho}{\rho\sin\alpha} \Rightarrow \frac{d\alpha}{dz} = \frac{1}{\sin\alpha} \qquad dz = \sin\alpha\, d\alpha$$

Ekv. för fältlinjerna

$$\int d\rho = \int\cos\alpha\, d\alpha \Rightarrow \rho = \sin\alpha + C_1$$

$$\int dz = \int\sin\alpha\, d\alpha \Rightarrow z = -\cos\alpha + C_2$$

punkt: $(\rho = 3,\ \alpha = \frac{\pi}{2},\ z = 2)$

$3 = \sin\frac{\pi}{2} + C_1 \Rightarrow C_1 = 2$

$2 = -\cos\frac{\pi}{2} + C_2 \Rightarrow C_2 = 2$

$$\begin{cases}\rho = \sin\alpha + 2\\ z = -\cos\alpha + 2\end{cases}$$

$y = 0 \Rightarrow \alpha = \pi\cdot n\ ,\ n = 0, 1 \ldots$

jämna $n \Rightarrow \rho = 2\ ,\ z = 1$

udda $n \Rightarrow \rho = 2\ ,\ z = 3$

$\Rightarrow$ punkterna $(2,0,1)$ och $(-2,0,3)$

studuppg. 2   PLK 1.3.1:9

$$T = \frac{2+\cos\theta}{r^2}$$

$$P(r,\theta,\varphi) = \left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$$

a) $\hat{r} + \hat{\varphi} \Rightarrow v \doteq (1,0,1)(\hat{r},\hat{\theta},\hat{\varphi})$

normera $v = \frac{1}{\sqrt{2}}(1,0,1)$ ← OBS: skriv ej på detta sätt när det gäller polära koordinater!

$V = \frac{1}{2}(\hat{r} + \hat{\varphi})$

$$\nabla T = \left(\frac{\partial T}{\partial r}, \frac{1}{r}\frac{\partial T}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial T}{\partial \varphi}\right) =$$

$$= \left(-\frac{2}{r^3}(2+\cos\theta), \frac{-\sin\theta}{r^3}, 0\right)$$

$$\nabla T \cdot \mathbb{V} = -\frac{1}{\sqrt{2}}\left(\frac{2}{r^3}(2+\cos\theta)\right) = -\frac{1}{2\sqrt{2}}$$

b) $\nabla T(p) = \left(\frac{-2}{2^3}(2+0), \frac{-1}{2^3}, 0\right) = \left(-\frac{1}{2}, -\frac{1}{8}, 0\right)$

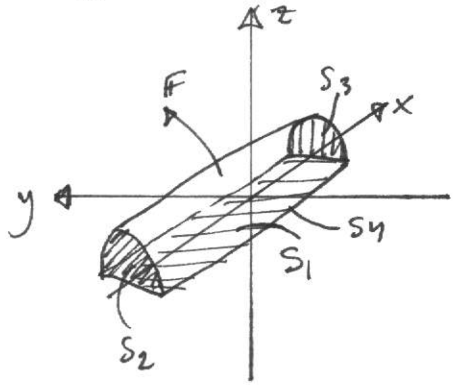
$$\left(-\frac{1}{2}, -\frac{1}{8}, 0\right)(\hat{r},\hat{\theta},\hat{\varphi})$$

$$|\nabla T \cdot p| = \sqrt{\left(\frac{1}{-2}\right)^2 + \left(\frac{1}{-8}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{64}} = \frac{\sqrt{17}}{8}$$

PLK
2.4.1:7

Låt S vara ytan $y^2+z^2=1 \quad -1\le x\le 1 \quad z\ge 0$

$\mathbb{F}=(x, x^2yz^2, x^2y^2z)$ Beräkna $\int_S \mathbb{F}\cdot d\mathbb{S}$



$$\int_{S_1}\mathbb{F}\cdot d\mathbb{S}_1+\int_{S_2}\mathbb{F}\cdot d\mathbb{S}_2+\int_{S_3}\mathbb{F}\cdot d\mathbb{S}_3+\int_{S_4}\mathbb{F}\cdot d\mathbb{S}_4=\int_V \nabla\mathbb{F}\,dV$$

For S_2 gäller: $\mathbb{n}=(-1,0,0)$, $\mathbb{F}=(-1, yz^2, y^2z)$

$$\int_S \mathbb{F}\cdot d\mathbb{S}_2=\int_{S_2}\mathbb{F}\cdot\mathbb{n}\cdot d\mathbb{S}=\int_S d\mathbb{S}_2=\frac{\pi}{2}$$

För S_3 gäller: $\mathbb{n}=(1,0,0)$, $\mathbb{F}=(1, yz^2, y^2z)$

$$\Rightarrow \int_{S_3}\mathbb{F}\,d\mathbb{S}_3=\frac{\pi}{2}$$

För S_4 gäller: $\mathbb{n}=(0,0,-1)$, $\int_{S_4}\mathbb{F}\,d\mathbb{S}_4=0$

För divergensen gäller

$$\int_V \nabla F dV = \int (1 + x^2 z^2 + x^2 y^2)\, dx\, dy\, dz =$$

mod. cylindrisk $\begin{cases} x = x \\ y = \rho\cos\theta \\ z = \rho\sin\theta \end{cases} \Rightarrow$

$$\int_{\theta=0}^{\pi}\int_{\rho=0}^{1}\int_{x=-1}^{1} (1 + \rho^2 x^2)\, d\rho\, d\theta\, dx = \frac{7\pi}{6}$$

Svar: $\int_{S_1} \mathbb{F} \cdot d\mathbb{S} = \frac{\pi}{6}$

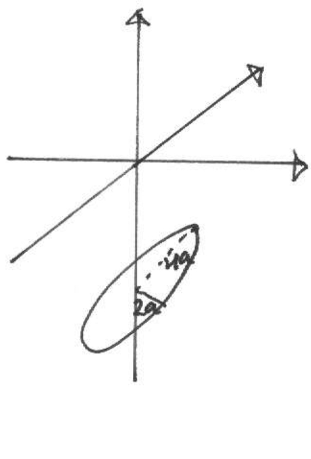
PLK 24.1:8

Trycket i en vätska: $p(\mathbb{r}) = p_0 - \rho g z + k z^2 = 0$

Kropp med volymen

$$V: \; 4\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + 4\left(\frac{z}{a}\right)^2 + 32\,\frac{z}{a} + 48 < 0$$

$$\left(\frac{x}{2a}\right)^2 + \left(\frac{y}{4a}\right)^2 + \left(\frac{z+4a}{2a}\right)^2 < 1 \quad \text{ellips}$$



Beräkna den totala kraften:

$$\mathbb{F} = -\int_S p\, d\mathbb{S} = \int \nabla p\, dV =$$

$$= -\int -\rho g \hat{z} + 2kz\hat{z}\, dV =$$

$$= \rho g \hat{z} \int_V dV - 2kz \int_V z\, dV =$$

$$= \rho g \hat{z}\, \frac{4\pi}{3} \cdot 2a \cdot 4a \cdot 2a - 2k\hat{z} \int_V z \underbrace{dx\,dy}_{= A(z)}\, dz$$

Hur ser $A(z)$ ut?

$$V = \frac{4\pi}{3} \cdot a \cdot b \cdot c \qquad A = \pi a b \quad y = 0$$

$$\left(\frac{x}{2a}\right)^2 + \left(\frac{z+4a}{2a}\right) = 1$$

$$x^2 = 4a^2\left(1 - \left(\frac{z+4a}{2a}\right)^2\right)$$

$$x = 0 \quad y^2 = 16a^2\left(1 - \left(\frac{z+4a}{2a}\right)^2\right)$$

$$A = \pi x y \Rightarrow A(z) = 8a^2\pi\left(1 - \left(\frac{z+4a}{2a}\right)^2\right)$$

$$F = \frac{64\pi a^3}{3}\rho g - 2k\hat{z} \int z\, 8a^2\pi\left(1 - \left(\frac{z+4a}{2a}\right)^2\right) dz$$

$$= \frac{64\pi a^3}{3}(\rho g + 8ka)\hat{z}$$

010917 studentuppg.
lr 3 PLK 2.2.1:5
övn 4

$$\int_c d\mathbf{r} \times \mathbf{B}$$

$$\mathbf{B} = \frac{B_0}{a}\frac{1}{x^2+y^2}\left[(x-y)a^2\hat{x} + (x+y)a^2\hat{y} + (x^2+y^2)z\hat{z}\right]$$

$$C: \quad \mathbf{r} = \left(a\cos\varphi,\ a\sin\varphi,\ \frac{a\varphi}{\pi}\right) \qquad 0 \le \varphi \le 2\pi$$

$$\mathbf{B} = \frac{B_0}{\cancel{a}}\cdot\frac{1}{\cancel{a^2}}\left[\cancel{a}(\cos\varphi - a\sin\varphi)\cancel{a^2},\ \cancel{a}(\cos\varphi+\sin\varphi)\cancel{a^2},\right.$$
$$\left., (\cancel{a^2}\cos^2\varphi + \cancel{a^2}\sin^2\varphi)\frac{\cancel{a}\varphi}{\pi}\right]$$

$$\int_c d\mathbf{r}\times\mathbf{B} = \int_0^{2\pi} \frac{d\mathbf{r}}{d\varphi}\times\mathbf{B}\,d\varphi$$

$$\frac{d\mathbf{r}}{d\varphi}\times\mathbf{B}\,d\varphi = B_0 a \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\varphi & \cos\varphi & \frac{1}{\pi} \\ \cos\varphi - \sin\varphi & \cos\varphi+\sin\varphi & \frac{\varphi}{\pi} \end{vmatrix} = \ldots =$$

$$= B_0 a\left[-\frac{1}{\pi}(\cos\varphi+\sin\varphi-\varphi\cos\varphi)\hat{x} + \frac{1}{\pi}(\varphi\sin\varphi+\cos\varphi - \right.$$
$$\left. -\sin\varphi)\hat{y} - (\cos\varphi(\cos\varphi-\sin\varphi) + \sin\varphi(\cos\varphi+\sin\varphi)\hat{z}\right]$$

$$I = B_0 a\int_0^{2\pi}\left(-\frac{1}{\pi}(\cos\varphi+\sin\varphi-\varphi\cos\varphi)\hat{x} + \frac{1}{\pi}(\varphi\sin\varphi+\cos\varphi-\sin\varphi)\hat{y} - \right.$$
$$\left. -\hat{z}\right)d\varphi$$

$$= B_0 a\left\{\frac{1}{\pi}\Big[\cos\varphi+\varphi\sin\varphi\Big]_0^{2\pi}\hat{x} + \frac{1}{\pi}\Big[\sin\varphi - \varphi\cos\varphi\Big]_0^{2\pi}\hat{y} - \Big[\varphi\Big]_0^{2\pi}\hat{z}\right\} =$$

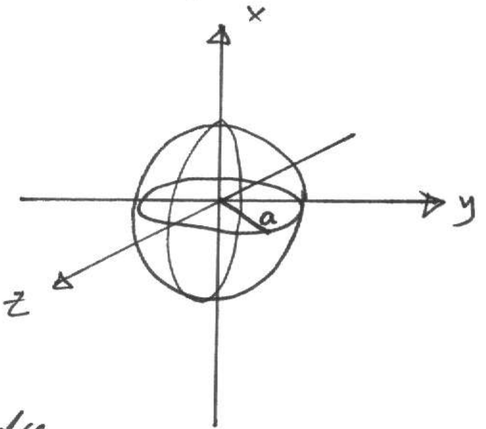
$$= B_a\left[\frac{1}{\pi}(-2\pi)\hat{y} - 2\pi\hat{z}\right]$$

studentuppgift

PLK 2.3.1:2

$$\iint_S \left(\frac{A}{r^2}\hat{r} + B\hat{z}\right) dS$$

$$\begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases}$$



$$dS = \left(\frac{\partial r}{\partial\theta} \times \frac{\partial r}{\partial\varphi}\right) d\theta d\varphi$$

$$\frac{\partial r}{\partial\theta} = a(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\frac{\partial r}{\partial\varphi} = a(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$\frac{\partial r}{\partial\theta} \times \frac{\partial r}{\partial\varphi} = a^2 \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{vmatrix} =$$

$$= a^2(\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \cos\theta\underline{\cos^2\varphi}\sin\theta +$$
$$+ \cos\theta\sin\theta\underline{\sin^2\varphi}) =$$

$$= a^2(\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \sin\theta\cos\theta)$$

$$F = \left(\frac{A}{r}\hat{r} + B\hat{z}\right)$$

$$r = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$F = \frac{A}{a^2}\left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta + \frac{Ba^2}{A}\right)$$

$$\int_0^{2\pi}\int_0^{\pi}\left(A\sin^3\theta\underline{\cos^2\varphi} + A\sin^3\theta\underline{\sin^2\varphi} + A\sin\theta\cos^2\theta + \right.$$
$$\left. + Ba^2\sin\theta\cos\theta\right)d\theta d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi}(A\sin\theta + Ba^2\sin\theta\cos\theta)d\theta d\varphi =$$

$$= \int_0^{2\pi}\left[-A\cos\theta - \frac{Ba^2\cos^2\theta}{2}\right]_0^{\pi} d\varphi = \int_0^{2\pi} A-(-A)\,d\varphi =$$

$$= 2A\int_0^{2\pi} d\varphi = 4\pi A$$

Stokes sats

$$\oint_L \mathbb{A}\, d\mathbb{r} = \int_S \nabla\times\mathbb{A}\, d\mathbb{S}$$

om $\nabla\times\mathbb{A} = 0 \Leftrightarrow$ $\mathbb{A}$ har en skalär potential
$A = \nabla\cdot\phi$

$\nabla\cdot\mathbb{B} = 0 \Leftrightarrow$ $\mathbb{B}$ har en vektorpotential
$\mathbb{B} = \nabla\times\mathbb{A}$

PLK 2.5.1:3

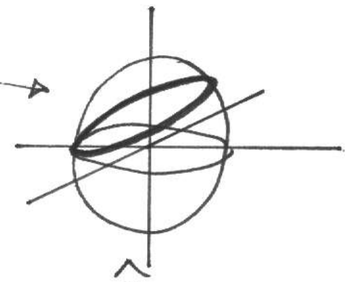
En partikel påverkas av kraftfältet:

$$\mathbb{F} = F_0\left[\left(\frac{\pi y}{a} + \sin\frac{\pi z}{a}\right)\hat{x} + \frac{x}{a}\hat{y} + \frac{\pi x}{a}\cos\frac{\pi z}{a}\hat{z}\right]$$

Vilket arbete utförs längs C: $x=z$, $x^2+y^2+z^2=a^2$

Kolla rotationen.

Projicera ned "snittet" på planet → $2x^2+y^2=a^2$



$$\nabla \times \mathbb{F} = F_0 \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\pi y}{a} + \frac{\sin z\pi}{a} & \frac{x}{a} & \frac{\pi x}{a}\cos\frac{\pi z}{a} \end{vmatrix} =$$

$$= \frac{(1-\pi)}{a} F_0 \hat{z}$$

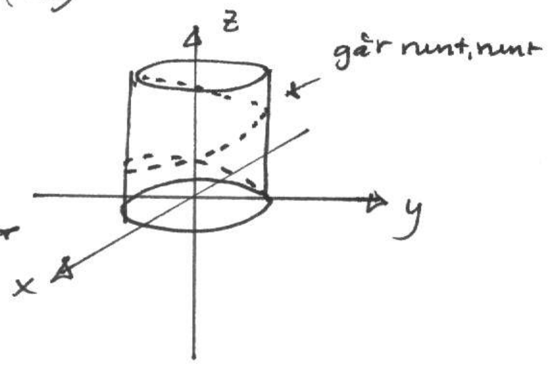
Beroende på riktning

Stoke ⇒ $\frac{1-\pi}{a} F_0 \int \hat{z}\, dS = \frac{1-\pi}{a} F_0 \frac{\pi a^2}{\sqrt{2}} \cdot (\pm 1)$

2.5.1:6 En kurva $\mathbb{r} = (a\cos\varphi, a\sin\varphi, b\varphi)$

samt ett vektorfält $\mathbb{B} = B_0 \left(\frac{x}{a}\right)^3 \hat{z}$ $\quad 0 \le \varphi \le 2\pi n$

Beräkna $\mathbb{F} = \int_C d\mathbb{r} \times \mathbb{B}$



Inte bra med Stokes, P.g.a utseendet

$d\mathbb{r} = (-a\sin\varphi d\varphi, a\cos\varphi d\varphi, b d\varphi)$

$$d\mathbb{r} \times \mathbb{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -a\sin\varphi d\varphi & a\cos\varphi d\varphi & b d\varphi \\ 0 & 0 & B_0 \cos^3\varphi \end{vmatrix} =$$

$= aB_0 \cos^4\varphi \, d\varphi \, \hat{x} + aB_0 \cos^3\varphi \sin\varphi \, d\varphi \, \hat{y}$

$$\mathbb{F} = \int_0^{2\pi n} aB_0 \cos^4\varphi \, d\varphi \, \hat{x} + aB_0 \cos^3\varphi \sin\varphi \, d\varphi \, \hat{y} =$$

$$= aB_0 \frac{3\pi n}{4} \hat{x}$$

010919 Stud. uppg

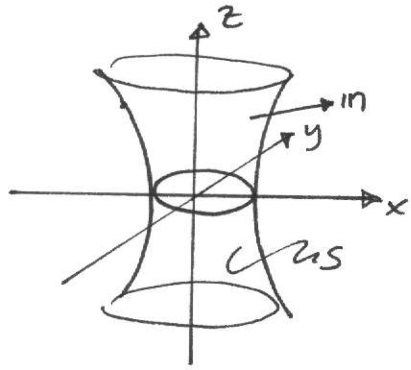
lv 3 PLK 2.4.1:9

övn 5

$$x^2+y^2-z^2=4 \Leftrightarrow x^2+y^2=\left(\sqrt{4+z^2}\right)^2$$

$$\mathbb{u} = (xz, yz, xy)\frac{1}{(x^2+y^2)}$$

$$a \le z \le b$$



Gauss sats ger jobbiga värden. Parametrisera istället

$$\iint \mathbb{u} \cdot \mathbb{n} \, dS$$

$$\mathbb{r} = \mathbb{r}(\varphi, z)$$

$$\mathbb{r} = (r\cos\varphi, r\sin\varphi, z)$$

$$\mathbb{r} = \sqrt{4+z^2}\cos\varphi, \sqrt{4+z^2}\sin\varphi, z)$$

$$\begin{cases} x = \sqrt{4+z^2}\cos\varphi \\ y = \sqrt{4+z^2}\sin\varphi \\ z = z \end{cases} \qquad \begin{matrix} 0 \le \varphi \le 2\pi \\ a \le z \le b \end{matrix}$$

$$\mathbb{u} = \frac{z\sqrt{4+z^2}\cos\varphi,\ z\sqrt{4+z^2}\sin\varphi,\ (4+z^2)\cos\varphi\sin\varphi}{(4+z^2)}$$

$$\hat{\mathbb{n}}\,dS = (\mathbb{r}_z' \times \mathbb{r}_\varphi')\,d\varphi\,dz$$

$$\frac{\partial \mathbb{r}}{\partial z} \times \frac{\partial \mathbb{r}}{\partial \varphi} = \left(\sqrt{4+z^2}\cos\varphi,\ \sqrt{4+z^2}\sin\varphi,\ -2z\right)$$

$$\mathbb{u}\cdot\hat{\mathbb{n}}\,dS = \frac{1}{(4+z^2)}\Big[z(4+z^2)\cos^2\varphi + z(4+z^2)\sin^2\varphi - $$
$$- 2z(4+z^2)\cos\varphi\sin\varphi\Big]$$

$$= z - 2z\cos\varphi\sin\varphi\,d\varphi\,dz = z(1-\sin 2\varphi)\,d\varphi\,dz$$

$$\iint_S \mathbb{u}\cdot\mathbb{n}\,dS = \int_a^b z\,dz \int_0^{2\pi}(1-\sin 2\varphi)\,d\varphi =$$

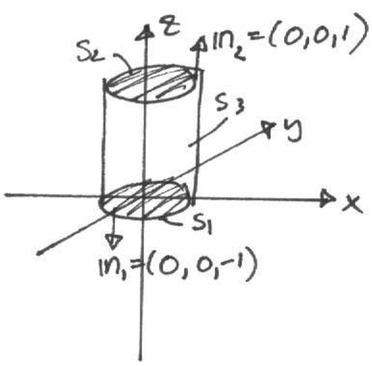
$$= \left[\frac{z^2}{2}\right]_a^b \cdot \left[\varphi + \frac{\cos 2\varphi}{2}\right]_0^{2\pi} = \pi(b^2-a^2)$$

Stud. uppg. PLK 2.4.1:10

$$\mathbb{u} = (3x^2yz,\ -xy^2z,\ x^3z)$$

$$S = x^2+y^2=1$$

$$\iint_{S_3} \underbrace{\nabla\times\mathbb{u}}_{\mathbb{F}}\cdot\mathbb{n}\,dS \qquad 0\le z\le 1$$



slut ytan med lock (S_2) och botten (S_1)

$$\nabla \times \mathbb{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} = (xy^2, 3x^2y - 3x^2z, -y^2z - 3x^2z)$$

$$\iint_{S_1+S_2+S_3} \mathbb{F}\cdot\mathbb{n}\,dS = [\text{Gauss}] = \iiint \operatorname{div}\mathbb{F}\,dxdydz = \iint_{S_1} \mathbb{F}\cdot\mathbb{n}_1\,dS +$$

$$+ \iint_{S_2} \mathbb{F}\cdot\mathbb{n}_2\,dS + \underbrace{\iint_{S_3} \mathbb{F}\cdot\mathbb{n}\,dS}_{\text{sökes}}$$

$$\operatorname{div}\mathbb{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\mathbb{F} = y^2 + 3x^2 - y^2 - 3x^2 = 0$$

$$\Rightarrow \iiint \operatorname{div}\mathbb{F}\,dxdydz = 0$$

$$\iint_{S_2} \mathbb{F}\cdot\mathbb{n}_2\,dS = \iint_{S_2} (-y^2z - 3x^2z)\,dS = [z=1] =$$

$$= -\iint_{S_2} y^2 + 3x^2\,dS = [\text{polära koord.}] = -\iint r^2(\sin^2\varphi + 3\cos^2\varphi)\,r\,dr\,d\varphi$$

$$= -\int_0^1 r^3\,dr \cdot \int_0^{2\pi} 1 + 2\cos^2\varphi\,d\varphi = -\frac{1}{4}\cdot\left[2\varphi + \frac{\sin 2\varphi}{2}\right]_0^{2\pi} =$$

$$= -\frac{1}{4}(2\pi) = -\frac{\pi}{2}$$

$$\iint_{S_1} \mathbb{F}\cdot\hat{\mathbb{n}}_1\,dS = \iint_{S_1} \mathbb{F}(0,0,-1)\,dS = \iint_{S_1} y^2z + 3x^2z\,dS = 0$$

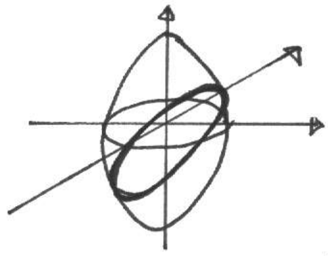
$$\iint_{S_3} \mathbb{F}\cdot\mathbb{n}\,dS = \pi$$

PKK 2.5.1:2

$$\mathbb{F}(x,y,z) = (x-y,\ y-z,\ z-x)$$

$$\text{ytor: } \begin{cases} S_1 : \ 4\rho^2 + z^2 = 20 \\ S_2 : \ \arctan \frac{z}{x+y} = \frac{\pi}{3} \end{cases}$$

$$\oint_C \mathbb{F} \times d\mathbb{r} \ ?$$



$$\oint \mathbb{F} \times d\mathbb{r} = -\iint (d\mathbb{S} \times \nabla) \times \mathbb{F} = -\int dS (\hat{n} \times \nabla) \times \mathbb{F}$$

Planets ekvation: $\sqrt{3}x + \sqrt{3}y - z = 0$ (bussigare variant av S_2)

$$\hat{n} = (+) \frac{(\sqrt{3}, \sqrt{3}, -1)}{\sqrt{7}}$$

- ger rätt riktning

$$\hat{n} \times \nabla = -\frac{1}{\sqrt{7}} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sqrt{3} & \sqrt{3} & -1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} =$$

$$= \frac{1}{\sqrt{7}} \left(\frac{\sqrt{3}\partial}{\partial z} - \frac{\partial}{\partial y},\ \frac{\partial}{\partial x} + \frac{\sqrt{3}\partial}{\partial z},\ -\frac{\sqrt{3}\partial}{\partial y} + \frac{\sqrt{3}\partial}{\partial x} \right)$$

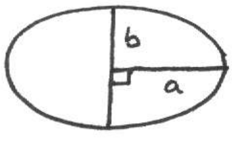
$$(\hat{n} \times \nabla) \times \mathbb{F} = \frac{1}{\sqrt{7}} (2\sqrt{3} - 1,\ 3\sqrt{3},\ \sqrt{3} - 2)$$ konstant vektor - bra!

Nu: $\int_S dS$?

planets ekv: $\sqrt{3}x + \sqrt{3}y - z = 0$
ju större x, y, desto större z
Tag fram ½ axlarna

$4x^2+4y^2+3(x+y)^2=20$ (ur ekv. för S_1)

$$x=y \Rightarrow \left.\begin{array}{l} x=1 \\ y=1 \\ z=2\sqrt{3} \quad (\text{ur } S_1) \end{array}\right\} a=\sqrt{1+1+12}=\sqrt{14}$$



$$x=-y \Rightarrow \left.\begin{array}{l} x=\sqrt{\frac{5}{2}} \\ y=\sqrt{\frac{5}{2}} \end{array}\right\} b=\sqrt{\frac{5}{2}+\frac{5}{2}}=\sqrt{5}$$

$$A=\pi\cdot\sqrt{14}\cdot\sqrt{5}=\pi\cdot\sqrt{70}$$

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pappersuppgift 2.

Visa att om $\mathbf{a}$ är en konstant vektor och $\mathbf{r}$ är ortsvektorn så gäller sambanden 1) $(\mathbf{a}\cdot\nabla)\mathbf{r}=\mathbf{a}$, 2) $\nabla(\mathbf{a}\cdot\mathbf{r})=\mathbf{a}$ samt 3) $\nabla\times(\mathbf{a}\times\mathbf{r})=2\mathbf{a}$

1) och 2) lätta att visa med index, 3) ngt svårare.

$a_i=(a_x,a_y,a_z)$

$$(\mathbf{a}\cdot\nabla)\mathbf{r}=\left(a_i\frac{\partial}{\partial x_i}\right)x_j\hat{x}_j=a_i\frac{\partial x_j}{\partial x_i}\hat{x}_j=a_i\overset{i=j}{\delta_{ij}}\hat{x}_j=$$
$$=a_i\hat{x}_i=\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{r})=\frac{\partial}{\partial x_j}\hat{x}_j(a_i x_i)=\hat{x}_j a_i\frac{\partial x_i}{\partial x_j}=\hat{x}_j a_i\delta_{ij}=\mathbf{a}$$

$$\nabla\times(\mathbf{a}\times\mathbf{r})=\varepsilon_{ijk}\frac{\partial}{\partial x_j}\hat{x}_i(a\times r)_k=\varepsilon_{ijk}\frac{\partial}{\partial x_j}\hat{x}_i\overbrace{(\varepsilon_{klm}a_l x_m)}^{\text{en komponent}}=$$
$$=\varepsilon_{ijk}\varepsilon_{klm}a_l\frac{\partial x_m}{\partial x_j}\hat{x}_i=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_l\delta_{mj}\hat{x}_i=$$
$$=a_i\hat{x}_i 3-a_i\hat{x}_i=2\mathbf{a}$$

pappersuppgift 3.

Den skalära funktionen $f(u)$ uppfyller diffekvationen

$$\nabla f(\mathbb{a}\cdot\mathbb{r}) = 2(\mathbb{a}\cdot\mathbb{r})\mathbb{a}$$

där $\mathbb{a}$ är en konstant vektor och $\mathbb{r}$ är ortsvektorn. Bestäm alla lösningar $f(u)$ till ekvationen.

kolla kedjeregeln: derivera funktionen · inre derivatan

$$\nabla f(\mathbb{a}\cdot\mathbb{r}) = f'(\mathbb{a}\cdot\mathbb{r})\cdot\nabla(\mathbb{a}\cdot\mathbb{r})$$

$$\nabla(\mathbb{a}\cdot\mathbb{r}) = \mathbb{a}$$ från förra uppgiften

$$\Rightarrow f'(\mathbb{a}\cdot\mathbb{r})\cdot\cancel{\mathbb{a}} = 2(\mathbb{a}\cdot\mathbb{r})\cancel{\mathbb{a}}$$

$$f'(\mathbb{a}\cdot\mathbb{r}) = 2(\mathbb{a}\cdot\mathbb{r}) \qquad \mathbb{a}\cdot\mathbb{r} = u$$

$$f'(u) = 2u$$

$$f(u) = u^2 + c$$

tips: Försök bli av med riktningen i början

010924
lv 4 PLK 4.2.1:2
övn. 6 Ett koordinatsystem uvz:

$$\begin{cases} x = uv \\ y = u^2 + \lambda v^2 \qquad u \geq 0 \\ z = z \end{cases}$$

$$\mathbf{r} = (uv,\ u^2 + \lambda v^2,\ z)$$

$\frac{\partial \mathbf{r}}{\partial u} = (v, 2u, 0)$ ligger i $\hat{u}$-led (cartesiska koord.)

$\frac{\partial \mathbf{r}}{\partial v} = (u, 2\lambda v, 0)$ ligger i $\hat{v}$-led

$$\frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1)$$

ortogonalt: skalärprodukten skall bli noll.

$$\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} = v \cdot u + 4\lambda uv = 0 \Rightarrow \lambda = -\frac{1}{4}$$

$$\left.\begin{aligned} h_u &= \sqrt{v^2 + 4u^2} \\ h_v &= \sqrt{u^2 - \frac{v^2}{4}} \\ h_z &= 1 \end{aligned}\right\} \text{normalisering}$$

Volymen: $\frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial z} \right)$ Vi är bara intresserade av tecknet på volymen

$$\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u & -\frac{v}{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{v}{2}\hat{x} - u\hat{y}$$

$$\frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial z} \right) = -\frac{v^2}{2} - 2u^2 < 0 \Rightarrow \text{Ett vänstersystem}$$

Beräkna $\nabla \times \mathbb{F}$

$$\mathbb{F} = \frac{2u^2 v \hat{u} + uv^2 \hat{v}}{\sqrt{4u^2+v^2}}$$

$$\nabla \times \mathbb{F} = \frac{1}{h_u h_v h_z} \begin{vmatrix} h_u \hat{u} & h_v \hat{v} & h_z \hat{z} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ h_u F_u & h_v F_v & h_z F_z \end{vmatrix} =$$

$$= \frac{2}{4u^2+v^2} \begin{vmatrix} \sqrt{4u^2+v^2}\, \hat{u} & \sqrt{u^2+\frac{v^2}{4}}\, \hat{v} & \hat{z} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ 2u^2 v & -\frac{uv^2}{2} & 0 \end{vmatrix} =$$

$$= \frac{2}{4u^2+v^2}\left(-\frac{v^2}{2} - 2u^2\right)\hat{z} = -\hat{z}$$

Pappersuppgift 2

Vektorfältet $\mathbb{F}(\rho, \phi, z) = \frac{z}{a}\nabla\rho + (a+\rho)\nabla\phi + \frac{\rho}{a}\nabla z$

där $\rho \phi z$ är cylinderkoordinater

samt ytorna S_1 och S_2

$$S_1 : x^2 + y^2 = 4a^2$$
$$S_2 : x^2 z^2 + y^2 z^2 - 4ayz^2 + a^2 z^2 - a^4 = 0$$

är givna.

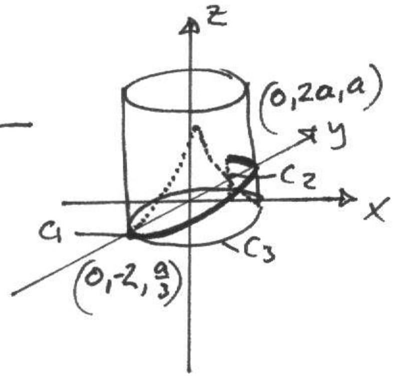
Låt C vara den del av skärningskurvan mellan S_1 och S_2 som går från punkten $(0, -2a, \frac{a}{3})$ till punkten $(0, 2a, a)$ genom positiva x

Bestäm tangentlinjeintegralen av $\mathbb{F}$ längs C.

$$\mathbb{F} = \frac{z}{a}\nabla\rho + (a+\rho)\nabla\phi + \frac{\rho}{a}\nabla z =$$

$$= \frac{z}{a}\frac{\partial}{\partial\rho}\hat{\rho}\rho + (a+\rho)\frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}\cdot\phi + \frac{\rho}{a}\frac{\partial}{\partial z}\hat{z}z =$$

$$= \frac{z}{a}\hat{\rho} + \left(\frac{a+\rho}{\rho}\right)\hat{\phi} + \frac{\rho}{a}\hat{z}$$



skriv om S_2: $z = \dfrac{a^2}{(x^2+(y-a)^2)^{\frac{1}{2}}}$

Stoke:

$$\oint \mathbb{F}\,d\mathbb{r} = \int \nabla\phi\,\mathbb{F}\,d\mathbb{S}$$

$$\nabla\times\mathbb{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \frac{z}{a} & a+\rho & \frac{\rho}{a} \end{vmatrix} = (a+2\rho)\hat{z} \Rightarrow \int \nabla\times\mathbb{F}\,d\mathbb{S} = 0$$

$$\int_{C_1} \mathbb{F}\cdot d\mathbb{r} = \int_0^{\frac{a}{3}} 2\,dz = \frac{2a}{3}$$

$$\int_{C_3} \mathbb{F}\cdot d\mathbb{r} = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3a}{2a}\,2a\,d\phi = -3\pi a$$

$$\int_{C} \mathbb{F}\cdot d\mathbb{r} = \left(3\pi + \frac{4}{3}\right)a$$

011001 studentuppgift

lv5 PLK 3.0.2:5

öv n.7

$\mathbf{a}\,\nabla u(f)$

$\mathbf{a}$ konstant vektor

$f = (\mathbf{a}\times\mathbf{r})^2$

$\mathbf{r}$ = ortsvektor

$\mathbf{a} = a\hat{z}$

$$\mathbf{a}\nabla u(f) = \frac{du}{df}\mathbf{a}\cdot\nabla f = \frac{du}{df}\mathbf{a}\cdot\nabla|\mathbf{a}\times\mathbf{r}|^2 =$$

$$= \frac{du}{df}\left(\mathbf{a}\cdot\nabla\left(a\sqrt{x^2+y^2}\right)^2\right) = \frac{du}{df}\left(\mathbf{a}\cdot\nabla\left(a^2(x^2+y^2)\right)\right) =$$

$$= \frac{du}{df}\left((0,0,a)\cdot(2a^2x, 2a^2y, 0)\right) = 0$$

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studentuppgift

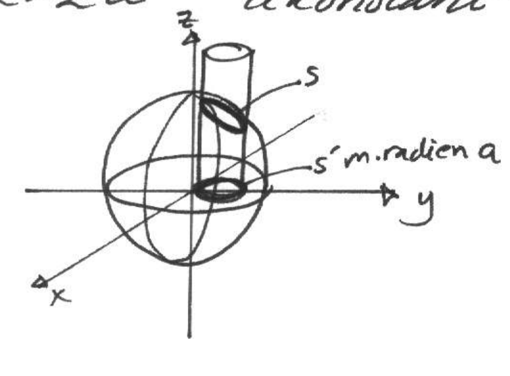
PLK 2.5.1:4 $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r}$

$\mathbf{F} = (x^2 - a(y+z)\hat{x} + (y^2 - az)\hat{y} + (z^2 - a(x+y))\hat{z})$

Γ skärning mellan cylinder: $(x-a)^2 + y^2 = a^2$ $z \geq 0$

och sfär: $x^2+y^2+z^2 = R^2$ $R > 2a$ a konstant

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_S \nabla\times\mathbf{F}\,d\mathbf{S}$$



$$\nabla \times \mathbb{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - a(y+z) & (y^2 - az) & z^2 - a(x+y) \end{vmatrix} = \ldots = a\hat{z}$$

projicera ned i xy-planet

$$\int_{S'} a\hat{z}\, d\mathbb{S}' = a \int d\mathbb{S} = a\pi a^2 = \pi a^3$$

4.3.1:10 koordinatsystem

$$\mathbb{r} = \left(uv\cos\varphi,\ uv\sin\varphi,\ \frac{u^2 - v^2}{2}\right)$$

En yta $S: u^4 + v^4 + u^2 v^2 = 1$

$$\phi(u, v, \varphi) = (u^2 - 1)^2 + (v^2 + 1)^2 + 2uv(\sin\varphi + uv) + (u^2 + v^2)^{-1}$$

$\oint \mathbb{F} \cdot d\mathbb{S}$?

Gauss: $\int_S \mathbb{F} \cdot d\mathbb{S} = \int \nabla \cdot \mathbb{F}\, dV = \int \nabla \cdot (\nabla\phi)\, dV =$

$= \int \nabla^2 \phi\, dV$

Laplace-operatorn

$$\frac{d\mathbb{r}}{du} = (v\cos\varphi,\ v\sin\varphi,\ u)$$

$$\frac{d\mathbb{r}}{dv} = (u\cos\varphi,\ u\sin\varphi,\ -v)$$

$$\frac{d\mathbb{r}}{d\varphi} = (-uv\sin\varphi,\ uv\cos\varphi,\ 0)$$

Skal faktorer $\begin{cases} h_u = (u^2+v^2)^{\frac{1}{2}} \\ h_v = (u^2+v^2)^{\frac{1}{2}} \\ h_\varphi = uv \end{cases}$

$$\nabla^2\phi = \frac{1}{h_u h_v h_\varphi} \sum_i \frac{\partial}{\partial u_i} \frac{h_u h_v h_\varphi}{h_i^2} \frac{\partial \phi}{\partial u_i} =$$

$$= \frac{1}{(u^2+v^2)uv}\left[\frac{\partial}{\partial u} uv \frac{\partial\phi}{\partial u} + \frac{\partial}{\partial v} uv \frac{\partial \phi}{\partial v} + \frac{\partial}{\partial \varphi} \frac{u^2+v^2}{uv} \frac{\partial}{\partial\varphi}\phi\right]$$

$$= \frac{1}{(u^2+v^2)uv}\left[24u^3v + 24uv^3 - \frac{8uv}{(u^2+v^2)^2} + \frac{8u^3v+8uv^3}{(u^2+v^2)^3}\right] =$$

$= 24$ Bra! Bra när Gauss ger 0 eller konstant.

ytan: $u^4+v^4+u^2v^2 =$ $\begin{cases} x = uv\cos\varphi \\ y = uv\sin\varphi \\ z = \frac{u^2-v^2}{2} \end{cases}$

$=$ [tag kvadraten på dessa och möblera om lite] $= 3x^2+3y^2+4z^2$

Volymen: $V = \frac{4\pi}{3} \cdot \frac{1}{3 \cdot 2}$

$\therefore \int \nabla^2\phi \, dV = 24 \int dV = \frac{16\pi}{3}$

Tillbaks till singulariteten:

$\phi = (u^2-1)^2 + (v^2+1)^2 + 2uv(\sin\varphi + uv) + (u^2+v^2)^{-1}$

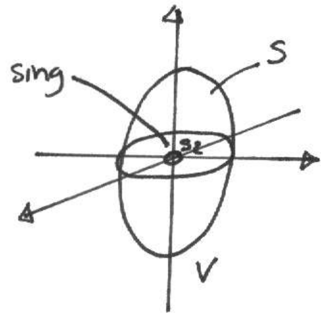
$= (u^2+v^2)^2 - 2u^2 + 2v^2 + 2 + 2uv\sin\varphi + \frac{1}{u^2+v^2}$

$= \underbrace{4r^2 \underbrace{- 4z + 2 + 2y}_{\text{cartesiskt } \phi_2} + \frac{1}{2r}}_{\text{sfäriskt } \phi_1}$

$$F_2 = \nabla \phi_2 = 2\hat{y} - 4\hat{z}$$

$$F_1 = \nabla \phi = \hat{r} \frac{\partial}{\partial r} \phi_1 = 8r - \frac{1}{2r^2}$$

borde vara minus



$$\int_S \mathbb{F} \cdot d\mathbb{S} = \int \nabla \cdot \mathbb{F} \, dV - \int \mathbb{F} \, d\mathbb{S}_2$$

$$\int \mathbb{F} \cdot d\mathbb{S}_2 = -\int F r^2 \sin\theta \, d\theta \, d\varphi = 2\pi$$

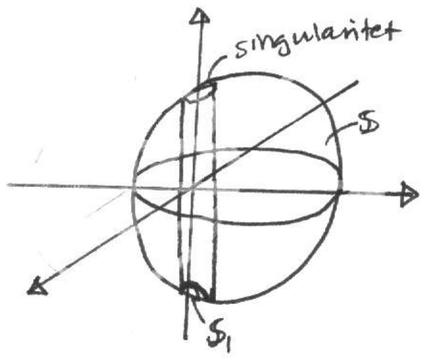
$$\Rightarrow \int \mathbb{F} \cdot d\mathbb{S} = \int \nabla \cdot \mathbb{F} \, dV - \int \mathbb{F} \cdot d\mathbb{S}_2 = \frac{16\pi}{3} - 2\pi = \frac{10\pi}{3}$$

PLK 4.3.1:11

$$\mathbb{F} = \mathbb{F}_1 + \mathbb{F}_2$$

$\mathbb{F}_1$ beskrivs av $\phi_1 = \dfrac{1}{\sqrt{(x-3)^2 + (y+1)^2 + z^2}} + xy^3$

$\mathbb{F}_2 = \dfrac{\rho^2 - az}{\rho} \hat{\rho}$ (cylindriska koordinater)



$$\int \mathbb{F}_2 \cdot d\mathbb{S} = \int \nabla \cdot \mathbb{F} \, dV - \int \mathbb{F}_2 \, d\mathbb{S}_1$$

$$\nabla \cdot \mathbb{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \left(\frac{\rho^2 - az}{\rho} \right) = 2$$

$$\int \nabla\cdot \mathbb{F}\cdot dV = 2\int dV = \frac{2\cdot 4\pi}{3}\cdot 3^3$$

$$-\int \mathbb{F}_2\, dS_1 = \int \lim_{\rho\to 0} \frac{\rho^2 - az}{\rho}\,\rho\, d\alpha\, dz = -a\int z\, d\alpha\, dz =$$

$$= -a 2\pi\left[\frac{z^2}{2}\right]_{z_{min}}^{z_{max}} = -\pi a\left(z_{max}^2 - z_{min}^2\right)$$

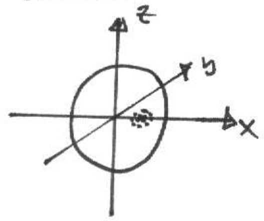
Sfärens ekv: $(x-2)^2+(y-1)^2+(z-1)^2=9$

$$x=y=0 \qquad 4+1+(z-1)^2=9$$

$$z = \begin{cases} 3 & \text{max} \\ -1 & \text{min} \end{cases}$$

$$\int \mathbb{F}_2\cdot dS = \int \nabla \mathbb{F}\, dV - \int \mathbb{F}_2\, dS_1 = 72\pi - 8\pi a$$

$$\phi = \underbrace{\frac{1}{\sqrt{(x-3)^2+(y+1)^2+z^2}}}_{\phi_0} + \overbrace{xy^3}^{\phi_3}$$



$\int \mathbb{F}\cdot dS = 4\pi$ bara arean överlever, ty $\mathbb{F} = \frac{1}{r^2}$

ϕ_3: $\mathbb{F}_3 = -\nabla\phi_3 = (-y^3, -3xy^2, 0)$

$$\int \mathbb{F}_1\cdot dS = \int \nabla\cdot \mathbb{F}_3\, dV = \int -6xy\, dV$$

Flyttar oss till singulariteten

$$\begin{cases} x-2 = x' \\ y-1 = y' \\ z = z' \end{cases}$$

$$= -6\int \underbrace{(x'+2)(y'+1)}\, dx'dy'dz' =$$

Pga symmetri och
udda funktioner
så "försvinner" x' och y'

$$= -12\int dV = -432\pi$$

$$\int \mathbb{F}\, dS = 4\pi - 432\pi + 72\pi - 8\pi a = -356\pi - 8\pi a$$

011001 (igen) lv5 övn 8 — studentuppgift

PLK 4.2.1:4

$$\mathbb{r} = a\left(uv\cos\varphi,\ uv\sin\varphi,\ \frac{u^2-v^2}{2}\right)$$

$0 \le u$ $\quad 0 \le \varphi \le 2\pi$

$v < \infty$ $\quad a = a[e]$

$$\frac{\mathbb{r}}{u} = a(v\cos\varphi, v\sin\varphi, u) \qquad e_u = \frac{\frac{\partial \mathbb{r}}{\partial u}}{\left|\frac{\partial \mathbb{r}}{\partial u}\right| = h_u}$$

$$\frac{\partial \mathbb{r}}{\partial v} = a(u\cos\varphi, u\sin\varphi, -v)$$

$$\frac{\partial \mathbb{r}}{\partial \varphi} = a(-uv\sin\varphi, uv\cos\varphi, 0)$$

$$h_u = a\sqrt{u^2+v^2}$$

$$h_v = a\sqrt{u^2+v^2}$$

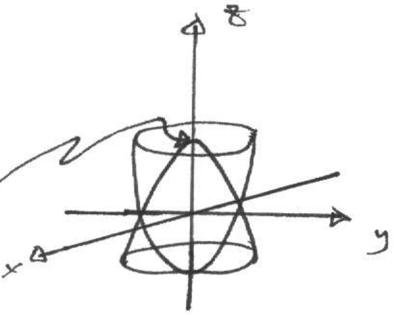
$$h_\varphi = auv$$

$$x^2+y^2 = a^2u^2v^2$$

$$\frac{2z}{a} = u^2-v^2 \Leftrightarrow v^2 = u^2 - \frac{2z}{a}$$

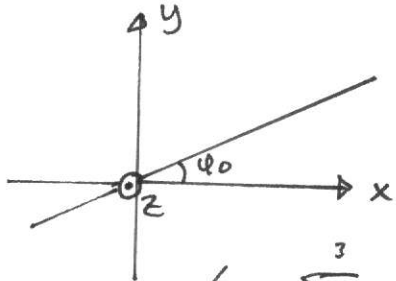
$$x^2+y^2 = a^2u_0^2\left(u_0^2 - \frac{2z}{a}\right)$$

$$\Leftrightarrow \quad z = -\frac{x^2+y^2}{2au_0^2} + \frac{au_0^2}{2}$$



$$x^2+y^2 = a^2v_0^2\left(\frac{2z}{a}+v_0^2\right) \Leftrightarrow z = \frac{x^2+y^2}{2av_0^2} - \frac{av_0^2}{2}$$

$$\frac{y}{x} = \tan\varphi_0$$



Allm. $$\nabla^2\phi(u_1,u_2,u_3) = \frac{1}{u_1u_2u_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\phi}{\partial u_i}\right)$$

$$\nabla^2\phi(u,v,\varphi) = \frac{1}{a^2uv(u^2+v^2)}\left(\frac{\partial}{\partial u}\frac{a^2uv\sqrt{u^2+v^2}}{a\sqrt{u^2+v^2}}\frac{\partial\phi}{\partial u}\right) +$$

$$+\frac{\partial}{\partial v}\left(\frac{a^2uv\sqrt{u^2+v^2}}{a\sqrt{u^2+v^2}}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial\varphi}\left(\frac{a^2(u^2+v^2)}{auv}\frac{\partial\phi}{\partial\varphi}\right) =$$

$$= \frac{1}{a^2uv(u^2+v^2)}\left(v\frac{\partial\phi}{\partial u} + uv\frac{\partial^2\phi}{\partial u^2} + u\frac{\partial\phi}{\partial v} + uv\frac{\partial^2\phi}{\partial v^2} + \right.$$

$$\left. + \frac{u^2+v^2}{uv}\frac{\partial^2\phi}{\partial\varphi^2}\right)$$

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PLK 4.3.1:5

$$(\mathbf{r}\cdot\nabla)\phi(r) = \nabla\cdot\frac{\mathbf{r}}{r}$$

$$\mathbf{r} = (x,y,z)$$

$$r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$$

$$\text{V.L} = (\mathbf{r}\cdot\nabla)\phi(r) = (\mathbf{r}\cdot\nabla\phi(r)) = \left(\mathbf{r}\cdot\frac{d\phi}{dr}\nabla r\right) =$$

$$= \left(\mathbf{r}\cdot\frac{d\phi}{dr}\frac{\mathbf{r}}{r}\right) = \frac{d\phi}{dr}\cdot\frac{1}{r}(\mathbf{r}\cdot\mathbf{r}) = \frac{d\phi}{dr}r$$

$$H.L = \nabla \cdot \frac{\mathbf{r}}{r} = \left(\nabla \frac{1}{r}\right)\cdot \mathbf{r} + \frac{1}{r}\left(\nabla \cdot \mathbf{r}\right) =$$

$$= \frac{-1}{r^2} \nabla r \cdot \mathbf{r} + \frac{1}{\mathbf{r}} \cdot 3 = -\frac{1}{r^2} \frac{1}{r} \mathbf{r} \cdot \mathbf{r} + \frac{3}{r} = \frac{2}{r}$$

$$r \frac{d\phi}{dr} = \frac{2}{r} \Rightarrow \frac{d\phi}{dr} = \frac{2}{r^2}$$

$$\phi(r) = -\frac{2}{r} + C$$

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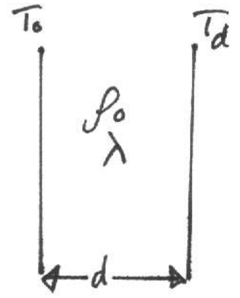


Bestäm temperaturfördelningen i plattans inre. (Värmeflöde från T_0 till T_d)

Fouriers lag

$$\int \mathbb{J}\, d\mathbb{S} = \left[J = -\nabla T\right] = \int -\lambda \nabla T \cdot d\mathbb{S} = \int -\lambda \nabla^2 T \, dV =$$

$$= \int \rho_0 \, dV \Rightarrow \nabla^2 T = -\frac{\rho_0}{\lambda}$$

Laplace ekvation:

$$\frac{\partial^2 T}{\partial x^2} = -\frac{\rho_0}{\lambda}$$

$$T(x) = -\frac{\rho_0}{\lambda} x^2 + Cx + D$$

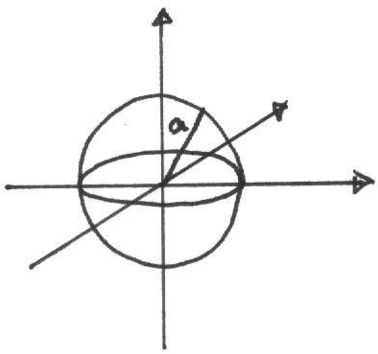
$$T(0) = T_0 \Rightarrow D = T_0$$

$$T(d) = T_d \Rightarrow C = \frac{T_d - T_0}{d} + \frac{\rho_0}{\lambda} d$$

$$T(x) = \frac{-\rho_0}{\lambda} x^2 + \frac{T_d - T_0}{d} x + \frac{\rho_0 d}{\lambda} x + T_0$$

$$T'(x) = \frac{2x\rho_0}{\lambda} + \frac{T_d - T_0}{d} + \frac{\rho_0 d}{\lambda} = 0 \qquad x = \frac{d}{2}$$

pappersuppgift



$$T(a,\theta,\varphi) = T_0\left(1+\frac{\cos\theta}{2}\right)$$

Dela upp och lös en i taget

Problemet är att finna lösning till $\nabla^2 T = 0$

$\nabla^2 T = 0$

$T(a,\theta,\varphi) = T_0$

Ansätt $T(r) = \alpha r^\beta$ Bara r-delen i Laplaceoperatorn

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} T(r) = T''(r) + \frac{2T'(r)}{r} = 0$$

$$\alpha\beta(\beta-1)r^{\beta-2} + 2\alpha\beta r^{\beta-2} = 0$$

$$\beta(\beta-1) + 2\beta = 0$$

$$\beta = \begin{cases} 0 \\ -1 \end{cases}$$

$T(r) = A + \frac{B}{r}$ $\quad B = 0$ Fysikaliskt måste denna försvinna

$T(a,\theta,\varphi) = T_0 \Rightarrow A = T_0$

$$T(a,\theta,\varphi) = \frac{T_0}{2}\cos\theta$$

Ansätt: $T = f(r)\cos\theta$

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 f(r)\cos\theta + \frac{1}{r^2\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial\theta} f(r)\cos\theta$$

$$= f''(r)\cos\theta + \frac{2}{r}f'(r)\cos\theta - \frac{f(r)}{r^2}2\cos\theta = 0$$

$$\alpha\beta(\beta-1)r^{\beta}+2\alpha\beta r^{\beta}-2\alpha r^{\beta}=0$$

$$\beta=\begin{cases}1\\-2\end{cases}$$

$$T(r)=Ar\quad\theta+\frac{B}{r^2}\sin\theta$$

$$T(r,\theta,\varphi)=T_0+\frac{T_0}{2a}r\sin\theta$$

superpositionering gäller för alla linjära fall

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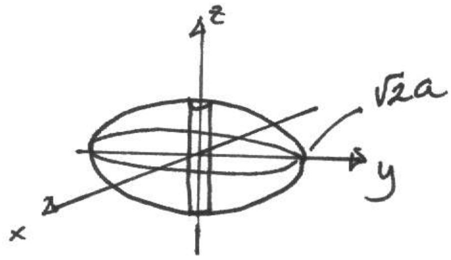
lv5 PLK 4.3.1:12
övn. 9

$$\mathbb{A}(r,\theta,\varphi) = A_0\left[\hat{r}\left(\frac{a}{r} + \frac{2r}{a}\cos^2\theta\right) + \hat{\theta}\left(\frac{a}{r}\cot\theta - \frac{r}{a}\sin 2\theta\right)\right]$$

$$S: r^2(1+\cos^2\theta) = 2a^2 \Rightarrow x^2+y^2+2z^2 = 2a^2$$

ellipsoid

$$\oint_S \mathbb{A}\,d\mathbb{S}\ ?$$



$$\text{Gauss}: \oint_S \mathbb{A}\,d\mathbb{S} = \int_V \nabla\cdot\mathbb{A}\,dV$$

$$\nabla\cdot\mathbb{A} = A_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right] =$$

$$= \ldots = \frac{2A_0}{a}$$

$$\int_V \nabla\cdot\mathbb{A}\,dV = \frac{2A_0}{a}\int dV = \frac{4\pi}{3}2a^2\cdot\frac{\cancel{a}A_0\cdot 2}{\cancel{a}} = \frac{16\pi A_0 a^3}{3}$$

OBS: singularitet i $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sin\theta = 0$ för $\theta = n\pi$

$$\frac{a}{r}\cot\theta = \left[\begin{matrix}\text{Cartesiska}\\\text{koordinater}\end{matrix}\right] = \frac{a}{\sqrt{x^2+y^2+z^2}}\cdot\frac{z}{\sqrt{x^2+y^2}} =$$

$$= \left[\begin{matrix}\text{cylindriska}\\\text{koordinater}\end{matrix}\right] = \frac{a}{\sqrt{\rho^2+z^2}}\cdot\frac{z}{\rho}$$

$$\Rightarrow \int_{S_1} A_\theta \, dS = -A_0 \int_S \frac{a}{\sqrt{\rho^2+z^2}} \cdot \frac{z}{\rho} \cdot \rho \, d\varphi dz \underset{\rho \to 0}{=}$$

$$= -A_0 \int \frac{az}{z} d\varphi dz = -4\pi A_0 a^2$$

$$\oint_S \mathbb{A} \, d\mathbb{S} = \int_V \nabla \cdot \mathbb{A} \, dV - \int_{S_2} A_\theta \, dS = \frac{16\pi A_0 a^2}{3} + 4\pi A_0 a^2 =$$

$$= \frac{28\pi A_0 a^2}{3}$$

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studentuppgift pappersuppgift [ev5, mentorstillfälle 9]

$$\mathbb{B}(x,y,z) = \frac{B_0}{a} \frac{1}{y^2+z^2}\left(x(y^2+z^2)\hat{x} - (y-z)a^2\hat{y} - \right.$$

$$\left. - (y+z)a^2\hat{z}\right)$$

$$C: \mathbb{r}(a\varphi, a\cos\varphi, a\sin\varphi) \qquad 0 \le \varphi \le 2\pi$$

$$\int_C d\mathbb{r} \times \mathbb{B}$$

$$\mathbb{B} = \frac{B_0}{a^3}\left(a^3\varphi\hat{x} - (a\cos\varphi - a\sin\varphi)a^2\hat{y} - (a\cos\varphi + a\sin\varphi)a^2\hat{z}\right)$$

$$= B_0\left(\varphi\hat{x} - (\cos\varphi - \sin\varphi)\hat{y} - (\cos\varphi + \sin\varphi)\hat{z}\right)$$

$$\frac{d\mathbb{r}}{d\varphi} = (a, -a\sin\varphi, a\cos\varphi)$$

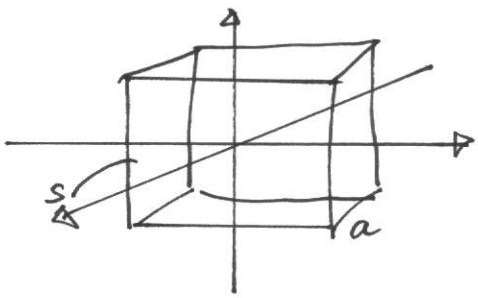
$$d\mathbf{r} \times \mathbf{B} = B_0 a \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -\sin\varphi & \cos\varphi \\ \varphi & -(\cos\varphi - \sin\varphi) & -(\cos\varphi + \sin\varphi) \end{vmatrix} d\varphi =$$

$$= B_0 a (1, \varphi\cos\varphi + \cos\varphi + \sin\varphi, \sin\varphi - \cos\varphi + \varphi\sin\varphi)\, d\varphi$$

$$\int_C d\mathbf{r} \times \mathbf{B} = B_0 a \Big[\varphi, \varphi\sin\varphi - \sin\varphi + \sin\varphi - \cos\varphi,$$

$$-\varphi\cos\varphi + \cos\varphi - \cos\varphi - \sin\varphi\Big]_0^{2\pi} = B_0 a(2\pi, 0, -2\pi)$$

Pappersuppgift [Lv5, mentorstillfälle 9]



$$\mathbf{E}(\mathbf{r}) = \frac{\rho_0 a}{\varepsilon_0}\left(\frac{x^2}{a^2}, \frac{y^2}{b^2}, \frac{z^2}{c^2}\right)$$

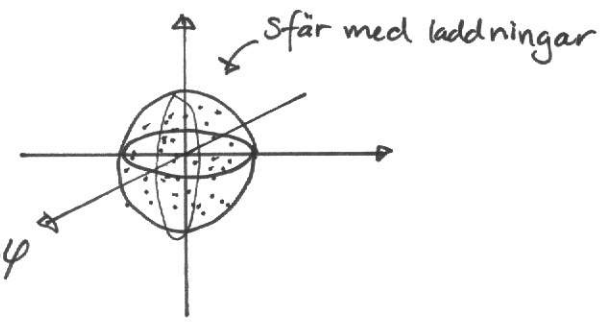
Vad är laddningen Q?

$$\int \mathbf{E}\, d\mathbf{S} = \int \nabla \cdot \mathbf{E}\, dV = \int \frac{\rho_0}{\varepsilon_0} dV = \frac{1}{\varepsilon_0}\int \rho_0\, dV = \frac{Q}{\varepsilon_0}$$

$$\int \mathbf{E}\, d\mathbf{S} = \int \nabla \cdot \mathbf{E}\, dV = \int \frac{\rho_0 a}{\varepsilon_0}\left(\frac{2x}{a^2} + \frac{2y}{b^2} + \frac{2z}{c^2}\right) dx\,dy\,dz$$

$$= 0$$

PLK 6.1.1:10



$\rho(r,\theta,\varphi) = \frac{\rho_r}{a}\sin\theta\cos\varphi$

$\phi(a\theta\varphi) = \phi_0$

Laplace-ekvation!

$$\nabla^2\phi = \frac{-\rho_0 r}{a\varepsilon_0}\sin\theta\cos\varphi$$

$$\nabla^2\phi = 0$$

$$\phi = \phi_0$$

Ansätt $\phi = f(r)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}f(r) = f''(r) + \frac{2}{r}f'(r) = 0$$

$$f(r) = \alpha r^\beta \Rightarrow \beta(\beta-1) + 2\beta = 0 \quad \beta = \begin{cases} 0 \\ -1 \end{cases}$$

$$\phi = A + \frac{B}{r} \quad B = 0 \Rightarrow \phi(a) = \phi_0 \Rightarrow A = \phi_0$$

Fysikaliskt måste $B=0$

vinkelberoende $\rightarrow$ $\nabla^2\phi = -\frac{\rho_0 r}{\varepsilon_0 a}\sin\theta\cos\varphi$

$$\phi(a) = 0$$

Ansätt $\phi(r,\theta,\varphi) = f(r)\sin\theta\cos\varphi$

vinkel beroende → $\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} f(r)\sin\theta\cos\varphi +$

$$+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} f(r)\sin\theta\cos\varphi - \frac{f(r)\sin\theta\cos\varphi}{r^2\sin^2\theta}$$

$$= f''(r)\sin\theta\cos\varphi + \frac{2}{r}f'(r)\sin\theta\cos\varphi +$$

$$+ \frac{f(r)\cos\varphi}{r^2\sin\theta}(\cos^2\theta - \sin^2\theta) - \frac{f(r)\cos\varphi}{r^2\sin\theta}$$ peta in en trigg-etta för att komma vidare

$$= f''(r)\sin\theta\cos\varphi + \frac{2}{r}f'(r)\sin\theta\cos\varphi - \frac{2}{r^2}f(r)\sin\theta\cos\varphi =$$

$$= -\frac{\rho_0 r}{\varepsilon_0 a}\sin\theta\cos\varphi$$

$$f''(r) + \frac{2}{r}f'(r) - \frac{2}{r^2}f(r) = -\frac{\rho_0 r}{\varepsilon_0 a}$$

$$f(r) = \alpha r^\beta$$

$$\alpha\beta(\beta-1)r^{\beta-2} + 2\alpha\beta r^{\beta-2} - 2\alpha r^{\beta-2} = -\frac{\rho_0 r}{\varepsilon_0 a}$$

$\beta = 3$ måste $\quad 6\alpha + 6\alpha - 2\alpha = \frac{-\rho_0}{\varepsilon_0 a}$

$$\alpha = -\frac{\rho_0}{10\varepsilon_0 a}$$

$$f(r) = A + \frac{B}{r^2} - \frac{\rho_0}{10\varepsilon_0 a}r^3$$

$$Aa - \frac{\rho_0 a^3}{10\varepsilon_0 a} = 0 \Rightarrow A = \frac{\rho_0 a}{10\varepsilon_0}$$

$$\phi(r,\theta,\varphi) = \phi_0 + \frac{\rho_0 a}{10\varepsilon_0}\left(r - \frac{r^3}{a^3}\right)\sin\theta\cos\varphi$$

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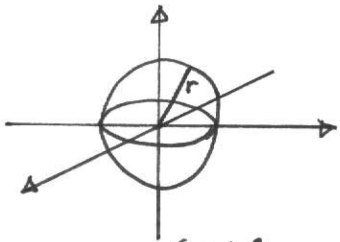
lv 6 pappersuppgift studentuppgift
övn 10

2. I sfären $r=a$ finns en rymdladdning med tätheten $\rho(r,\theta,\varphi)=\rho_0\left(\frac{r}{a}\right)^3\sin\theta\sin\varphi$

och på sfären gäller $\phi(a,\theta,\varphi)=\theta_0$

Bestäm den elektriska potentialen ϕ i sfären.



$$\rho(r,\theta,\varphi)=\rho_0\left(\frac{r}{a}\right)^3\sin\theta\sin\varphi=\rho_0\frac{r^2}{a^3}y$$

$$\phi(a,\theta,\varphi)=\phi_0$$

$$\begin{cases} x' = r\sin\theta\sin\varphi \\ y' = r\cos\theta \\ z' = r\sin\theta\cos\varphi \end{cases}$$

$$\nabla^2\phi = -\frac{\rho_0}{\varepsilon_0} = \frac{-\rho_0}{\varepsilon_0}\frac{r^3}{a^3}\cos\theta$$

Ansatz: $\phi(r,\theta) = A(r)+B(r)\cos\theta$

Homo: $\nabla^2\phi = \ldots =$

$$=\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A'(r)\right)}_{①}+\cos\theta\underbrace{\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B'(r)\right)-\frac{2B(r)}{r^2}\right)}_{②}=0$$

① $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A'(r)\right)=0$

...

ofysikaliskt

$A(r) = -\frac{C}{r}+D$ $\quad A=D \quad \because A$ konstant

② $\frac{1}{r^2}\left(2rB'(r) + r^2B''(r) - 2B(r)\right) = 0$

Ansatz: $B_{homo}(r) = \alpha r^{\beta}$

$$\beta = \begin{cases} 1 & \beta \neq -2 \\ -2 & \end{cases}$$

lösningen blir singulär i origo

$\therefore B_{homo}(r) = \alpha r$

Partikulärlösning

$$\nabla^2\phi = \frac{-\rho_0}{\varepsilon_0}\left(\frac{r}{a}\right)^3\cos\theta$$

$$② \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B'(r)\right) - 2B(r) = \frac{-\rho_0}{\varepsilon_0}\left(\frac{r}{a}\right)^3$$

$$r^2B''(r) + 2rB'(r) - 2B(r) = \frac{-\rho_0}{\varepsilon_0}\frac{r^5}{a^3}$$

Ansatz: $B_{part}(r) = \gamma r^5$

...

$$\gamma = -\frac{\rho_0}{a^3\varepsilon_0} \qquad \therefore B_{part}(r) = \frac{\rho_0 r^5}{28\varepsilon_0 a^3}$$

Randvillkor: $\phi(r,\theta) = A(r) + B(r)\cos\theta$

$$\phi(a,\theta) = \phi_0 \Rightarrow \begin{cases} A(a) = \phi_0 = A \\ B(a) = 0 \end{cases}$$

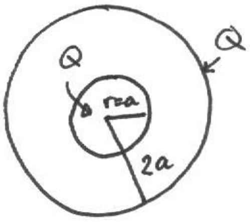
$$B(a) = 0 \Rightarrow -\frac{\rho_0 a^5}{28\varepsilon_0 a^3} + \alpha a = 0$$

$$\alpha = \frac{\rho_0 a}{28\varepsilon_0}$$

$$\phi(r,\theta,\varphi) = \phi_0 + \frac{\rho a}{28\varepsilon_0}\left(r - \frac{r^5}{a^4}\right)\cos\theta$$

Pappersuppg. 1

En elektrisk laddning Q är jämnt fördelad i en sfär med radien a. Den omges av ett tunt sfäriskt skal med radien 2a och laddningen Q. Bestäm det elektriska fältet $\mathbb{E}(r)$ och potentialen $\phi(r)$ överallt.



$$\mathbb{E}(r) = -\nabla\phi$$

Gauss lag $\quad \varepsilon_0 \oint_S \mathbb{E}\cdot d\mathbb{S} = Q$

Gör ett smart antagande: $\mathbb{E} = \mathbb{E}(r)\hat{r}$

$r>2a$ $\quad \varepsilon_0 \oint_S E(r)\hat{r}\cdot r^2\sin\theta\, d\theta\, d\varphi\, \hat{r} = 2Q$

$$\varepsilon_0 E(r) r^2 4\pi = 2Q$$

$$\mathbb{E}(r) = \frac{2Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$2a > r > a \qquad \mathbb{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

$r<a$ $\quad \varepsilon_0 \int \mathbb{E}(r)\cdot d\mathbb{S} = \rho\,\frac{4\pi r^3}{3}$

$$\rho = \frac{Q}{\frac{4\pi a^3}{3}}$$

$$\varepsilon_0 E(r) r^2 4\pi = Q\frac{r^3}{a^3} \qquad \mathbb{E}(r) = \frac{Qr}{4\pi\varepsilon_0 a^3}$$

Pappersuppgift

Det elektriska fältet $\mathbb{E}(r) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$ som svarar mot en punktladdning i origo är källfritt för $r \neq 0$ och har följaktligen en vektorpotential $\mathbb{A}$. Beräkna den ~~källfria~~ vektorpotentialen $\mathbb{A}$ till fältet $\mathbb{E}(r)$ som i ett sfäriskt koordinatsystem har formen

$$\mathbb{A} = f(r, \theta, \varphi)\hat{\varphi}$$

där $f(2, \frac{\pi}{2}, \varphi) = \frac{3}{2}$. Ange de punkter där $\mathbb{A}$ ej är definierad.

$$\mathbb{E}(r) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$ källfritt för $r \neq 0$

$$\Rightarrow \nabla \cdot E = 0$$

$\nabla \cdot E = 0 \Leftrightarrow$ E-fältet har en vektorpotential $\mathbb{A}$, $\mathbb{E} = \nabla \times \mathbb{A}$

$\nabla \cdot \mathbb{A} = 0$ (i denna uppgift)

$\mathbb{A} = f(r,\theta,\varphi)\hat{\varphi}$ $\qquad f(2, \frac{\pi}{2}, \varphi) = \frac{3}{2}$

$$\mathbb{E} = \nabla \times \mathbb{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & r\sin\theta f(r,\theta,\varphi) \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} r\sin\theta f(r,\theta,\varphi)\hat{r} - \frac{r}{r^2\sin\theta}\frac{\partial}{\partial r} r\sin\theta f(r,\theta,\varphi)\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta f(r,\theta,\varphi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r} r f(r,\theta,\varphi)\hat{\theta}$$

$$\begin{cases} \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta f(r,\theta,\varphi) = \frac{Q}{4\pi\varepsilon_0 r^2} \\ -\frac{1}{r}\frac{\partial}{\partial r} r f(r,\theta,\varphi) = 0 \quad (\text{har inget } \hat{\theta}) \end{cases}$$

$$\Rightarrow r f(r,\theta,\varphi) = A(\theta,\varphi)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A(\theta,\varphi) = \frac{Q}{4\pi\varepsilon_0}$$

$$\sin\theta A(\theta,\varphi) = -\frac{Q\cos\theta}{4\pi\varepsilon_0} + B(\varphi)$$

$$\nabla\cdot\mathbb{A} = 0 \Rightarrow \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} f(r,\theta,\varphi) = 0 \Rightarrow B \text{ konstant}$$

(beror ej på φ)

$$f(r,\theta,\varphi) = \frac{B}{r\sin\theta} - \frac{Q\cos\theta}{4\pi\varepsilon_0 r\sin\theta}$$

$$f(2,\tfrac{\pi}{2},\varphi) = \frac{3}{2}$$

$$\Rightarrow \frac{B}{2} = \frac{3}{2} \Rightarrow B = 3$$

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lv6 studentuppgift PLK 6.1.1:14
övn 11

$$g = -\nabla\phi$$

$$\nabla^2\phi = \gamma\rho$$

$$\nabla^2\phi = \int \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right) = \int \gamma\rho r^2$$

$$\int r^2 \frac{\partial\phi}{\partial r} = \int \frac{\gamma\rho r^3}{3r^2} + \frac{A}{r^2}$$

$$\phi = \frac{\gamma\rho r^2}{6} - \frac{A}{r} + B$$

$$g = \nabla\phi = \frac{\gamma\rho r}{3} + \frac{A}{r^2}$$

$\underline{0<r<R}$, $\rho = \rho_0$

$$g_1 = -\frac{\gamma\rho_0 r}{3}\hat{r}$$

$\underline{r>R}$ $$g_2 = -\frac{A}{r^2}\hat{r}$$

$\underline{r=R}$ $$\frac{\gamma\rho_0 R}{3} = \frac{A}{R^2}$$

$$A = \frac{\gamma\rho_0 R^3}{3}$$

$$g_1 = \begin{cases} -\dfrac{\gamma\rho_0 r}{3}\hat{r} \\ \dfrac{\gamma\rho_0 R^3}{3r^2}\hat{r} \end{cases}$$

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Bestäm den begränsade lösningen till Poissons ekvation

$$\nabla^2\psi(\rho,\varphi) = -\psi_0 R^3\rho^{-5}\cos\varphi$$

utanför cirkeln $\rho = R$ som uppfyller randvillkoret $\psi(R,\varphi) = \psi_0 \sin\varphi$

Dela upp lösningen i två delar

$$\begin{cases} \psi_1(\rho)\sin\varphi = \psi_0 \sin\varphi \\ \psi_2(\rho)\cos\varphi = 0 \end{cases}$$

$$\nabla^2\psi_1(\rho)\sin\varphi = 0$$

$$\nabla^2\psi_2(\rho)\cos\varphi = -\psi_0 R^3\rho^{-5}\cos\varphi$$

$$\nabla^2\psi_1(\rho)\sin\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi_1}{\partial\rho}\right)\sin\varphi + \frac{\psi_1}{\rho_2}\frac{\partial^2}{\partial\varphi^2}(\sin\varphi) =$$

$$= \sin\varphi\left(\psi_1'' + \frac{\psi_1'}{\rho} - \frac{\psi_1}{\rho^2}\right) = 0$$

$$\Rightarrow \quad \psi_1'' + \frac{\psi_1'}{\rho} - \frac{\psi_1}{\rho^2} = 0$$

$$\psi_1 = \alpha\rho^\beta \Rightarrow \alpha\beta(\beta-1)\rho^{\beta-2} + \frac{\alpha\beta\rho^{\beta-1}}{\rho} - \frac{\alpha\rho}{\rho^2} = 0$$

$$\beta^2 - 1 = 0 \qquad \beta = \pm 1$$

$$\psi_1 = \frac{\alpha_1}{\rho} + \alpha_2\rho$$

$$\psi_1 = \frac{\alpha_1}{\rho}$$

$$\psi_1(R)\sin\varphi = \psi_0 \sin\varphi$$

$$\frac{\alpha_1}{R}\sin\varphi = \psi_0 \sin\varphi$$

$\Rightarrow \alpha_1 = \psi_0 R \qquad \psi_1(\rho)\sin\varphi = \frac{R\psi_0}{\rho}\sin\varphi$

pss $\nabla^2(\rho)\cos\varphi = \cos\varphi\left(\psi_2'' + \frac{\psi_2'}{\rho} - \frac{\psi_2}{\rho^2}\right) =$

$$= -R^3\psi_0\rho^{-5}\cos\varphi$$

homo: $\begin{cases} \psi_{2h} = \alpha\rho^{\beta} \\ \psi_{2h} = \frac{\alpha_1}{\rho} \end{cases}$

partikulär $\begin{cases} \psi_{2p} = \alpha\rho^{\beta} \\ \psi_{2p} = \alpha_2\rho^{\beta} \end{cases}$

$\Rightarrow \alpha_2\left(\beta(\beta-1)\rho^{\beta-2} + \frac{\beta\rho^{\beta-1}}{\rho} - \frac{\rho^{\beta}}{\rho^2}\right) = -R^3\psi_0\rho^{-5}$

$\beta = -3$ förkorta bort ρ:na

$$\alpha_2(\beta(\beta-1) + \beta - 1) = -R^3\psi_0$$

$$\alpha_2(-3\cdot(-4) - 3 - 1) = -R^3\psi_0$$

$$\alpha_2 = \frac{-R_3\psi_0}{8}$$

$$\Psi_2(\rho) = \frac{-R^3\Psi_0}{8\rho^3} + \frac{\alpha_1}{\rho}$$

$$\Psi_2(R)\cos\varphi = \left(\frac{-R^3\Psi_0}{8R^3} + \frac{\alpha_1}{R}\right)\cos\varphi = 0$$

$$\frac{\alpha_1}{R} = \frac{\Psi_0}{8} \Rightarrow \alpha_1 = \frac{\Psi_0 R}{8}$$

$$\Psi_2(\rho)\cos\varphi = \frac{-R^3\Psi_0}{8\rho^3} + \frac{R\Psi_0}{8\rho}$$

$$\Psi_1(\rho)\sin\varphi + \Psi(\rho)\cos\varphi = \frac{R\Psi_0}{\rho}\sin\varphi + \frac{\Psi_0}{8}\left(\frac{R}{\rho} - \frac{R^3}{\rho^3}\right)\cos\varphi$$

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pappersuppgift

För en endimensionell elektrisk ledare med längden L gäller att konduktiviteten

$$\sigma = \sigma_0\left[1 + \left(\frac{x}{L}\right)^2\right]$$

Man kopplar ledarens vänstersida $x=0$ till en likströmsgenerator, som ger en potential ϕ_0 medan den högra änden jordas ($\phi = 0$). Ohms lag ger att sambandet mellan strömtätheten J och σ är $J = -\sigma\nabla\phi$. Bestäm potentialen i ledaren. (Ledning: strömtätheten J är konstant

J konstant $\Rightarrow$ $C = -\sigma \phi'(x)$

$$\phi'(x) = \frac{-C}{\sigma_0 \left[1 + \left(\frac{x}{L}\right)^2\right]}$$

$$\phi(x) = -\frac{C}{\sigma_0} L \arctan\left(\frac{x}{L}\right) + B$$

Randvillkor $\quad \phi(0) = \phi_0$

$\qquad\qquad\qquad \phi(L) = 0$

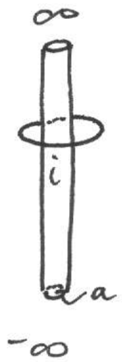
$\phi(0) = \phi_0 \Rightarrow B = \phi_0$

$\phi(L) = 0 \Rightarrow \dfrac{C \cdot L \pi}{\sigma_0 4} + \phi_0 = 0$

$$\Rightarrow C = -\frac{4 \sigma \phi_0}{\pi \cdot L}$$

$$\phi(x) = -\frac{4\phi_0}{\pi} \arctan\left(\frac{x}{L}\right) + \phi_0$$

pappersuppg. En oändligt lång rak ledare har cirkulärt tvärsnitt med radien a och leder en likström med strömstyrkan i. Använd Amperes lag för att härleda magnetfältet i och kring ledaren, om materialet i det antas homogent och isotropt.
(Ledning: Det är ett empiriskt faktum att $\mathbb{B}$-fältet kring en ström i $\hat{z}$-riktningen är riktat i $\hat{\varphi}$-riktningen)



Amperes lag: $\frac{1}{\mu_0}\oint \mathbb{B}\cdot d\mathbb{r} = i$

vi är i vakuum

avståndet från ledaren

$\mathbb{B} = B(\rho)\hat{\varphi}$

Vad blir $\mathbb{B}$?

$r > a$: $\frac{1}{\mu_0}\oint_C B(\rho)\hat{\varphi}\,\rho\, d\varphi\,\hat{\varphi} = \frac{1}{\mu_0}\oint_C B(\rho)\rho\, d\varphi = i$

$\frac{1}{\mu_0} B(\rho)\rho\cdot 2\pi = i \Rightarrow B(\rho) = \frac{i\mu_0}{2\pi\rho}$

$r < a$: $\frac{1}{\mu_0}\oint_C B(\rho)\hat{\varphi}\,\rho\, d\varphi\,\hat{\varphi} = \frac{1}{\mu_0}\oint_C B(\rho)\rho\, d\varphi =$

$= \frac{i}{\pi a^2}\cdot \pi\rho^2 \Rightarrow \frac{B(\rho)}{\mu_0}\rho\, 2\pi = \frac{i\rho^2}{a^2}$

$\Rightarrow B(\rho) = \frac{i\mu_0}{2\pi a^2}\rho$

Pappersuppgift

För ett tidsberoende magnetfält gäller Faradays lag, som i differentialform lyder:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

Amperes lag i differentialform lyder:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

För $\mathbf{B}$ gäller $\nabla \cdot \mathbf{B} = 0$ och strömtätheten $\mathbf{J}$ uppfyller Ohms lag $\mathbf{J} = \sigma \mathbf{E}$. Kombinera dessa ekvationer till en fältekvation för $\mathbf{B}$. För stort σ ger denna fältekvation en relevant modell för magnetfältet i en god ledare.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 \sigma \mathbf{E}$$

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \sigma \nabla \times \mathbf{E}$$

$$-\nabla^2 \mathbf{B} + \nabla(\nabla \cdot \mathbf{B}) = -\mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t}$$

$$\Rightarrow \nabla^2 \mathbf{B} = \mu_0 \frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

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övning 12

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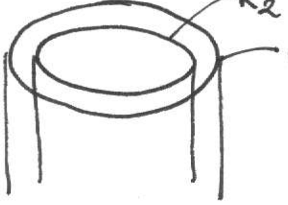

$$\nabla^2\phi = 0$$

ansätt: $\phi(r)$ $\quad \nabla^2 f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r f'(r)\right) = 0$

$$\Leftrightarrow \quad r f'(r) = C \quad \Leftrightarrow \quad f'(r) = \frac{C}{r}$$

$$f(r) = C \cdot \ln r + D$$

$$f(R_1) = \phi_1 \Rightarrow f(R_1) = C \ln R_1 + D = \phi_1 \Rightarrow$$

$$D = \phi - C \ln R_1$$

$$f(R_2) = \phi_2 \Rightarrow C \ln R_1 + \phi_1 - C \ln R_1 = \phi_2$$

$$\Rightarrow C = \frac{\phi_2 - \phi_1}{\ln \frac{R_2}{R_1}}$$

$$\phi(r) = \frac{\phi_2 - \phi_1}{\ln \frac{R_2}{R_1}} \cdot \ln r + \phi_1 \cdot \frac{\phi_2 - \phi_1}{\ln \frac{R_2}{R_1}} \ln R_1$$

$$\phi(r) = \frac{\phi_2 - \phi_1}{\ln \frac{R_2}{R_1}} \left(\ln \frac{r}{R_1}\right) + \phi_1$$

$$E(r) = -\nabla\phi$$

$$\nabla\phi = \frac{\phi_2 - \phi_1}{\ln\left(\frac{R_2}{R_1}\right)} \cdot \frac{1}{r}\hat{r} \qquad E(r) = \frac{\phi_1 - \phi_2}{\ln \frac{R_2}{R_1}} \cdot \frac{1}{r}\hat{r}$$

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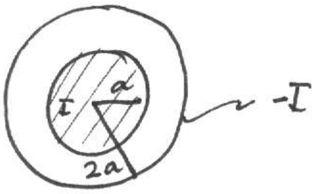
En koaxialkabel består av en cirkulär ledare med radien 2a. Genom den cirkulära ledaren går det en ström I, och genom det yttre skalet går det en returström $-I$. Beräkna magnetfältet i och omkring koaxialkabeln.



ledare i mitten, massiv med strömmen I

Antag lång kabel

Antag $\mathbf{B}(\mathbf{r}) = B(r)\hat{\theta}$

Amperes lag: $\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J}\, d\mathbf{S}$

Arean $\pi a^2 \Rightarrow \mathbf{J} = \frac{I}{\pi a^2}\hat{z}$

<u>Fall $r<a$</u>

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \oint_C B(r)\hat{\theta}\hat{\theta}\, dr =$$

$$= B(r)\cdot 2\pi r = \mu_0 \int_S \mathbf{J}\, d\mathbf{S}$$

OBS: μ_0 (prop. konst.) som endast gäller i vakuum.

$$= \frac{\mu_0 I}{\pi a^2}\cdot \pi r^2 \Rightarrow B(r) = \frac{\mu_0 I r}{2\pi a^2}$$

<u>Fall $a<r<2a$</u>

$$\oint \mathbf{B}\, d\mathbf{r} = B(r)\cdot 2\pi = \mu_0 I \Rightarrow B(r) = \frac{\mu_0 I}{2\pi r}$$

<u>Fall $r>2a$</u> $\oint \mathbf{B}\cdot d\mathbf{r} = I + (-I) = 0$

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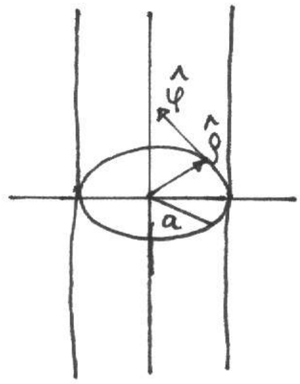
lv 7
övning 13 Ett magnetfält genereras av en elektrisk ström som i cylindriska koordinater kan skrivas

$$\mathbb{J} = \begin{cases} J_0 \hat{z} & \rho < a \\ 0 & \rho > a \end{cases}$$

Bestäm en vektorpotential $A = A\hat{z}$ som beskriver magnetfältet



(ρ, φ, z)

Använd Amperes lag i vakuum:

$$\frac{1}{\mu_0} \oint \mathbb{B}\, d\mathbb{r} = i$$

Vet att: $\nabla \times \mathbb{A} = \mathbb{B}$

$$\nabla \times A(\rho, \varphi, z) = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & A \end{vmatrix} =$$

Har endast komponent i z-led

$$= \frac{1}{\rho}\Big(\underbrace{\frac{\partial A}{\partial \varphi}}_{=0} \hat{\rho} - \frac{\partial A}{\partial \rho} \rho \hat{\varphi} \Big)$$

ingen komponent i $\hat{\rho}$-led för strömmen fall $r>a$:

$$\frac{1}{\mu}\oint \mathbf{B}\,d\mathbf{r} = \frac{-1}{\mu_0}\oint \underbrace{\frac{\partial}{\partial\rho}A\hat{\varphi}}_{=\mathbf{B}} \cdot \underbrace{\rho\, d\varphi\,\hat{\varphi}}_{=d\mathbf{r}} = i$$

$$= \frac{-1}{\mu_0}\frac{\partial}{\partial\rho}A\rho\int_0^{2\pi} d\varphi = i$$

uttryck för i:

$$(*)\quad i = \int_S \mathbf{J}\cdot d\mathbf{S}$$

$$(❀)\quad \frac{\partial}{\partial\rho}A = \frac{-\mu_0 i}{2\pi\rho}$$

$$(*) \Rightarrow \int (0,0,J_0)(0,0,1)\,dS = J_0\pi a^2$$

Nu har vi ett uttryck för i, stoppa in i (❀) och lös ut A

$$\frac{\partial A}{\partial\rho} = \frac{-\mu_0 J_0 a^2}{2\rho}$$

$$A = \frac{-\mu_0 J_0 a^2}{2}\ln\rho + C(z)$$

Då $r<a$: samma - bara i:et ändras

Separabel diff. ekvation:

$$\partial A = \frac{-\mu_0 J_0\rho^2}{2}\frac{1}{\rho}\,d\rho$$

$$A = \frac{-\mu_0 J_0\rho^2}{4} + D(z)$$

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Härled ur Maxwells ekvationer vågekvationerna för en elektromagnetisk våg som rör sig genom ett elektriskt ledande medium med fria laddningar. Ställ upp vågekvationerna för både $\mathbb{E}$- och $\mathbb{B}$-fälten.

Maxwell
$$\begin{cases} \nabla \cdot \mathbb{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \times \mathbb{E} = -\dfrac{\partial \mathbb{B}}{\partial t} \\ \nabla \times \mathbb{B} = \mu_0 \mathbb{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbb{E}}{\partial t} \end{cases}$$

$$\nabla \times \nabla \times \mathbb{E} = -\nabla \times \frac{\partial \mathbb{B}}{\partial t}$$

$$\nabla \cdot (\nabla \cdot \mathbb{E}) - \nabla^2 \mathbb{E} = -\frac{\partial \nabla}{\partial t} \times \mathbb{B}$$

utveckla

$$-\frac{\partial}{\partial t} \nabla \times \mathbb{B} = -\frac{\partial}{\partial t}\left(\mu_0 \mathbb{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t}\right)$$

$$\nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \mathbb{E} = -\frac{\partial}{\partial t}(\mu_0 \mathbb{J}) - \mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t}$$

$$\nabla^2 \mathbb{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbb{E}}{\partial t^2} + \frac{\partial}{\partial t}(\mu_0 \mathbb{J}) + \nabla\left(\frac{\rho}{\varepsilon_0}\right)$$

$$\nabla \times \nabla \times \mathbb{B} = \nabla \times \left(\mu_0 \mathbb{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbb{E}}{\partial t}\right)$$

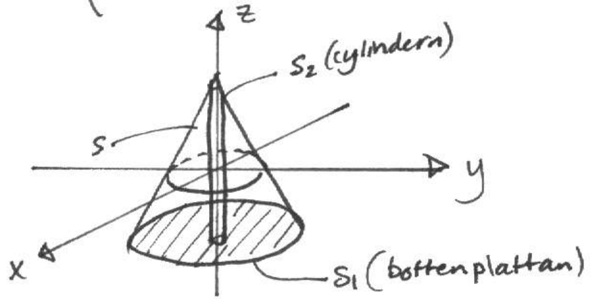
$$\nabla \cdot (\nabla \cdot \mathbb{B}) \cdot \nabla^2 \mathbb{B} = \nabla \times \mu_0 \mathbb{J} + \varepsilon_0 \mu_0 \frac{\partial \nabla \times \mathbb{E}}{\partial t}$$

$$\nabla^2 \mathbb{B} = -\mu_0 \nabla \times \mathbb{J} + \varepsilon_0 \mu_0 \frac{\partial^2 \mathbb{B}}{\partial t^2}$$

PLK 7.2 $\quad \displaystyle\int \mathbb{F}\cdot d\mathbb{S}$?

$$S:\ x^2+y^2=(z-2)^2 \qquad -2\le z\le 2$$

$$\mathbb{F}(r) = F_0\left(\frac{\rho\hat{\rho}+z\hat{z}}{(\rho^2+z^2)^{\frac{3}{2}}}+\rho\hat{\alpha}\right)$$



Gauss sats:

$$\int_V \nabla\cdot\mathbb{F}\,dV = \int_S \mathbb{F}\cdot d\mathbb{S} + \int_{S_1}\mathbb{F}\cdot d\mathbb{S} + \int_{S_2}\mathbb{F}\cdot d\mathbb{S}$$

$$\nabla\cdot\mathbb{F} = \ldots = 0$$

$\underline{S_1}$: $$\int_{S_1}\mathbb{F}\cdot d\mathbb{S} = \int_{S_1}\mathbb{F}\,dS(0,0,-1) = \iint -F_0\frac{2}{(\rho^2+4)^{\frac{3}{2}}}\rho\,d\rho\,d\alpha =$$

ur Beta

$$\stackrel{\downarrow}{=} 4\pi F_0\left(\frac{1}{2}-\frac{1}{2\sqrt{5}}\right)$$

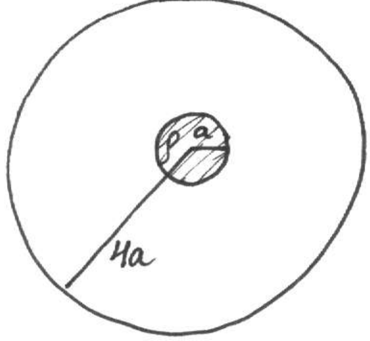
$\underline{S_2}$: $$\int \mathbb{F}\cdot d\mathbb{S}_2 = \int F_z\,dS(\hat{\rho}) = \int_{-2}^{2}\int_0^{2\pi} -F_0\frac{\rho}{(\rho^2+z^2)^{\frac{3}{2}}}\rho\,d\alpha\,dz =$$

Ur Beta

$$\stackrel{\downarrow}{=} \lim_{\rho\to 0} -2\pi F_0\rho^2\left[\frac{z^2}{\rho^2\sqrt{\rho^2+z^2}}\right]_{-2}^{2} = -4\pi F_0$$

$$\Rightarrow \int_S \mathbb{F}\cdot d\mathbb{S} = 2\pi F_0\left(1+\frac{1}{\sqrt{5}}\right)$$

PLK 7.4



$- \frac{a\rho_0}{48}$

Gauss lag $\varepsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = Q$

$r<a$ $\varepsilon_0 \int E(r) r^2 \sin\theta \, d\varphi \, d\theta = \varepsilon 4\pi E(r) r^2 = \rho \frac{4\pi}{3} r^3$

$\Rightarrow \mathbf{E}(r) = \frac{\rho r}{3\varepsilon_0} \hat{r}$

$a<r<4a$

$$\varepsilon E(r) r^2 4\pi = \rho \frac{4\pi a^3}{3}$$

$$\mathbf{E}(r) = \frac{\rho_0 a^3}{3\varepsilon_0 r^2} \hat{r}$$

$4a<r$ $\varepsilon E(r) \cdot r^2 4\pi = \rho \frac{4\pi}{3} a^3 - \frac{a\rho_0}{48} \cdot 4\pi (4a)^2 = 0$

$$W = \frac{\varepsilon_0}{2} \int |E(r)|^2 dV$$

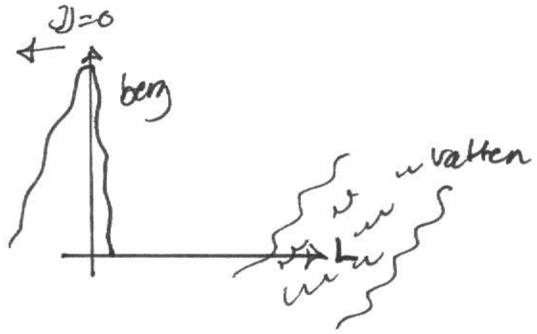
$$W_{r<a} = \frac{\varepsilon_0}{a} \int_0^a \int_0^\pi \int_0^{2\pi} \frac{\rho_0^2 r^2}{9\varepsilon_0} r^2 \sin\theta \, d\theta \, d\varphi \, dr = \frac{4\pi}{18} \frac{\rho_0^2}{\varepsilon_0} \frac{a^5}{5}$$

$$W_{a<r<4a} = \frac{\varepsilon_0}{2} \int_a^{4a} \int_0^\pi \int_0^{2\pi} \frac{\rho_0^2 a^6}{9\varepsilon_0^2 r^4} r^2 \sin\theta \, d\theta \, d\varphi \, dr = \frac{4\pi}{18\varepsilon_0} \rho_0^2 a^5 \frac{3}{4}$$

$$W_{tot} = W_{r<a} + W_{a<r<4a} = \frac{19}{90} \frac{\pi \rho_0^2 a^5}{\varepsilon_0}$$

PLK 7.5

$$\frac{dn}{dt} = R\cdot n - \nabla\cdot J = \underbrace{Rn + k\nabla^2 n}_{\text{lös denna}}$$



$$\frac{\partial^2 n}{\partial x^2} + \frac{R}{k} n = 0$$

$$n(x) = A\cos\sqrt{\frac{R}{k}}\,x + B\sin\sqrt{\frac{R}{k}}\,x$$

$$n'(0) = 0 \Rightarrow B = 0$$

$$n(L) = 0 \Rightarrow A\cos\sqrt{\frac{R}{k}}\,L = 0 \qquad A \neq 0$$

$$\Rightarrow \sqrt{\frac{R}{k}}\,L = \frac{\pi}{2}$$