Föreläsning 5/12-13

FUNKTIONSFÖLJDER OCH FUNKTIONSSERIER
LIKFORMIG KONVERGENS

$f_1(x), f_2(x), \dots, f_n(x), \dots$

$f_n : [a,b] \longrightarrow \mathbb{R}$

$x = x_0$: numerisk följd (talföljd)
$f_1(x_0), \dots, f_n(x_0), \dots$

Konvergent/divergent?

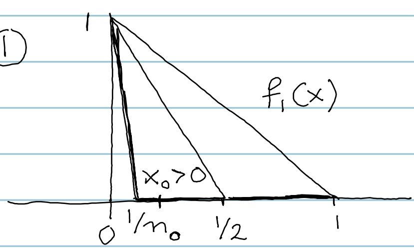
konvergent: $\lim_{n\to\infty} f_n(x_0) = l_{x_0}$

Om $\{f_n(x)\}_{n=1}^{\infty}$ konvergerar $\forall x \in [a,b]$, mot

l_x, då kallar vi $l_x = f(x)$, och vi säger att

funktionsföljden $\longrightarrow f(x)$ punktvis

Exempel ①



$f_n(x) \xrightarrow[n\to\infty]{} 0$ $\quad x \neq 0$
$\quad x \in (0,1]$

→

$\forall n > n_0 \quad \frac{1}{n} < \frac{1}{n_0} < x_0$

$f_n(x_0) = 0 \qquad \forall n > n_0$

$f_n(0) = 1 \longrightarrow 1$

$\Rightarrow \quad f_n \longrightarrow f$ punktvis

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases}$$

f_n kontinuerliga $\forall n$

<u>Men</u> f ej kontinuerlig

def. $\{f_n(x)\}_{n=1}^{\infty}$ konvergerar mot $f(x)$ punktvis i $[a,b]$, om

$$\forall x \in [a,b] \quad \forall \varepsilon > 0 \quad \exists N_{\varepsilon,x} : \quad \forall n > N$$

$$|f_n(x) - f(x)| < \varepsilon$$

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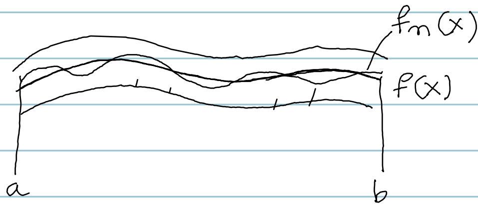
def $\{f_n(x)\}_{n=1}^{\infty}$ konvergerar likformigt mot $f(x)$ i $[a,b]$ om

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon : \forall n > N \quad \forall x \in [a,b]$$

$$|f_n(x) - f(x)| < \varepsilon$$

$\longrightarrow$



(uniform convergence)

"jämnhet"

Likformig konvergens:

$f_n \longrightarrow f$ likformigt $\Longleftrightarrow$ $\forall \varepsilon > 0 \ \exists N_\varepsilon : \forall n > N_\varepsilon$

$\sup_{x \in [a,b]} |f_n(x) - f(x)| < \varepsilon$

Om f_n, f kontinuerliga så $\sup_{[a,b]} = \max_{[a,b]}$

Exempel ② $f_n(x) = x^n - x^{n+1}, \ x \in [0,1]$

$0 \le x < 1 : \quad x^n \longrightarrow 0, \ x^{n+1} \longrightarrow 0$
$f_n(1) = 0$
$\Longrightarrow f_n(x) \longrightarrow 0$ punktvis i $[0,1]$

$\sup_{[0,1]} \big| \underbrace{(x^n - x^{n+1})}_{\ge 0} - 0 \big| = \max_{[0,1]} (x^n - x^{n+1})$
(max)

$x = 0; 1: \quad 0$

$(f_n - f)'(x) = n x^{n-1} - (n+1)x^n = 0$

$x \neq 0 \qquad \cancel{x^{n-1}}(n - (n+1)x) = 0$

$\longrightarrow$

$$x_n = \frac{n}{n+1} \qquad \left.(x^n - x^{n+1})\right|_{x=\frac{n}{n+1}} =$$

$$= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} = \left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right) =$$

$$= \underbrace{\frac{1}{\left(1+\frac{1}{n}\right)^n}}_{\to 1/e} \cdot \underbrace{\frac{1}{n+1}}_{\to 0} \xrightarrow[n\to\infty]{} 0$$

$$\Longrightarrow \sup_{[0,1]} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

$\Longrightarrow f_n \to f$ likformigt på $[0,1]$

Exempel (3) $\quad f_n(x) = x^n - x^{2n}, \quad [0,1]$

$f_n \to 0$ punktvis i $[0,1]$

$$\sup_{[0,1]} |(x^n - x^{2n}) - 0| = \max_{[0,1]} (x^n - x^{2n})$$

$x = 0;1: \quad x^n - x^{2n} = 0$

$x \in (0,1) \quad (f_n - f)'(x) = nx^{n-1} - 2n x^{2n-1} = 0$

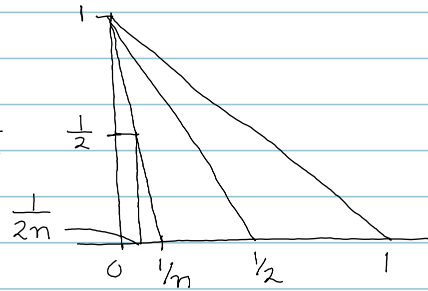
$$nx^{n-1}(1-2x^n) = 0$$

$$x_n = \frac{1}{\sqrt[n]{2}}$$

$$\left.(x^n - x^{2n})\right|_{x=\frac{1}{\sqrt[n]{2}}} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \not\to 0$$

$\Longrightarrow x^n - x^{2n} \to 0$ punktvis i $[0,1]$, men ej likformig

Tillbaka till exempel 1



$$\sup_{[0,1]} |f_n(x) - f(x)| \geq \frac{1}{2}$$

$$\Rightarrow \not\to 0$$

$$f_n\left(\frac{1}{2n}\right) = \frac{1}{2} \qquad f\left(\frac{1}{2n}\right) = 0$$

Likformig konvergens $\Longrightarrow$ punktvis konvergens

$$|f_n(x) - f(x)| \leq \sup_{[a,b]} |f_n(x) - f(x)| < \varepsilon$$

SATS: $\left.\begin{array}{l} f_n : [a,b] \to \mathbb{R} \\ f_n \text{ kontinuerliga i } [a,b] \end{array}\right\} f_n \in C'[a,b]$

$f_n \to f$ likformigt i $[a,b]$

$\Longrightarrow$ f kontinuerlig i $[a,b]$

BEVIS: Tag $\varepsilon > 0$, godtyckligt

$x_0 \in [a,b]$, godtyckligt

$$|f(x) - f(x_0)| \overset{?}{<} \varepsilon \quad \text{för} \quad |x - x_0| < \text{ngt } \delta$$

$|f(x) - f_n(x)| < \varepsilon$ OK pga konvergensen

$|f_n(x) - f_n(x_0)| < \varepsilon$ OK ty f_n kontinuerlig

$|f(x_0) - f_n(x_0)| < \varepsilon$ OK pga konvergensen

Ordentligt:

$$|f(x) - f(x_0)| = |f(x) - f_M(x) + f_M(x) - f_M(x_0) +$$

$$+ f_M(x_0) - f(x_0)| \leq$$

$$\leq |f(x) - f_M(x)| + |f_M(x) - f_M(x_0)| + |f_M(x_0) - f(x_0)|$$

$f_n \longrightarrow f$ likformigt $\Longrightarrow \exists N_\varepsilon : \forall n > N_\varepsilon$

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in [a,b]$$

Välj och fixera $M > N_\varepsilon$

f_M är en fix funktion

kontinuerlig

$\Longrightarrow \exists \delta_{\varepsilon, M} > 0 : \forall x \in [a,b] : |x - x_0| < \delta_\varepsilon$

fixt

$$|f_M(x) - f_M(x_0)| < \varepsilon$$

$\Longrightarrow \forall x \in [a,b] : |x - x_0| < \delta_\varepsilon$

$$|f(x) - f(x_0)| \leq |f(x) - f_M(x)| +$$

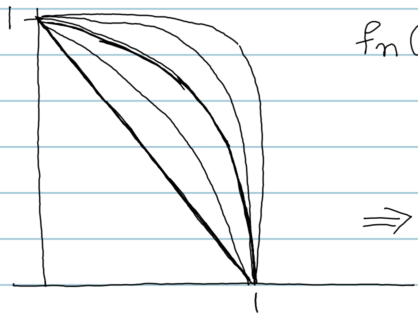
$$+ |f_M(x) - f_M(x_0)| + |f_M(x_0) - f(x_0)|$$

$$< 3\varepsilon$$

$\Rightarrow$ f kontinuerlig i x_0

$\Rightarrow$ f kontinuerlig i $[a,b]$

Exempel (4)

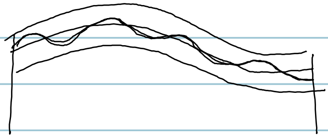


$$f_n(x) = 1 - x^n \longrightarrow \begin{cases} 1 & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

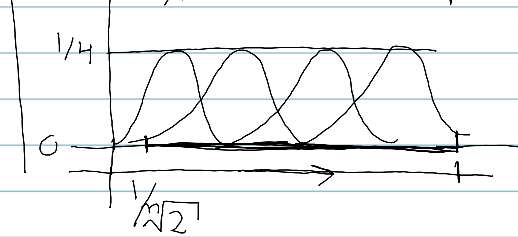
ej kont.

$\Rightarrow f_n \to f$ pktvis

ej likformigt



$x^n - x^{2n} \to 0$ pktvis



SATS: $\{f_n\}_{n=1}^{\infty}$, $f_n : [a,b] \to \mathbb{R}$

$f_n \in C^1[a,b]$

$f_n \to f$ likformigt i $[a,b]$

$$\Rightarrow \int_a^b f_n(x)\,dx \xrightarrow[n\to\infty]{} \int_a^b f(x)\,dx$$

talföljd

$$\int_a^x f_n(t)\,dt \xrightarrow[n\to\infty]{} \int_a^x f(t)\,dt \quad \text{likformigt i } [a,b]$$

funktionsföljd

BEVIS $\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| =$

$$= \left| \int_a^b (f_n(x) - f(x))\,dx \right| \leq \int_a^b |f_n(x) - f(x)|\,dx \leq$$

$$\leq \int_a^b \sup_{t\in[a,b]} |f_n(\overset{t}{\cancel{x}}) - f(\overset{t}{\cancel{x}})|\,dx =$$

$$= \sup_{[a,b]} |f_n(t) - f(t)| \underbrace{\int_a^b 1\,dx}_{=b-a} =$$

$$= \underbrace{\sup_{[a,b]} |f_n(t) - f(t)|}_{<\varepsilon\ \forall n > N_\varepsilon} \cdot (b-a) < \varepsilon(b-a)$$

$< \varepsilon\ \forall n > N_\varepsilon$, där $\varepsilon > 0$ godtyckligt valt

$$\Rightarrow \int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$$

OBS! (1) $f_n \in C[a,b] \Rightarrow \begin{cases} \exists \int_a f_n(x)\,dx \\ f \in C[a,b] \Rightarrow \exists \int_a^b f \end{cases}$

(2) $\ldots < \varepsilon(b-a)$ det måste vara ett ändligt intervall

(bevis forts) $\int_a^x f_n(t)\,dt \longrightarrow \int_a^x f(t)\,dt$ likformigt i $[a,b]$

$$\sup_{x\in[a,b]} \left| \int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt \right| < \ldots <$$

$$< \sup_{x\in[a,b]} \varepsilon(x-a) = \varepsilon(b-a)$$

$\Longrightarrow$ påståendet följer

Dominerad konvergens: $f_n : I \to \mathbb{R}$

$f_n \longrightarrow f$ punktvis i I

$\int_I f_n$ och $\int_I f$ finns

$\exists\, g : I \to \mathbb{R}$ s.a. $\exists \int_I g(x)\,dx$

$|f_n(x)| \leq g(x) \quad \forall n\,(> n_0),\ \forall x \in I$

$\Longrightarrow \int_I f_n(x)\,dx \longrightarrow \int_I f(x)\,dx$

SATS: $f_n \in C^1[a,b]$

$f_n \longrightarrow f$ punktvis i $[a,b]$

$f_n' \longrightarrow g$ likformigt i $[a,b]$

$\Longrightarrow f_n \longrightarrow f$ likformigt

och $\exists\, f' = g$

Visa!

Exempel: $f_n(x) = \frac{1}{n}\sin nx \xrightarrow[n\to\infty]{} 0$ likformigt $x \in [0,\pi]$

$f_n'(x) = \cos x \quad \nexists \lim_{n\to\infty}$

BEVIS $f_n' \longrightarrow g$ likformigt

$$\underbrace{\int_{x_0}^{x} f_n'(t)\,dt}_{=} \longrightarrow \int_{x_0}^{x} g(t)\,dt$$ likformigt (enl. föregående sats)

$f_n' \in C[a,b] \Rightarrow g \in [a,b] \Rightarrow \int_{x_0}^{x} g(t)\,dt$

$$\Longrightarrow f_n(x) - f_n(x_0) \longrightarrow \int_{x_0}^{x} g(t)\,dt$$ likformigt

$$\Longrightarrow f_n(x) \xrightarrow[n\to\infty]{} f(x_0) + \int_{x_0}^{x} g(t)\,dt$$ likformigt

$f_n \longrightarrow f$ pktvis

$$\Longrightarrow f(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt; \quad f_n \to f \text{ likf.}$$

$$\Longrightarrow \exists\, f'(x) = g(x)$$ (enl. analysens huvudsats)

(Räcker att $\exists\, x_0 \in [a,b]$: $\{f_n(x_0)\}$ konvergent)

FUNKTIONSSERIER

$$\sum_{k=1}^{\infty} f_k(x) = \underbrace{f_1(x) + \ldots + f_n(x)}_{= S_n(x)} + \ldots$$

$$S_n(x) = f_1(x) + \ldots + f_n(x)$$

def. Serien konvergerar punktvis i $[a,b]$ om

$\exists\ S(x)$, def i $[a,b]$, s.a. $S_n(x) \xrightarrow[\text{pktvis}]{} S(x)$

Serien konvergerar likformigt om

$S_n(x) \longrightarrow S(x)$ likformigt

Följd $f_1, \ldots, f_n, \ldots$

$$g_n = f_n - f_{n-1} \qquad \sum_{k=1}^{n} g_k = (f_n - \cancel{f_{n-1}}) + (\cancel{f_{n-1}} - \cancel{f_{n-2}}) +$$

$$+ \ldots + \cancel{f_1} = f_n$$

Weierstaß majorantsats

$$\sum_{n=1}^{\infty} f_n(x), \quad f_n(x) : \underbrace{[a,b]}_{I} \longrightarrow \mathbb{R}$$

$$|f_n(x)| \le \underset{\in \mathbb{R}}{a_n} \qquad \forall x \in I, \quad \forall n\ (> n_0)$$

$\sum_{n=1}^{\infty} a_n$ konvergent serie

$\longrightarrow$

$\Rightarrow$ serien $\sum_{n=1}^{\infty} f_n(x)$ likformigt konvergent

BEVIS ? punktvis konvergens

Tag $x_0 \in I$, godtyckligt

$(0 \leq) \left| f_n(x_0) \right| \leq a_n$

$\Rightarrow \sum_{n=1}^{\infty} \left| f_n(x_0) \right|$ konvergent enl. jämförelsekriteriet

$\Rightarrow \sum_{n=1}^{\infty} f_n(x_0)$ absolutkonvergent $\Rightarrow$ konvergent

x_0 godtyckligt valt i I $\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ pktvis konvergent i I

$S(x) = \sum_{n=1}^{\infty} f_n(x)$ (pktvis)

$S_n(x) = f_1(x) + \dots + f_n(x)$

$\sigma_n = a_1 + \dots + a_n \xrightarrow[n \to \infty]{} \sigma$

$\underbrace{\left| \sigma_n - \sigma \right|}_{= \sum_{k=n+1}^{\infty} a_k} < \varepsilon \quad \forall n > N_\varepsilon$

Triangelolikheten för serier: $\left| \sum_{1}^{\infty} f_n \right| \overset{?}{\leq} \underbrace{\sum_{n=1}^{\infty} \left| f_n \right|}_{\text{konv.}}$

$$\left|\sum_{k=1}^{n} f_k\right| \le \underbrace{\sum_{k=1}^{n} |f_k|}_{\le \sum_{k=1}^{\infty} |f_k|}$$

("vanliga" triangelolikheten)

$$\Rightarrow \left|\sum_{k=1}^{n} f_k\right| \le \underbrace{\sum_{k=1}^{\infty} |f_k|} \quad \forall n\,(>n_0) \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\left(\sum^{n}\right) \le \sum_{k=1}^{\infty} |f_k|$$

$$|S_n(x) - S(x)| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \le \sum_{k=n+1}^{\infty} |f_k(x)| \le$$

$$\le \sum_{k=n+1}^{\infty} a_k < \varepsilon$$

$$\forall n > N_\varepsilon$$

$\Rightarrow$ $S_n(x) \to S(x)$ likformigt