0. Introduktion

Övning 0.1 (Sid. 1)

Lösning

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ = de naturliga talen.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ = de hela talen.

$\mathbb{Q} = \{\frac{a}{b} : a, b \text{ heltal}, b \neq 0\}$ = de rationella talen.

Resten är irrationella; rationella och irrationella tillsammans utgör de reella talen.

$\frac{6}{2} = 3$ är naturligt (tal); 0 är naturligt; 3 är naturligt; -3 är heltal; $\frac{3}{0{,}1} = 30$ är naturligt; $\frac{3}{5}$ är rationellt, $\frac{5}{3}$ är rationellt; $\sqrt{2}$ är reellt, $-\frac{0{,}3}{0{,}02} = \frac{30}{-2} = -15$ är heltal; $\frac{0}{5}$ är naturligt; π är slutligen reellt.

Anm. Alla naturliga tal är heltal och alla heltal är rationella. Dessa är i sin tur reella.

Resultat: a) Naturliga är $\frac{6}{2}, 0, 3, \frac{3}{0{,}1}$ och $\frac{0}{5}$;

b) Heltal är $\frac{6}{2}, 0, 3, -3, \frac{3}{0{,}1}, -\frac{0{,}3}{0{,}02}$ och $\frac{0}{5}$;

c) Rationella är $\frac{6}{2}, 0, 3, -3, \frac{3}{0{,}1}, \frac{3}{5}, \frac{5}{3}, -\frac{0{,}3}{0{,}02}$ och $\frac{0}{5}$ och …

Övning 0.2 (Sid. 1)

Lösning

Rationella och irrationella är "disjunkta" eller med ett annat ord oförenliga. Sammansättning: (här subtraktion) av ett irrationellt och ett rationellt är faktiskt irrationellt. Båda de givna talen är således irrationella.

Svar: Nej.

Övning 0.3 (Sid. 1)

Lösning

a) $(x+3)(x-3) - (x+3)^2 = x^2 - 3^2 - (x^2 + 6x + 9) = x^2 - 9 - x^2 - 6x - 9 =$
$= \underline{-6x - 18}.$

b) $(x+3)(x-3) - (x-3)^2 = x^2 - 9 - (x^2 - 6x + 9) = x^2 - 9 - x^2 + 6x - 9 =$
$= \underline{6x - 18}.$

c) $(3x+5)^2 - (3x-5)^2 = \underbrace{(3x+5-3x+5)(3x+5+3x-5)}_{\text{enl. konjugatregeln}} = 10 \cdot 6x = \underline{60x}.$

Övning 0.4 (Sid. 1)

Lösning

Se nästföljande sida.

De reella talen multipliceras två i taget, s.a.

$(a-b)^3=(a-b)\cdot(a-b)^2=(a-b)\cdot(a^2-2ab+b^2)=a(a^2-2ab+b^2)-$

$-b(a^2-2ab+b^2)=a^3-2a^2b+ab^2-a^2b+2a^2b-b^3=\underline{a^3-3a^2b+}$

$\underline{+3ab^2-b^3}.$

Anm. För reella x,y och z gäller

$x\cdot y=y\cdot x$ (den kommutativa lagen för $\cdot$)

$x(y+z)=xy+xz$ (den distributiva lagen).

$x\cdot y\cdot z=x\cdot(y\cdot z)=(x\cdot y)\cdot z$ (den associativa lagen).

Övning 0.5 (Sid. 1)

Lösning

$a^{32}-b^{32}=(a^{16})^2-(b^{16})^2=(a^{16}-b^{16})(a^{16}+b^{16})$;

$a^{16}-b^{16}=(a^8)^2-(b^8)^2=(a^8-b^8)(a^8+b^8)$;

$a^8-b^8=(a^4)^2-(b^4)^2=(a^4-b^4)(a^4+b^4)$;

$a^4-b^4=(a^2)^2-(b^2)^2=(a^2-b^2)(a^2+b^2)$;

$a^2-b^2=(a-b)(a+b)$;

$\therefore\ a^{32}-b^{32}=(a^{16}+b^{16})(a^{16}-b^{16})=$

$=(a^{16}+b^{16})(a^8+b^2)(a^2-b^8)=$

$=(a^{16}+b^{16})(a^8+b^8)(a^4+b^4)(a^4-b^4)=$

$=(a^{16}+b^{16})(a^8+b^8)(a^4+b^4)(a^2+b^2)(a^2-b^2)=$

$=(a^{16}+b^{16})(a^8+b^8)(a^4+b^4)(a^2+b^2)(a+b)(a-b)\Leftrightarrow$

$\Leftrightarrow(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})=\dfrac{a^{32}-b^{32}}{a-b}$, VSV.

Övning 0.6 (Sid. 1)

Lösning

Lösning

$x^2-7x+xy-7y=x(x-7)+y(x-7)=\underline{(x+y)(x-7)}.$

Övning 0.7 (Sid. 1)

Lösning

$a^6-a^4+a^2-1=a^4(a^2-1)+1\cdot(a^2-1)=(a^4+1)(a^2-1)=\underline{(a^4+1)(a+1)(a-1)}.$

Övning 0.8 (Sid. 1)

Lösning

$x^2y+2x^2y-y-2=x^2(y+2)-(y+2)=(x^2-1)(y+2)=\underline{(x-1)(x+1)(y+2)}.$

Övning 0.9 (Sid. 1)

Lösning

$7x^5+7xy^4-14x^3y^2=7x(x^4+y^4-2x^2y^2)=7x(x^2-y^2)=$

$= 7x((x-y)(x+y))^2 = \underline{7x(x-y)^2(x+y)^2}.$

Övning 0.10 (Sid. 1)

Lösning

$a^2-(b+c)^2 = (a-(b+c))(a+(b+c)) = \underline{(a-b-c)(a+b+c)}.$

Övning 0.11 (Sid. 1)

Lösning

$(x^2+y^2)^2-(2xy)^2 = (x^2+y^2-2xy)(x^2+y^2+2xy) = \underline{(x-y)^2(x+y)^2}.$

Övning 0.12 (Sid. 1)

Lösning

$(x^2+y^2-z^2)^2-4x^2y^2 = (x^2+y^2-z^2)^2-(2xy)^2 =$

$= (x^2+y^2+2xy-z^2)(x^2+y^2-2xy-z^2) =$

$= ((x+y)^2-z^2)\cdot((x-y)^2-z^2) =$

$= \underline{(x+y+z)(x+y-z)(x-y+z)(x-y-z)}.$

Övning 0.13 (Sid. 1)

Lösning

$\frac{1}{60}+\frac{1}{108}+\frac{1}{72} = \frac{1}{12\cdot 5}+\frac{1}{12\cdot 9}+\frac{1}{12\cdot 6} = \frac{1}{12}\cdot\frac{1}{5}+\frac{1}{12}\cdot\frac{1}{9}+\frac{1}{12}\cdot\frac{1}{6} =$

$= \frac{1}{12}\left(\frac{1}{5}+\frac{1}{9}+\frac{1}{6}\right) = (mgn = 90) = \frac{1}{12}\cdot\frac{18+10+15}{90} = \frac{1}{12}\cdot\frac{43}{90} = \underline{\frac{43}{1080}}.$

b) $\frac{3}{4}-\frac{5}{6}+\frac{1}{9} = (mgn = 36) = \frac{27}{36}-\frac{30}{36}+\frac{4}{36} = \frac{27-30+4}{36} = \underline{\frac{1}{36}}.$

c) $\frac{1}{35}-\frac{1}{25}+\frac{1}{63}-\frac{1}{245} = \frac{1}{63}-\left(\frac{1}{25}-\frac{1}{35}+\frac{1}{245}\right) = \frac{1}{63}-\frac{1}{5}\left(\frac{1}{5}-\frac{1}{7}+\frac{1}{49}\right) =$

$= \frac{1}{63}-\frac{1}{5}\left(\frac{1}{5}-\frac{7}{49}+\frac{1}{49}\right) = \frac{1}{63}-\frac{1}{5}\left(\frac{1}{5}-\frac{6}{49}\right) = \frac{1}{63}-\frac{1}{5}\cdot\frac{49-30}{245} =$

$= \frac{1}{63}-\frac{1}{5}\cdot\frac{19}{245} = \frac{1}{63}-\frac{19}{1225} = \frac{1}{7}\left(\frac{1}{9}-\frac{19}{175}\right) = \frac{1}{7}\cdot\frac{175-9\cdot 19}{175\cdot 9} = \underline{\frac{4}{11025}}.$

Övning 0.14 (Sid. 2)

Lösning

$\frac{2}{3x+9}+\frac{x}{x^2-9}-\frac{1}{2x-6} = \frac{2}{3(x+3)}+\frac{x}{(x+3)(x-3)}-\frac{1}{2(x-3)} =$

$= \frac{1}{6}\left(\frac{4}{x+3}+\frac{6x}{(x-3)(x+3)}-\frac{3}{x-3}\right) = (mgn = (x-3)(x+3)) =$

$= \frac{1}{6}\left(\frac{4(x-3)}{(x+3)(x-3)}+\frac{6x}{(x+3)(x-3)}-\frac{3(x+3)}{(x-3)(x+3)}\right) =$

$= \frac{1}{6}\cdot\frac{4x-12+6x-3x-9}{(x-3)(x+3)} = \frac{7x-21}{6(x^2-9)} = \frac{7(x-3)}{6(x^2-9)} = \underline{\frac{7}{6(x+3)}}.$

Övning 0.15 (Sid. 2)

Lösning

$\frac{3x-y}{x^2-2xy+y^2}-\frac{2}{x-y}-\frac{2y}{(x-y)^2} = \frac{3x-y}{(x-y)^2}-\frac{2(x-y)}{(x-y)^2}-\frac{2y}{(x-y)^2} = (x+y) =$

$= \frac{3x-y-2(x-y)-2y}{(x-y)^2} = \frac{3x-y-2x+2y-2y}{(x-y)^2} = \frac{x-y}{(x-y)^2} = \underline{\frac{1}{x-y}}.$

Övning 0.16 (Sid. 2)

Lösning

Se nästa sida.

$$\frac{a}{a^2+4ab+4b^2}+\frac{2b}{a^2-4b^2}=\frac{a}{(a+2b)^2}+\frac{2b}{(a+2b)(a-2b)}=$$
$$=\frac{1}{a+2b}\left(\frac{a}{a+2b}+\frac{2b}{a-2b}\right)=\frac{1}{a+2b}\cdot\left(\frac{a(a-2b)}{(a+2b)(a-2b)}+\frac{2b(a+2b)}{(a+2b)(a-2b)}\right)$$
$$=\frac{1}{a+2b}\cdot\frac{a(a-2b)+2b(a+2b)}{(a+2b)(a-2b)}=\frac{1}{a+2b}\cdot\frac{a^2-2ab+2ab+4b^2}{(a+2b)(a-2b)}=$$
$$=\underline{\frac{a^2+4b^2}{(a+2b)^2(a-2b)}}.$$

Övning 0.17 (Sid. 2)

Lösning

$$\frac{3+\sqrt{5}}{2+\sqrt{5}}=\frac{(3+\sqrt{5})(2-\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})}=\frac{6-3\sqrt{5}+2\sqrt{5}-5}{2^2-5}=\frac{1-\sqrt{5}}{-1}=\underline{\sqrt{5}-1}.$$

Övning 0.18 (Sid. 2)

Lösning

a) $$\frac{1+2\sqrt{2}}{3-\sqrt{2}}=\frac{(1+2\sqrt{2})(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}=\frac{3+\sqrt{2}+6\sqrt{2}+4}{3^2-2}=\frac{7+7\sqrt{2}}{7}=\underline{1+\sqrt{2}}.$$

b) $$\frac{1}{\sqrt{13}+\sqrt{11}}=\frac{\sqrt{13}-\sqrt{11}}{(\sqrt{13}+\sqrt{11})\cdot(\sqrt{13}-\sqrt{11})}=\frac{\sqrt{13}-\sqrt{11}}{13-11}=\underline{\frac{1}{2}(\sqrt{13}-\sqrt{11})}.$$

c) $$\frac{2}{\sqrt{x+1}+\sqrt{x-1}}=\frac{2(\sqrt{x+1}-\sqrt{x-1})}{(\sqrt{x+1}+\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})}=\frac{2(\sqrt{x+1}-\sqrt{x-1})}{(\sqrt{x+1})^2-(\sqrt{x-1})^2}=$$
$$=\frac{2(\sqrt{x+1}-\sqrt{x-1})}{x+1-(x-1)}=\frac{\cancel{2}(\sqrt{x+1}-\sqrt{x-1})}{\cancel{2}}=\underline{\sqrt{x+1}-\sqrt{x-1}}.$$

Övning 0.19 (Sid. 2)

Lösning

$$\frac{\frac{3}{5a}-\frac{a}{15}}{\frac{1}{a}-\frac{1}{3}}=\frac{15a\left(\frac{3}{5a}-\frac{a}{15}\right)}{15a\left(\frac{1}{a}-\frac{1}{3}\right)}=\frac{9-a^2}{15-5a}=\frac{\cancel{(3-a)}(3+a)}{5\cancel{(3-a)}}=\underline{\frac{a+3}{5}}.$$

Övning 0.20 (Sid. 2)

Lösning

$$\frac{\frac{x}{y}-\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}-2}=\frac{xy\left(\frac{x}{y}-\frac{y}{x}\right)}{xy\left(\frac{x}{y}+\frac{y}{x}-2\right)}=\frac{x^2-y^2}{x^2+y^2-2xy}=\frac{(x-y)(x+y)}{(x-y)^2}=\underline{\frac{x+y}{x-y}}.$$

Övning 0.21 (Sid. 2)

Lösning

a) $$\frac{16x^4}{81}-y^4=\left(\frac{2x}{3}\right)^4-y^4=\left(\left(\frac{2x}{3}\right)^2\right)^2-(y^2)^2=\left(\left(\frac{2x}{3}\right)^2+y^2\right)\left(\left(\frac{2x}{3}\right)^2-y^2\right)=$$
$$=\left(\frac{4x^2}{9}+y^2\right)\cdot\left(\frac{2x}{3}+y\right)\left(\frac{2x}{3}-y\right)=\frac{(4x^2+9y^2)(2x-3y)}{27}\cdot\left(\frac{2x}{3}+y\right)$$
$$\Leftrightarrow\left(\frac{16x^4}{81}-y^4\right)\Big/\left(\frac{2x}{3}+y\right)=\underline{\frac{1}{27}(4x^2+9y^2)(2x-3y)}.$$

b) $$\frac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x+1}}=\frac{(x^2-1)\left(\frac{1}{x+1}+\frac{1}{x-1}\right)}{(x^2-1)\left(\frac{1}{x-1}-\frac{1}{x+1}\right)}=\frac{x-1+x+1}{x+1-(x-1)}=\frac{2x}{2}=\underline{x}.$$

Övning 0.22 (Sid. 3)

Lösning

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}=\frac{1}{1+\frac{1}{1+1/2}}=\frac{1}{1+\frac{1}{3/2}}=\frac{1}{1+2/3}=\frac{1}{5/3}=\underline{\frac{3}{5}}.$$

Övning 0.23 (Sid. 3)

Lösning

Se nästföljande sida.

a) Faktorsatsen tillämpas direkt och vi får

$(x-1)(x-2)(x-3)=0 \Leftrightarrow \underline{x=1} \vee \underline{x=2} \vee \underline{x=3}.$

b) $x(x^2-4)=x(x-2)(x+2)=0 \Leftrightarrow \underline{x=0} \vee \underline{x=2} \vee \underline{x=-2}.$

Övning 0.30 (Sid. 3)

Lösning

a) $x^2+10x+24=0 \Leftrightarrow (x^2+10x+25)-1=(x+5)^2-1=(x+5)^2-1^2=$

$=(x+5+1)(x+5-1)=(x+6)(x+4)=0 \Leftrightarrow x+6=0 \vee x+4=0$

$\Leftrightarrow \underline{x=-6} \vee \underline{x=-4}.$

b) $x^2+10x+25=0 \Leftrightarrow (x+5)^2=0 \Leftrightarrow \underline{x=x_1=x_2=-5}.$

Övning 0.31 (Sid. 3)

Lösning

a) $x^3+10x^2+24x=x(x^2+10x+24)=x(x+6)(x+4)=0 \Leftrightarrow$

$\Leftrightarrow x=0 \vee x+6=0 \vee x+4=0 \Leftrightarrow \underline{x=0} \vee \underline{x=-6} \vee \underline{x=-4}.$

b) $x^4+10x^3+25x^2=x^2(x^2+10x+25)=x^2(x+5)^2=0 \Leftrightarrow$

$\Leftrightarrow x^2=0 \vee (x+5)^2=0 \Leftrightarrow \underline{x=x_1=x_2=0} \vee \underline{x=x_3=x_4=-5}.$

Övning 0.32 (Sid. 3)

Lösning

Se nästföljande sida.

$\underline{\sqrt{x+2}=x}$; $VL \geq 0 \Rightarrow x \geq 0.$ (*) (Villkor på x.)

$x+2=x^2 \Leftrightarrow x^2-x-2=0 \overset{(*)}{\Leftrightarrow} x=\frac{1}{2}+\sqrt{\frac{1}{4}+2}=\frac{1}{2}+\frac{3}{2}=\underline{2}.$

Övning 0.33 (Sid. 3)

Lösning

$\underline{\sqrt{x+2}=-x}$; $VL \geq 0 \Rightarrow -x \geq 0 \Leftrightarrow \underline{x \leq 0}$ (**)

$x+2=x^2 \Leftrightarrow x^2-x-2=0 \overset{(**)}{\Leftrightarrow} x=\frac{1}{2}-\frac{3}{2}=\underline{-1}.$

Övning 0.34 (Sid. 4)

Lösning

$\sqrt{3x+2}=\sqrt{2x+1} \Leftrightarrow 3x+2=2x+1 \Leftrightarrow x=-1.$ (Ingen rot)

Övning 0.35 (Sid. 4)

Lösning

$(3+\sqrt{x})(3-\sqrt{x})=8\sqrt{x} \Leftrightarrow 3^2-(\sqrt{x})^2=8\sqrt{x} \Leftrightarrow (\sqrt{x})^2+8\sqrt{x}-9=0$

$\Leftrightarrow \sqrt{x}=-4+\sqrt{16+9}=-4+5=1 \Leftrightarrow \underline{x=1}$

Övning 0.36 (Sid. 4)

Lösning

Samtliga är sanna. $(x \leq y \Leftrightarrow x<y \vee x=y).$

a) $\sqrt{12}-\sqrt{3}=\sqrt{4\cdot 3}-\sqrt{3}=\sqrt{4}\sqrt{3}-\sqrt{3}=2\sqrt{3}-\sqrt{3}=\underline{\sqrt{3}}.$

b) $\dfrac{\sqrt{42}}{\sqrt{6}}=\sqrt{\dfrac{42}{6}}=\underline{\sqrt{7}}.$

c) $\sqrt{3}\cdot\sqrt{12}=\sqrt{3\cdot 12}=\sqrt{36}=\underline{6}.$

d) $\sqrt{3^2+4^2}-3-4=\sqrt{25}-7=5-7=\underline{-2}.$

e) $\sqrt{5^2+12^2}=\sqrt{169}=\underline{13}.$

Övning 0.24 (Sid. 3)

Lösning

$$\frac{1}{x-1}-\frac{1}{x-2}=\frac{1}{x-3}-\frac{1}{x-4} \Leftrightarrow \frac{x-2-(x-1)}{(x-1)(x-2)}=\frac{x-4-(x-3)}{(x-3)(x-4)} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{x^2-3x+2}=\frac{1}{x^2-7x+12} \Leftrightarrow x^2-3x+2=x^2-7x+12 \Leftrightarrow$$

$$\Leftrightarrow 4x=10 \Leftrightarrow \underline{x=\frac{5}{2}}.$$

Övning 0.25 (Sid. 3)

Lösning

$$\frac{1}{x+2}-\frac{x+2}{x-2}=\frac{x^2}{4-x^2} \Leftrightarrow \frac{x-2}{x^2-4}-\frac{(x+2)^2}{x^2-4}=\frac{-x^2}{x^2-4} \wedge \underline{x\neq\pm 2} \Leftrightarrow$$

$$\Leftrightarrow \frac{x-2-(x+2)^2}{x^2-4}=-\frac{x^2}{x^2-4} \Leftrightarrow x-2-(x+2)^2=-x^2 \Leftrightarrow -3x-6=0$$

$$\Leftrightarrow 3x=-6 \Leftrightarrow x=-2 \text{ (ingen rot).}$$

Man räknar och räknar och sen … ingenting.

$$x\neq\pm 2 \Rightarrow (x^2-4)\cdot\left(\frac{1}{x+2}-\frac{x+2}{x-2}+\frac{x^2}{x^2-4}\right)=x-2+x^2-(x+2)^2\neq 0$$

Övning 0.26 (Sid. 3)

Lösning

$$x^2+ax=(x+b)^2+c \Leftrightarrow x^2+ax=x^2+2bx+b^2+c \Leftrightarrow$$

$$\Leftrightarrow ax=2bx+b^2+c \Leftrightarrow (2b-a)x+b^2+c=0 \Leftrightarrow \begin{cases} 2b-a=0 \\ c=-b^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \underline{b=\tfrac{1}{2}a} \wedge \underline{c=-\tfrac{1}{4}a^2}.$$ ($\wedge$ utläses "och").

Övning 0.27 (Sid. 3)

Lösning

a) $x^2+6x+7=(x^2+6x)+7=(x^2+2\cdot x\cdot 3+3^2-3^2)+7=$

$=(x^2+6x+9)-9+7=\underline{(x+3)^2-2}.$

b) $x^2-7x+13=(x^2-7x)+13=(x^2-2\cdot x\cdot\frac{7}{2}+\frac{49}{4}-\frac{49}{4})+13=$

$=(x^2-2\cdot x\cdot\frac{7}{2}+\frac{49}{4})-\frac{49}{4}+13=\underline{(x-\frac{7}{2})^2+\frac{3}{4}}.$

Övning 0.28 (Sid. 3)

Lösning

$$x^2+5x=x^2+2\cdot x\cdot\frac{5}{2}=x^2+2\cdot x\cdot\frac{5}{2}+\frac{25}{4}-\frac{25}{4}=\underline{(x+\frac{5}{2})^2-\frac{25}{4}}.$$

Övning 0.29 (Sid. 3)

Lösning

Se nästföljande sida.

Övning 0.37 (Sid. 4)

Lösning

$$\left.\begin{aligned} &\frac{2}{0{,}02}=\frac{100\cdot 2}{100\cdot 0{,}02}=\frac{200}{2}=100\\ &\frac{31}{0{,}2}=\frac{10\cdot 31}{10\cdot 0{,}2}=\frac{310}{2}=155\\ &\frac{0{,}00009}{0{,}000006}=\frac{0{,}000001\cdot 90}{0{,}000001\cdot 6}=\frac{90}{6}=15 \end{aligned}\right\} \Rightarrow \frac{0{,}00009}{0{,}000006}<\frac{2}{0{,}02}<\frac{31}{0{,}2}.$$

Övning 0.38 (Sid. 4)

Lösning

$$\frac{3x+1}{x+2}<2 \overset{!}{\Leftrightarrow} \begin{cases} 3x+1<2(x+2) \wedge x+2>0 \\ 3x+1>2(x+2) \wedge x+2<0 \end{cases} \Leftrightarrow \begin{cases} x<3 \wedge x>-2 \\ x>3 \wedge x<-2 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \underline{-2<x<3}.$

Anm. När man multiplicerar en olikhet med ett negativt tal kastar man om olikhetstecknet.

Övning 0.39 (Sid. 4)

Lösning

a) $\frac{x^2+1}{x}<x \Leftrightarrow x+\frac{1}{x}<x \Leftrightarrow \frac{1}{x}<0 \Leftrightarrow \underline{x<0}.$

b) $\frac{2x^2}{x+2}<x-2 \Leftrightarrow \frac{2x^2}{x+2}-(x-2)<0 \Leftrightarrow \frac{2x^2-(x-2)(x+2)}{x+2}<0 \Leftrightarrow$

$\Leftrightarrow \frac{2x^2-(x^2-4)}{x+2}<0 \Leftrightarrow \frac{x^2+4}{x+2}<0 \overset{!}{\Leftrightarrow} x+2<0 \Leftrightarrow \underline{x<-2}.$

Anm. I $\overset{!}{\Leftrightarrow}$ underförstås $x^2+2^2>0$

c) $\frac{x^2+2}{x^2+1}=\frac{x^2+1+1}{x^2+1}=1+\frac{1}{x^2+1}>1 \Leftrightarrow \frac{1}{x^2+1}>0$; detta uppfylls av alla reella x.

Övning 0.40 (Sid. 4)

Lösning

a) $x^2<4 \Leftrightarrow x^2-2^2=(x+2)(x-2)<0$; $f(x)=(x+2)(x-2)$;

x		-2		2	
sgn$(x+2)$	$-$	0	$+$	$+$	$+$
sgn$(x-2)$	$-$	$-$	$-$	0	$+$
sgn$(f(x))$	$+$	0	$-$	0	$+$

$x^2<4 \Leftrightarrow f(x)<0 \Leftrightarrow \underline{-2<x<2}.$

Anm. sgn$(f(x))$ utläses "signum f av x" eller "tecknet av f av x".

b) Med samma teckentabell får vi $x<-2$ el. $x>2$.

c) $(x+1)^2>(x+5)^2 \Leftrightarrow (x+5)^2-(x+1)^2<0 \Leftrightarrow (x+5+x+1)(x+5-x-1)<0$

$\Leftrightarrow (2x+6)\cdot 4<0 \Leftrightarrow 8(x+3)<0 \Leftrightarrow x+3<0 \Leftrightarrow \underline{x<-3}.$

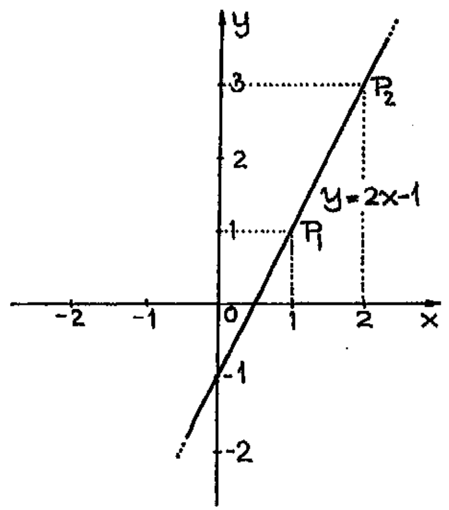
Övning 0.41 (Sid. 4)

Lösning

En rät linje i planet bestäms av 2 punkter.

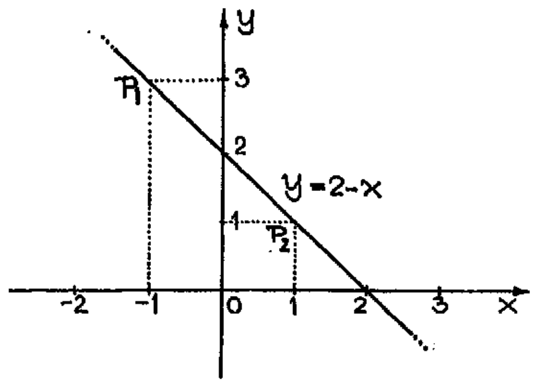
a) $y = 2x - 1$

$x = 1 \Rightarrow y = 1$; $P_1 : (1,1)$. $x = 2 \Rightarrow y = 3$; $P_2 : (2,3)$.



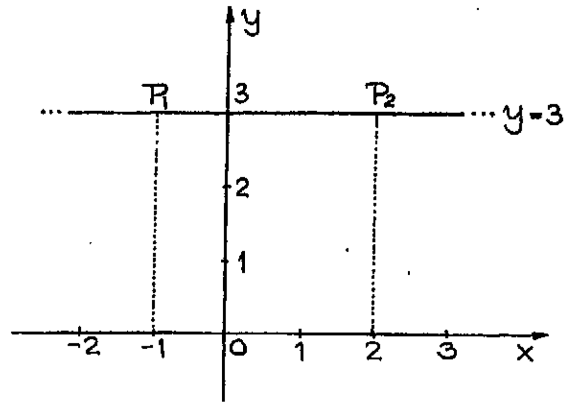
b) $y = 2 - x$

$x = -1 \Rightarrow y = 3$; $P_1 : (-1,3)$. $x = 1 \Rightarrow y = 1$; $P_2 : (1,1)$.



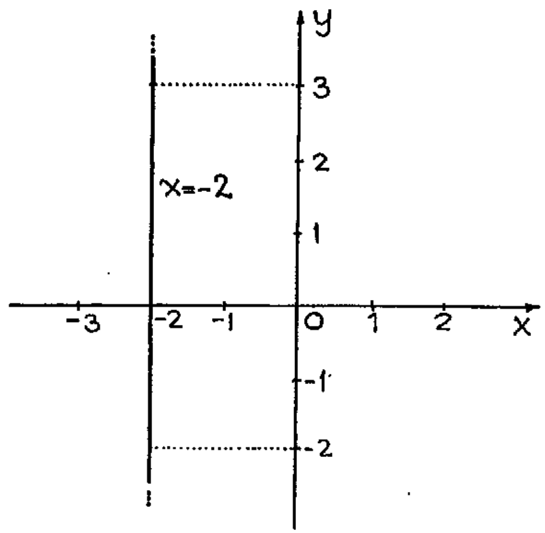
c) $y = 3$

$x = -1 \Rightarrow y = 3$; $P_1 : (-1,3)$. $x = 2 \Rightarrow y = 3$; $P_2 : (2,3)$.



d) $x = -2$

$x = -2 \Rightarrow y = -2$; $P_1 : (-2,-2)$: $x = -2 \Rightarrow y = 3$; $P_2 : (-2,3)$.



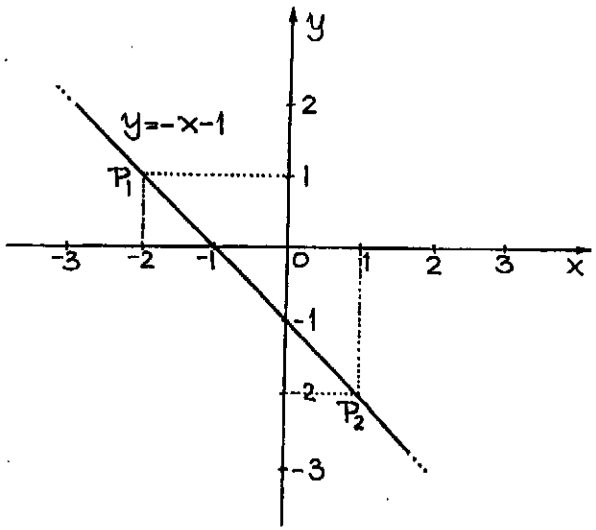
Övning 0.42 (Sid. 4)

Lösning

Se nästföljande sida.

a) $\begin{cases} P_1:(-2,1) \\ P_2:(1,-2) \end{cases} \Rightarrow k=\frac{y_2-y_1}{x_2-x_1}=\frac{-2-1}{1-(-2)}=\frac{-3}{3}=-1;$

$y-y_1=k(x-x_1) \Rightarrow y-1=(-1)(x+2)=-x-2 \Leftrightarrow \underline{y=-x-1.}$



b) $\begin{cases} P_1:(-1,2) \\ P_2:(2,2) \end{cases} \Rightarrow k=\frac{y_2-y_1}{x_2-x_1}=\frac{2-2}{2-(-1)}=0 \Rightarrow y-2=0\cdot(x+1) \Leftrightarrow \underline{y=2.}$

c) Låt oss byta plats mellan x och y.

$\begin{cases} P_1:(1,0) \\ P_2:(1,2) \end{cases} \Rightarrow \kappa=\frac{x_2-x_1}{y_2-y_1}=\frac{1-1}{2-0}=0 \Rightarrow x-1=0\cdot(y-0) \Leftrightarrow \underline{x=1.}$

Anm. I normalform är linjens ekvation given av

$$Ax+By+C=0.$$

Denna kan lösas m.a.p. x eller y, nämligen

$y=\frac{C-Ax}{B}$ eller $x=\frac{C-By}{A}$; vid $B=0$ har vi $x=\frac{C}{A}$ och vid $A=0$ har vi $y=\frac{C}{B}$.

Övning 0.43 (Sid. 4)

Lösning

a) $P_0:(0,2),\ k=-1.$

$y-y_0=k(x-x_0) \Rightarrow y-2=(-1)(x-0) \Leftrightarrow \underline{y=2-x.}$

b) $P_0:(2,1),\ k=3.$

$y-y_0=k\cdot(x-x_0) \Rightarrow y-1=3(x-2)=3x-6 \Leftrightarrow \underline{y=3x-5.}$

c) $P_0:(a,b),\ k=k$

$y-y_0=k\cdot(x-x_0) \Rightarrow y-b=k(x-a)=kx-ka \Leftrightarrow \underline{y=kx+b-ka.}$

d) $P_1:(a,b),\ P_2:(a+1,b+1).$

$k=\frac{y_2-y_1}{x_2-x_1}=\frac{b+1-b}{a+1-a}=1$; $y-b=1\cdot(x-a) \Leftrightarrow \underline{y=x+b-a.}$

Övning 0.44 (Sid. 5)

Lösning

$\frac{a}{x}=\frac{x}{b} \Leftrightarrow x^2=ab \Leftrightarrow x=\sqrt{ab}$; $x \le r \Leftrightarrow \sqrt{ab} < \frac{a+b}{2}.$

Det geometriska medelvärdet av a och b, $G=\sqrt{ab}$, överstiger aldrig deras aritmetiska medelvärdet.

Övning 0.45 (Sid. 5)

Lösning

$$\begin{cases} A = ax^{k-1} \\ B = ax^{k+1} \end{cases} \Rightarrow AB = ax^{k-1} \cdot ax^{k+1} = a^2x^{2k} \Leftrightarrow \sqrt{AB} = ax^k;$$

detta är termen mellan A och B.

Övning 0.46 (Sid. 5)

Lösning

$$\begin{cases} A = a+(k-1)d \\ B = a+(k+1)d \end{cases} \Rightarrow \frac{A+B}{2} = \frac{a+kd-d+a+kd+d}{2} = \frac{2a+2kd}{2} =$$

$= \frac{2(a+kd)}{2} = a+kd$, termen mellan A och B.

Övning 0.47 (Sid. 5)

Lösning

Påstående: $\sqrt{3}$ är irrationellt.

Bevis: Beviset ges i övningsboken.

Övning 0.48 (Sid. 5)

Lösning

Ett godtyckligt heltal n kan skrivas på formen $n = 5k, 5k+1, 5k+2, 5k+3$ eller $5k+4$, för något heltal k. Om n^2 är delbart med 5, så är n delbart med 5, dvs. $n = 5k$. Detta kan visas på följande sätt:

$$\begin{cases} n = 5k \Rightarrow n^2 = 25k^2 = 5(5k) & \text{(delbart med 5).} \\ n = 5k+1 \Rightarrow n^2 = 5(5k^2+2k)+1 & \text{(ej delbart med 5).} \\ n = 5k+2 \Rightarrow n^2 = 5(5k^2+4k)+4 & \text{(ej delbart med 5).} \\ n = 5k+3 \Rightarrow n^2 = 5(5k^2+6k+1)+4 & \text{(ej delbart med 5).} \\ n = 5k+4 \Rightarrow n^2 = 5(5k^2+8k+3)+1 & \text{(ej delbart med 5).} \end{cases}$$

Påstående: $\sqrt{5}$ är irrationellt.

Bevis: Antag motsatsen, dvs. antag att $\sqrt{5}$ är rationellt. Det finns då heltal a och b s.a. $\sqrt{5} = \frac{a}{b}$ och s.a. $\frac{a}{b}$ är förkortat så långt som möjligt. Kvadrering ger $a^2 = 5b^2$. Det innebär, enligt utredningen ovan, att a är en multipel av 5, ty a^2 är det. Det innebär i sin tur att $a = 5c$. Detta i kombination med $a^2 = 5b^2$ ger $25c^2 = 5b^2 \Leftrightarrow 5c^2 = b^2$. Alltså även

b är en multipel av 5. Jag har kommit fram till att 5 är en faktor i a och b, vilket strider mot antagandet att $\frac{a}{b}$ var förkortat så långt som möjligt. Det finns således inga heltal a och b som satisfierar ekvationen $\sqrt{5} = \frac{a}{b}$; $\sqrt{5}$ är inte rationellt, det är irrationellt.

Övning 0.49 (Sid. 5)

Lösning

$\frac{4}{14} = \frac{\cancel{2}\cdot 2}{\cancel{2}\cdot 7} = \frac{2}{7} = \frac{2\cdot 12}{7\cdot 12} = \frac{24}{84} = \frac{2\cdot 24}{2\cdot 84} = \frac{48}{168}.$

Övning 0.50 (Sid. 5)

Lösning

$\frac{\sqrt{216}}{3\sqrt{2}} = \frac{\sqrt{6\cdot 36}}{3\sqrt{2}} = \frac{\sqrt{6}\cdot\sqrt{36}}{3\sqrt{2}} = \frac{\sqrt{6}\cdot 6}{3\sqrt{2}} = \frac{\sqrt{6}\cdot 2}{\sqrt{2}} = \frac{\sqrt{6}\cdot\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = \sqrt{6}\sqrt{2} =$

$= \sqrt{3\cdot 2}\cdot\sqrt{2} = \sqrt{3}\cdot\sqrt{2}\cdot\sqrt{2} = \sqrt{3}\cdot(\sqrt{2})^2 = 2\sqrt{3}$ (rätt).

$\frac{\sqrt{108}}{3} = \frac{\sqrt{3\cdot 36}}{3} = \frac{\sqrt{3}\cdot\sqrt{36}}{3} = \frac{\sqrt{3}\cdot 6}{3} = 2\sqrt{3}$ (rätt).

$\sqrt{12} = \sqrt{3\cdot 4} = \sqrt{3}\cdot\sqrt{4} = \sqrt{3}\cdot 2 = 2\sqrt{3}$ (rätt).

Anm. I grundskolan och gymnasieskolan uppmanas man att förenkla så långt som möjligt...

Övning 0.51 (Sid. 5)

Lösning

$x + \frac{4}{x} = 5 \Leftrightarrow x\left(x + \frac{4}{x}\right) = 5x \Leftrightarrow x^2 + 4 = 5x \Leftrightarrow x^2 - 5x + 4 = 0$

$\Leftrightarrow x = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 4} = \frac{5 \pm 3}{2} \Leftrightarrow x = 4 \vee x = 1.$

Övning 0.52 (Sid. 6)

Lösning

$x + \frac{4}{x} > 5 \Leftrightarrow x + \frac{4}{x} - 5 > 0 \Leftrightarrow \frac{x^2 - 5x + 4}{x} = \frac{(x-1)(x-4)}{x} > 0.$

Jag sätter $VL = f(x) = \frac{(x-1)(x-4)}{x}.$

x		0		1		4	
sgn(x)	−		+	+	+	+	+
sgn$(x-1)$	−		−	0	+	+	+
sgn$(x-4)$	−		−	−	−	0	+
sgn$f(x)$	−		+	0	−	0	+

Resultat: $0 < x < 1 \vee x > 4.$

Övning 0.53 (Sid. 6)

Lösning

Det finns ingen övre begränsning hos de reella talen; antagandet är således felaktigt.

Övning 0.54 (Sid. 6)

forts.

$$\frac{1-x^4}{1-(x^2+1)^2}<1 \Leftrightarrow \frac{1-x^4}{(1+(x^2+1))(1-(x^2+1))}<1 \Leftrightarrow \frac{1-x^4}{-x^2(x^2+2)}<1 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 1-x^4>-x^2(x^2+2) \\ x\neq 0 \end{cases} \Leftrightarrow \begin{cases} 1-x^4>-x^4-2x^2 \\ x\neq 0 \end{cases} \Leftrightarrow \begin{cases} -2x^2<1 \\ x\neq 0 \end{cases} \Leftrightarrow x\neq 0.$$

Svar: Olikheten är giltig för alla $x\neq 0$.

Övning 0.55 (Sid. 6)

Lösning

$$x\neq\pm 1 \Rightarrow \frac{5}{x-1}+\frac{8}{x+1}-\frac{3x+7}{x^2-1}=\frac{5(x+1)}{(x-1)(x+1)}-\frac{8(x-1)}{(x-1)(x+1)}-\frac{3x+7}{x^2-1}=$$

$$=\frac{5(x+1)+8(x-1)-(3x+7)}{x^2-1}=\frac{5x+5+8x-8-3x-7}{x^2-1}=\frac{10x-10}{x^2-1}=$$

$$=\frac{10\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}=\frac{10}{x+1}.$$

Övning 0.56 (Sid. 6)

Lösning

$x=\sqrt{x}+2 \Leftrightarrow \sqrt{x}=x-2\geq 0 \Leftrightarrow x\geq 2$ (villkor på x). (*)

$x-2=\sqrt{x} \overset{(*)}{\Leftrightarrow} (x-2)^2=x \Leftrightarrow x^2-4x+4=x \Leftrightarrow x^2-5x+4=0$

$\overset{(*)}{\Leftrightarrow} x=\frac{5}{2}\pm\frac{3}{2}=4.$

Övning 0.57 (Sid. 6)

Lösning

Jag antar att $x\neq 0, \pm 1$ om räkningarna ska ha någon mening; på gymnasialnivå räknar man formellt.

$$\frac{2}{x-\frac{1}{x}}-\frac{1/x}{1-\frac{1}{x}}=\frac{2x}{x^2-1}-\frac{1}{x-1}=\frac{2x}{x^2-1}-\frac{x+1}{x^2-1}=\frac{2x-(x+1)}{x^2-1}=\frac{x-1}{x^2-1}=\frac{1}{x+1}.$$

Övning 0.58 (Sid. 6)

Lösning

$x\neq 2,3 \Rightarrow \frac{1}{x-2}\neq\frac{1}{x-3} \Rightarrow$ rötter saknas.

Övning 0.59 (Sid. 6)

Lösning

a) $\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{\sqrt{2}}{2}.$

b) $\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}=\frac{3+1+2\sqrt{3}}{3-1}=\frac{4+2\sqrt{3}}{2}=\frac{\cancel{2}(2+\sqrt{3})}{\cancel{2}}=2+\sqrt{3}.$

c) $\frac{1}{\sqrt{a+b}+\sqrt{a}}=\frac{\sqrt{a+b}-\sqrt{a}}{(\sqrt{a+b}+\sqrt{a})(\sqrt{a+b}-\sqrt{a})}=\frac{\sqrt{a+b}-\sqrt{a}}{(\sqrt{a+b})^2-(\sqrt{a})^2}=$

$=\frac{\sqrt{a+b}-\sqrt{a}}{a+b-a}=\frac{\sqrt{a+b}-\sqrt{a}}{b}.$

Övning 0.60 (Sid. 6)

Lösning

a) $\frac{x^2+5x}{x+1}\leq 3 \Leftrightarrow \frac{x^2+5x}{x+1}-3\leq 0 \Leftrightarrow \frac{x^2+5x}{x+1}-\frac{3(x+1)}{x+1}=\frac{x^2+2x-3}{x+1}\leq 0$

$\Leftrightarrow \frac{(x-1)(x+3)}{x+1}\leq 0$; $f(x)=\frac{(x+3)(x-1)}{x+1}$ studeras.

forts.

		-3		-1		1		x
sgn(x+3)	$-$	0	$+$		$+$	$+$	$+$	
sgn(x+1)	$-$	$-$	$-$		$+$	$+$	$+$	
sgn(x−1)	$-$	$-$	$-$		$-$	0	$+$	
sgn(f(x))	$-$	0	$+$		$-$	0	$+$	

Resultat: $x \leq -3 \lor -1 < x \leq 1.$

b) $\frac{2x^2+3x-6}{x^2-2} > 4 \Leftrightarrow \frac{2x^2+3x-6}{x^2-2} - 4 = \frac{2x^2+3x-6-2(x^2-4)}{x^2-2} =$

$= \frac{2x^2+3x-6-4x^2+8}{x^2-2} = \frac{-2x^2+3x+2}{x^2-2} = \frac{-2(x^2-3x/2-1)}{x^2-2} =$

$= \frac{-2(x+1/2)(x-2)}{x^2-2} < 0 \Leftrightarrow \frac{(x+1/2)(x-2)}{(x-\sqrt{2})(x+\sqrt{2})} > 0\ ;\ f(x) = \frac{(x+1/2)(x-2)}{(x-\sqrt{2})(x+\sqrt{2})}$

		$-\sqrt{2}$		$-1/2$		$\sqrt{2}$		2		x
sgn(x+√2)	$-$		$+$	$+$	$+$		$+$	$+$	$+$	
sgn(x−1/2)	$-$		$-$	0	$+$		$+$	$+$	$+$	
sgn(x−√2)	$-$		$-$	$-$	$-$		$+$	$+$	$+$	
sgn(x−2)	$-$		$-$	$-$	$-$		$-$	0	$+$	
sgn(f(x))	$+$		$-$	0	$+$		$-$	0	$+$	

Resultat: $-\sqrt{2} < x < -\frac{1}{2} \lor \sqrt{2} < x < 2.$

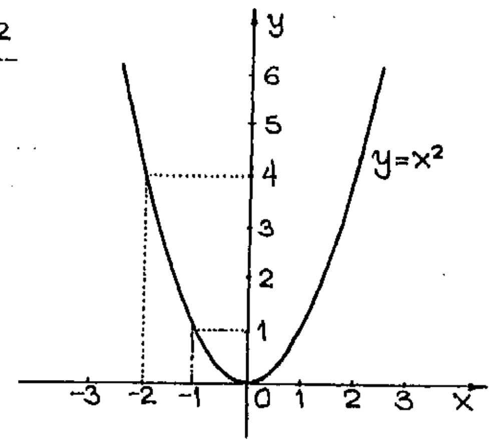
2. # Funktioner

Övning 1.1 (Sid. 16)

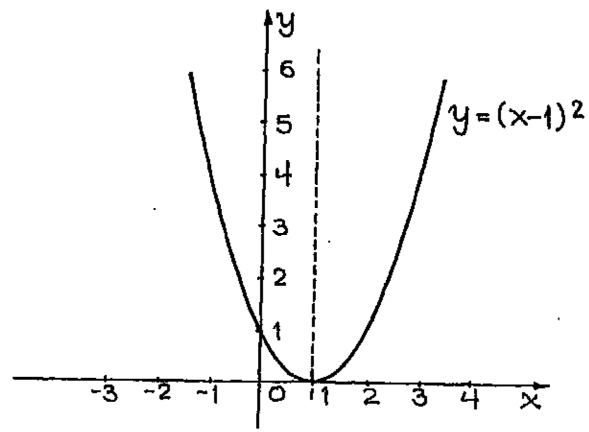
Lösning

$f(x) = x^2$

a) $y = f(x) = x^2$

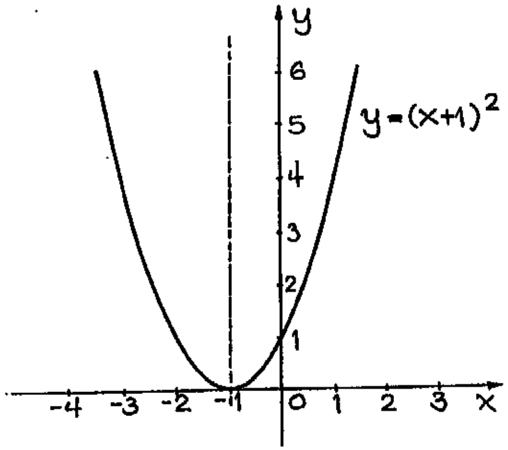


b) $y = f(x-1) = (x-1)^2$



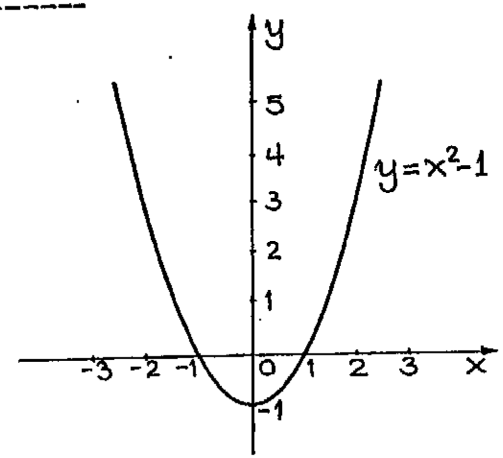
Man får kurvan $y = (x-1)^2$ genom förskjutning av kurvan $y=x^2$ 1 enhet åt höger.

c) $y = f(x+1) = (x+1)^2$



Man får kurvan $y=(x+1)^2$ genom förskjutning av kurvan $y=x^2$ 1 enhet åt vänster.

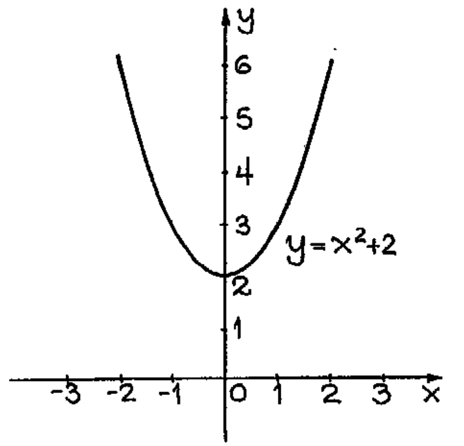
d) $y = f(x)-1 = x^2-1.$



Man får kurvan $y=x^2-1$ genom förskjutning

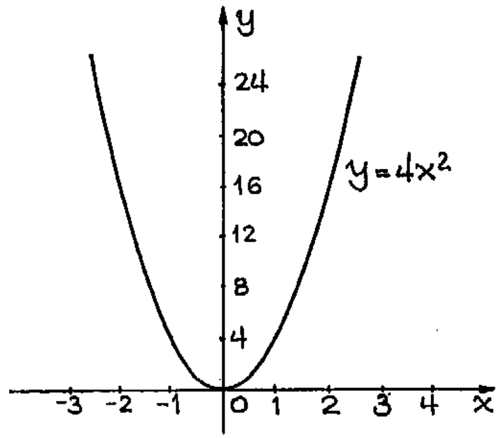
av kurvan $y=x^2$ 1 enhet nedåt.

e) $y = f(x)+2 = x^2+2.$



Kurvan $y=x^2+2$ är kurvan $y=x^2$ förskjuten 2 enheter uppåt.

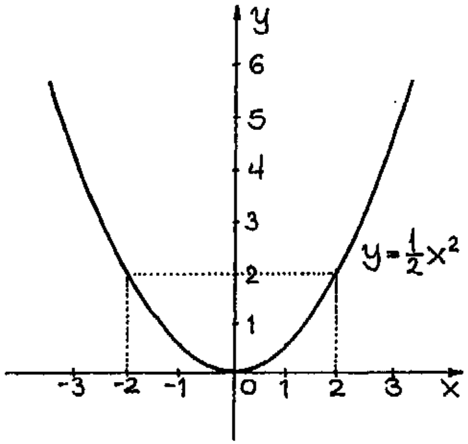
f) $y = f(2x) = 4x^2.$



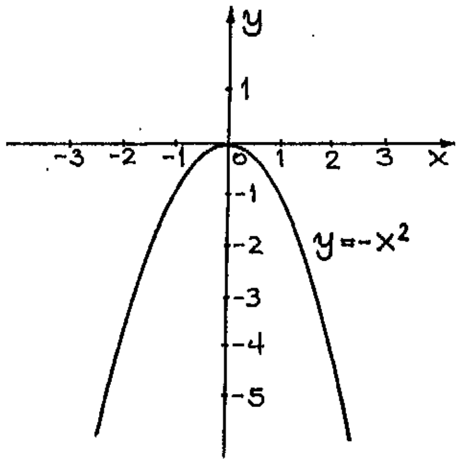
forts.

Man "fryser" kurvan genom lämplig axelgradering. Detta är vanligt vid mätning med oscilloskop.

g) $y = \frac{1}{2}f(x) = \frac{1}{2}x^2$



h) $y = -f(x) = -x^2$

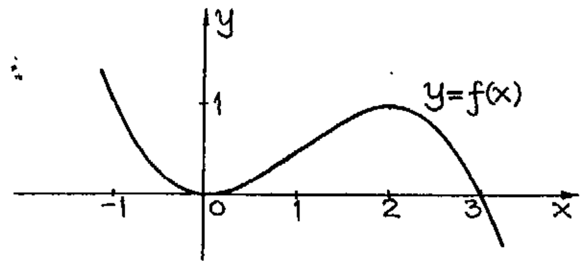


Kurvan $y=-x^2$ är kurvan $y=x^2$ speglad i x-axeln. Symmetrilinjen är densamma.

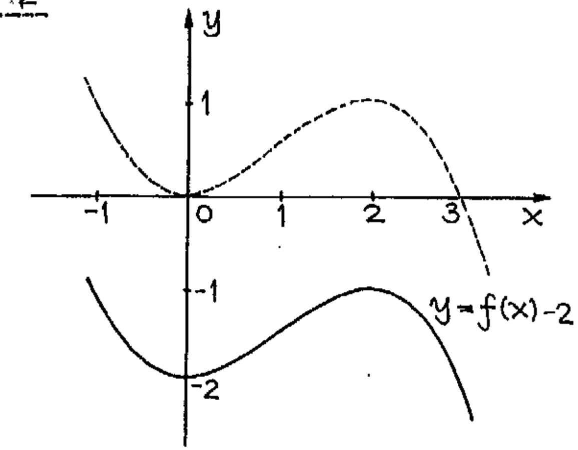
i) $y = f(-x) = (-x)^2 = x^2 = f(x)$. (Se under a)).

Övning 1.2 (Sid. 16)

Lösning



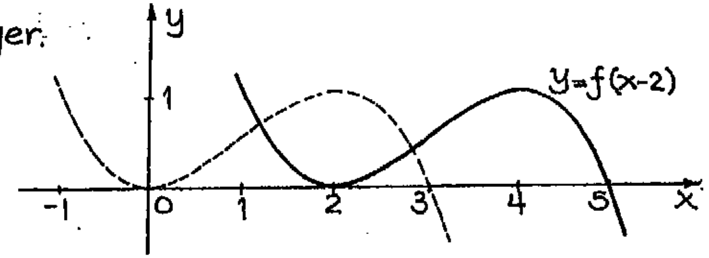
a) $y = f(x) - 2$



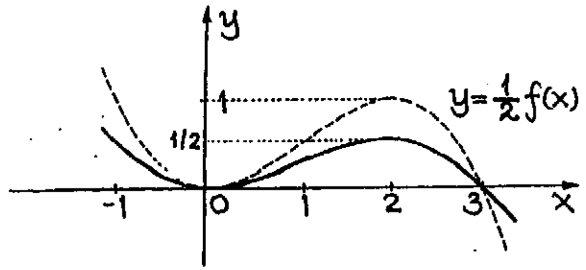
Man får kurvan $y=f(x)-2$ genom förskjutning av kurvan $y=f(x)$ 2 enheter nedåt.

b) $y = f(x-2)$

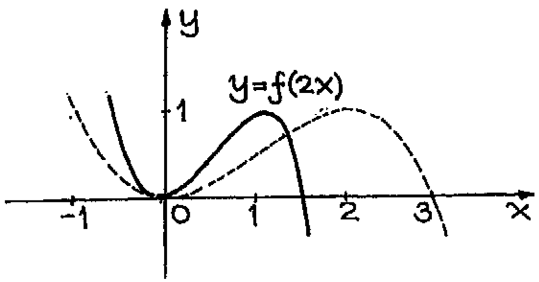
Man får kurvan $y=f(x-2)$ genom translation (stel förflyttning) av kurvan $y=f(x)$ 2 enheter åt höger.



c) $\underline{y=\frac{1}{2}f(x)}$.



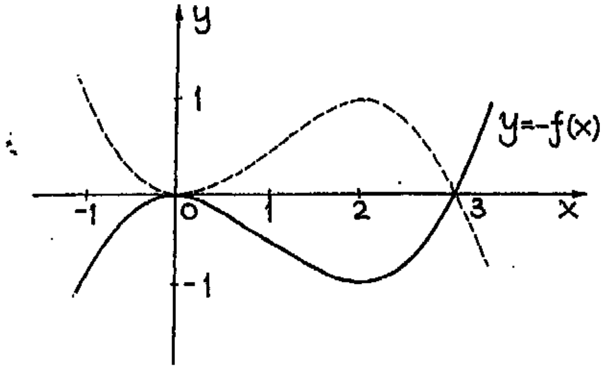
d) $\underline{y=f(2x)}$



Man får kurvan $y=f(2x)$ genom att "pressa samman" kurvan $y=f(x)$ till hälften i sidled. Nollställena är $x=0$ och $x=3/2$.

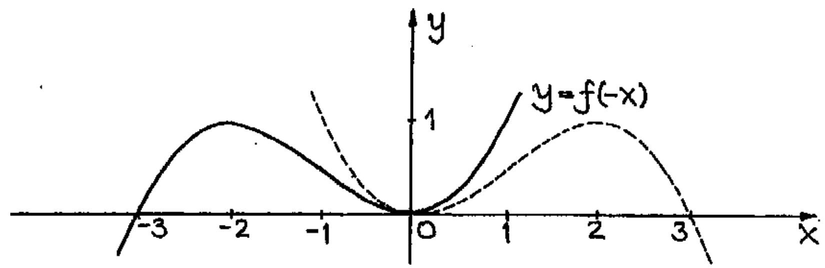
e) $\underline{y=-f(x)}$

Kurvan $y=-f(x)$ är spegelbilden av $y=f(x)$ i x-axeln (se fig. nedan).



f) $\underline{y=f(-x)}$

Man får kurvan $y=f(-x)$ genom spegling av $y=f(x)$ i y-axeln.

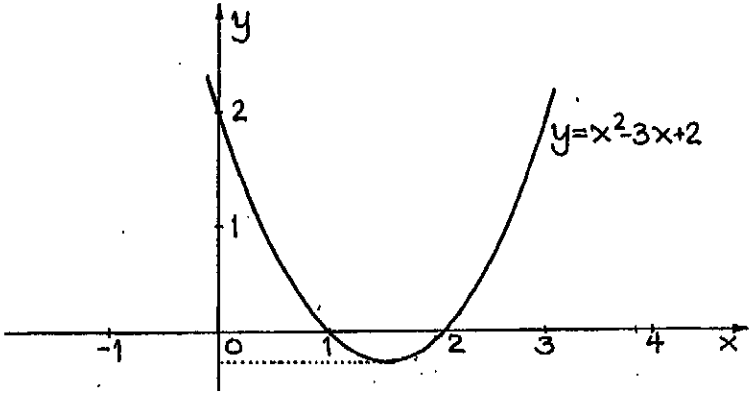


Övning 1.3 (Sid. 16)

Lösning

a) $x^2-3x+2=x^2-2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{1}{4}=\underline{(x-\frac{3}{2})^2-\frac{1}{4}}$.

b) $y=x^2-3x+2=(x-\frac{3}{2})^2-\frac{1}{4}$, symmetrilinje $x=\frac{3}{2}$ och minimipunkt $(\frac{3}{2}, -\frac{1}{4})$.



c) $x^2-3x+2=0 \Leftrightarrow \underline{x=1} \vee \underline{x=2}.$ (Se fig.)

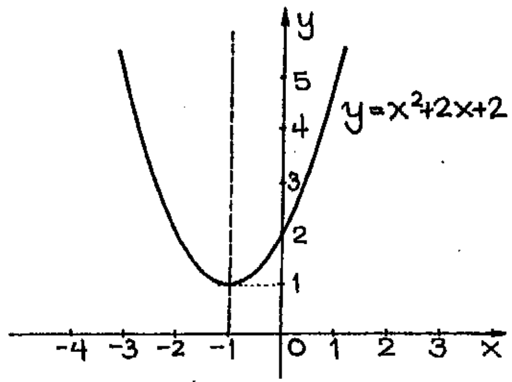
d) $f_{min} = f(\frac{3}{2}) = -\frac{1}{4}.$ (Se fig.)

e) $f(x) \geq 0 \Leftrightarrow \underline{x \leq 1} \vee \underline{x \geq 2}$ (Se fig.)

Övning

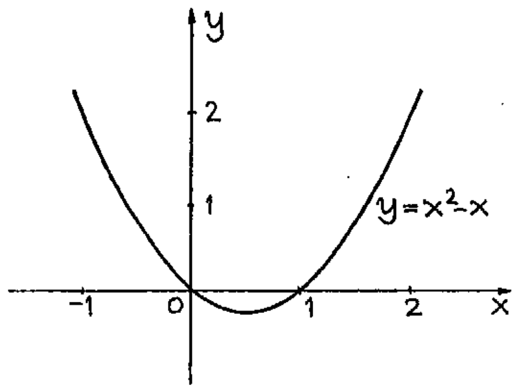
Lösning

a) $f(x) = x^2+2x+2 = \underline{(x+1)^2+1}.$



$f(x) \geq 1 > 0 \Rightarrow$ nollställen saknas. $f_{min} = f(-1) = 1.$

b) $f(x) = x^2-x = (x-\frac{1}{2})^2-\frac{1}{4} \geq -\frac{1}{4}.$

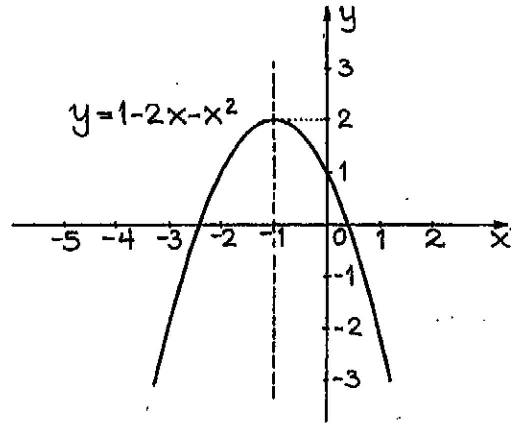


$f(x)=0 \Leftrightarrow x=0 \vee x=1$ (Se fig.)

$f_{min} = f(\frac{1}{2}) = -\frac{1}{4}.$

$f(x) \geq 0 \Leftrightarrow \underline{x \leq 0} \vee \underline{x \geq 1}.$

c) $f(x) = 1-2x-x^2 = -(x^2+2x)+1 = -(x^2+2x+1)+2 = \underline{2-(x+1)^2}.$

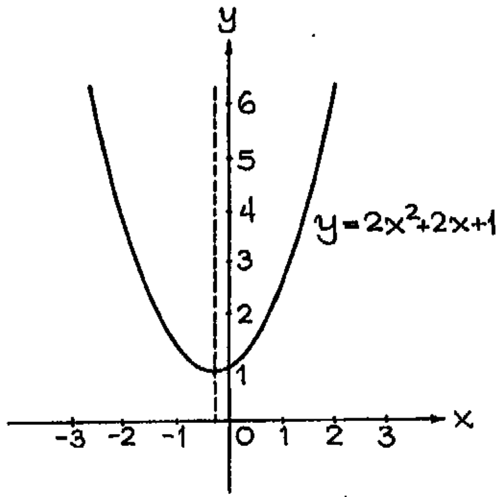


$f(x)=0 \Leftrightarrow (x+1)^2=2 \Leftrightarrow x+1=\pm\sqrt{2} \Leftrightarrow x=-1-\sqrt{2} \vee x=-1+\sqrt{2}$

f_{min} existerar inte; $f_{max} = f(-1) = 2$.

$f(x) \geq 0 \Leftrightarrow -1-\sqrt{2} \leq x \leq -1+\sqrt{2}$.

d) $f(x) = 2x^2+x+1 = 2(x^2+\frac{1}{2}x)+1 = \underline{2(x+\frac{1}{4})^2+\frac{7}{8}}$.



$f(x) \geq \frac{7}{8} > 0 \Rightarrow$ nollställen saknas; $f_{min} = f(-\frac{1}{4}) = \frac{7}{8}$.

Övning 1.5 (Sid. 16)

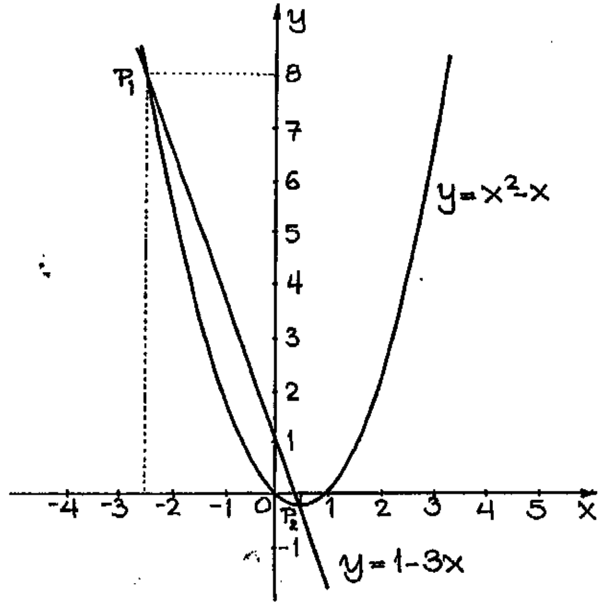
Lösning

$\underline{f(x) = x^2-x}$, $\underline{g(x) = 1-3x}$.

$f(x) = g(x) \Rightarrow x^2-x = 1-3x \Leftrightarrow x^2+2x = 1 \Leftrightarrow x = -1 \pm \sqrt{2}$.

Graferna skär varandra i punkterna

$\underline{P_1: (-1-\sqrt{2}, 2+3\sqrt{2})}$ och $\underline{P_2: (-1+\sqrt{2}, 4-3\sqrt{2})}$.

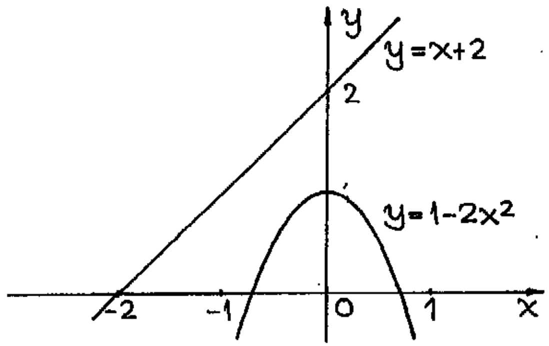


$f(x) > g(x) \Leftrightarrow \underline{x < -1-\sqrt{2}}$ V $\underline{x > -1+\sqrt{2}}$. (V = eller).

Övning 1.6 (Sid. 16)

Lösning

$\underline{f(x) = 1-2x^2,\ g(x) = 2+x}$.



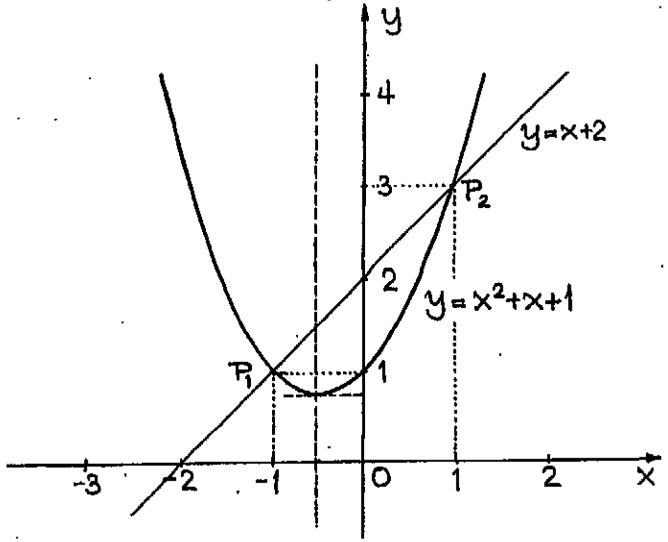
forts.

f:s och g:s grafer saknar gemensamma punkter; $f(x)>g(x)$ satisfieras inte av några x.

Övning 1.7 (Sid. 16)

Lösning

$f(x)=x^2+x+1=(x+\frac{1}{2})^2+\frac{3}{4}$.



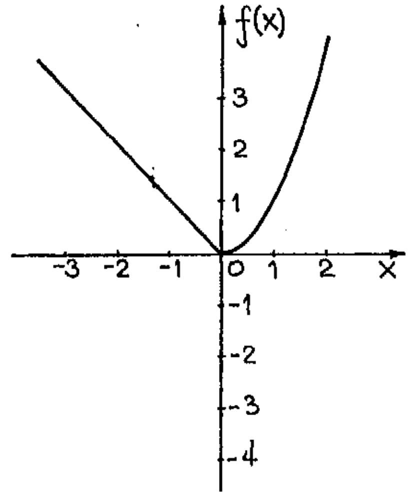
$$\begin{cases} P_1:(-1,1) \\ P_2:(1,3) \end{cases} \Rightarrow k=\frac{y_2-y_1}{x_2-x_1}=\frac{y-y_1}{x-x_1} \Leftrightarrow \frac{3-1}{1+1}=\frac{y-1}{x+1} \Leftrightarrow \underline{y=x+2}.$$

$f(x)<g(x) \Leftrightarrow -1<x<1$ (utläses direkt ur figuren).

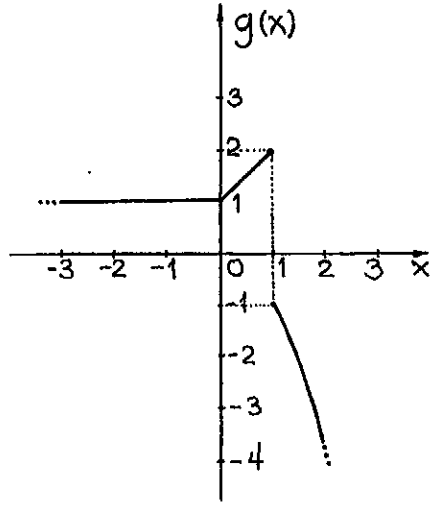
Övning 1.8 (Sid. 17)

Lösning

a) $f(x)=\begin{cases} x^2, & x\geq 0 \\ -x, & x<0 \end{cases}$;



b) $g(x)=\begin{cases} 1, & x<0 \\ 1+x, & 0\leq x<1 \\ -x^2, & x\geq 1 \end{cases}$.



Övning 1.9 (Sid. 17)

Lösning

$\underline{f(x)=x^2}$, $\underline{g(x)=3x}$

a) $f(2x)=g(-x) \Leftrightarrow (2x)^2=3(-x) \Leftrightarrow 4x^2=-3x \Leftrightarrow 4x^2+3x=0$

$\Leftrightarrow 4x(x+\frac{3}{4})=0 \Leftrightarrow x=0 \vee x+\frac{3}{4}=0 \Leftrightarrow \underline{x=0} \vee \underline{x=-\frac{3}{4}}$.

b) $g(x)>f(x) \Leftrightarrow 3x>x^2 \Leftrightarrow x^2-3x=x(x-3)<0 \Leftrightarrow \underline{0<x<3}$.

$$\begin{cases} f(x-1)=(x-1)^2=x^2-2x+1 \\ g(x)=3(x-1)=3x-3 \end{cases} \Rightarrow h(x)=(x-1)^2+3(x-1)=$$

$=(x-1)(x-1+3)=(x-1)(x+2)$; $\underline{x_1=1,\ x_2=-2}$.

Övning 1.10 (Sid. 17)

Lösning

En kurva är graf till en funktion $y=f(x)$ om en linje parallell med y-axeln skär kurvan i högst en punkt. Kurvorna i a), b) och c) är funktionsgrafer.

Övning 1.11 (Sid. 17)

Lösning

$$|x| \triangleq \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

a) $|3| = 3$ (ty $x=3>0$).

b) $|-3| = -(-3) = 3$ (ty $x=-3<0$).

c) $\sqrt{3^2} = |3| = 3$ (Obs! $\sqrt{x^2} = |x|$; se e) nedan.)

d) $\sqrt{(-3)^2} = |-3| = 3$.

e) $y = \sqrt{x^2} \geq 0 \Leftrightarrow y^2 = x^2 \Leftrightarrow y = \pm x = \left.\begin{cases} x, & x \geq 0 \\ -x, & x<0 \end{cases}\right\} = |x|$

f) $\sqrt{(-x^2)} = |x|$ (Se definitionen).

Anm. Symbolen $\triangleq$ utläses "är enligt definition lika med"; en annan beteckning är $:=$.

Övning 1.12 (Sid. 17)

Lösning

a) $|x| = 4 \Leftrightarrow \pm x = 4 \Leftrightarrow \underline{x=4} \vee \underline{x=-4}.$

b) $|x| = 0 \Leftrightarrow \pm x = 0 \Leftrightarrow \underline{x=0}.$

c) $|x| = -1$ saknar rötter, ty $VL = |x| \geq 0$, för alla x.

d) $|x-1| = 3 \Leftrightarrow \pm(x-1) = 3 \Leftrightarrow x-1=3 \vee x-1=-3 \Leftrightarrow \underline{x=4} \vee \underline{x=-2}.$

e) $|2x+1| = 1 \Leftrightarrow \pm(2x+1) = 1 \Leftrightarrow 2x+1=1 \vee 2x+1=-1 \Leftrightarrow 2x=0 \vee$ $\vee\ 2x=-2 \Leftrightarrow \underline{x=0} \vee \underline{x=-1}.$

f) $|1-x| = 1 \Leftrightarrow \pm(1-x) = 1 \Leftrightarrow 1-x=1 \vee 1-x=-1 \Leftrightarrow \underline{x=0} \vee \underline{x=2}.$

Anm. Blanda inte absolutbeloppet här och i de komplexa talen; där är absolutbeloppet mer generaliserad, tvådimensionellt.

Övning 1.13 (Sid. 17)

Lösning

a) $|x| \leq 1 \Leftrightarrow \pm x \leq 1 \Leftrightarrow \begin{cases} x \leq 1 \\ -x \leq 1 \end{cases} \Leftrightarrow \begin{cases} x \leq 1 \\ x \geq -1 \end{cases} \Leftrightarrow \underline{-1 \leq x \leq 1}.$

b) $|x| \geq 2 \Leftrightarrow \pm x \geq 2 \Leftrightarrow x \geq 2 \vee -x \geq 2 \Leftrightarrow \underline{x \geq 2} \vee \underline{x \leq -2}.$

c) $|x-1| < 2 \overset{a)}{\Leftrightarrow} -2 < x-1 < 2 \Leftrightarrow \underline{-1 < x < 3}.$

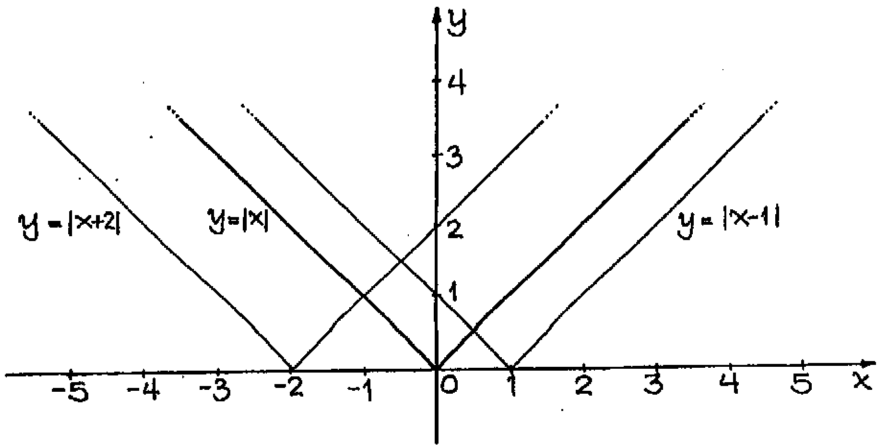
forts.

d) $|x+2|<1 \Leftrightarrow -1<x+2<1 \Leftrightarrow \underline{-3<x<-1}$.

Övning 1.14 (Sid. 17)

Lösning

a) $\underline{y=|x|}$ b) $\underline{y=|x-1|}$, c) $\underline{y=|x+2|}$.



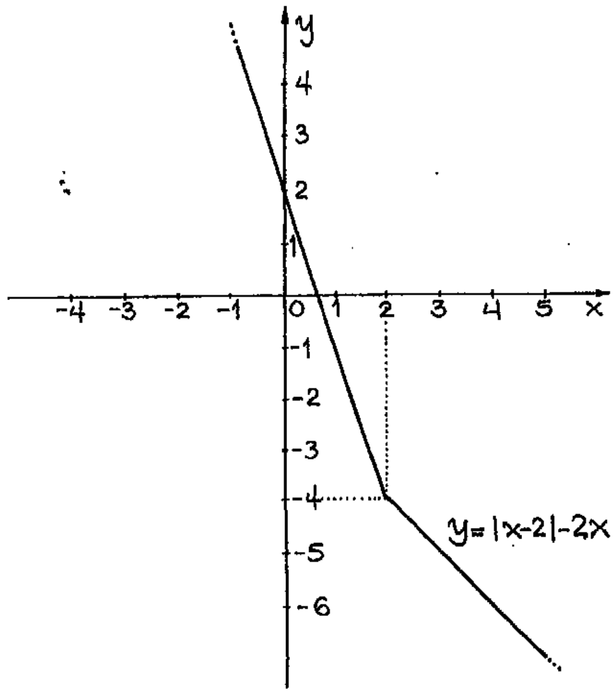
Man får "enhörningen" $y=|x-1|$ ($y=|x+2|$) gm stel förflyttning av "enhörningen" $y=|x|$ 1 enhet (2 enheter) åt höger (vänster).

Övning 1.15 (Sid. 17)

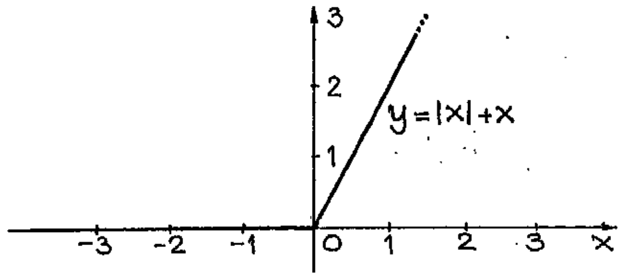
Lösning

a) $|x-2| = \begin{cases} x-2, & x-2 \geq 0 \\ -(x-2), & x-2<0 \end{cases} = \begin{cases} x-2, & x \geq 2 \\ -x+2, & x<2 \end{cases} \Rightarrow f(x)=|x-2|-2x =$

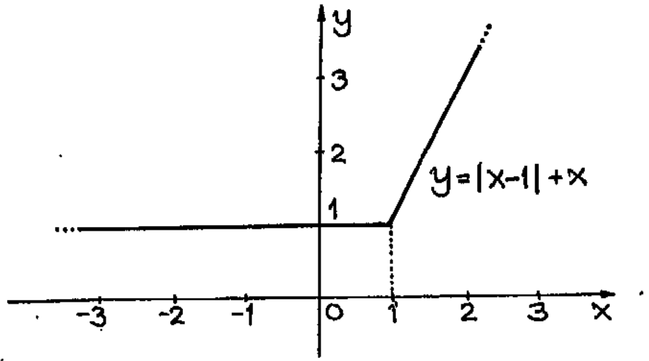
$= \begin{cases} x-2-2x, & x \geq 2 \\ -x+2-2x, & x<2 \end{cases} = \begin{cases} -x-2, & x \geq 2 \\ -3x+2, & x<2 \end{cases}$, (Se fig. nedan.)



b) $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x<0 \end{cases} \Rightarrow f(x) = \begin{cases} x+x, & x \geq 0 \\ -x+x, & x<0 \end{cases} = \begin{cases} 2x, & x \geq 0 \\ 0, & x<0 \end{cases}$.



c) $|x-1| = \begin{cases} x-1, & x-1 \geq 0 \\ -(x-1), & x-1<0 \end{cases} = \begin{cases} x-1, & x \geq 1 \\ -x+1, & x<1 \end{cases} \Rightarrow f(x) = |x-1|+x = \begin{cases} 2x-1, & x \geq 1 \\ 1, & x<1 \end{cases}$



Övning 1.16 (Sid. 17)

Lösning

a)

$$\begin{array}{rl}
 & x^2+3x-3 \\ \hline
 & x^5+3x^4-2x^3+0x^2+2x-1 \;\boxed{x^3+0x^2+x+1} \\
(-) & \underline{x^5+0x^4+x^3+x^2} \\
 & 3x^4-3x^3-x^2+2x-1 \\
(-) & \underline{3x^4+0x^3+3x^2+3x} \\
 & -3x^3-4x^2-x-1 \\
(-) & \underline{-3x^3+0x^2-3x-3} \\
 & -4x^2+2x+2
\end{array}$$

Kvoten är x^2+3x-3 och resten $-4x^2+2x+2$.

b) $x \neq 1 \Rightarrow 1+x+x^2+x^3+x^4+x^5 = \frac{x^6-1}{x-1}$ (geometrisk serie)

$\Leftrightarrow$ kvoten är $1+x+x^2+\ldots+x^5$ och resten 0.

Anm. Man kan även dividera...

c)

$$\begin{array}{rl}
 & x^2-2x+3 \\ \hline
 & x^4+2x^3+0x^2+0x+25 \;\boxed{x^2+4x+5} \\
(-) & \underline{x^4+4x^3+5x^2} \\
 & -2x^3-5x^2+0x+25 \\
(-) & \underline{-2x^3-8x^2-10x} \\
 & 3x^2+10x+25 \\
(-) & \underline{3x^2+12x+15} \\
 & -2x+10
\end{array}$$

Kvoten är x^2-2x+3 och resten $-2x+10$.

Övning 1.17 (Sid. 18)

Lösning

a) $x^2-4 = x^2-2^2 = \underline{(x-2)(x+2)}$.

b) $x^2+2x+1 = \underline{(x+1)^2}$.

c) $x^3-x = x(x^2-1) = \underline{x(x-1)(x+1)}$.

d) $x^2-3x+2 = (x-\frac{3}{2})^2-\frac{1}{4} = (x-\frac{3}{2})^2-(\frac{1}{2})^2 = (x-\frac{3}{2}-\frac{1}{2})(x-\frac{3}{2}+\frac{1}{2}) =$

$= \underline{(x-2)(x-1)}$.

e) $2-x-x^2 = -(x^2+x)+2 = -((x+\frac{1}{2})^2-\frac{1}{4})+2 = \frac{9}{4}-(x+\frac{1}{2})^2 =$

$= (\frac{3}{2})^2-(x+\frac{1}{2})^2 = (\frac{3}{2}+\frac{1}{2}+x)(\frac{3}{2}-\frac{1}{2}-x) = \underline{(2+x)(1-x)}$.

f) $x^4-2x^3+x^2 = x^2(x^2-2x+1) = \underline{x^2(x-1)^2}$.

Övning 1.18 (Sid. 18)

Lösning

Se nästa sida.

a) $x^2-1=\underline{(x-1)(x+1)}$.

b) $x^2+1>0$, så reella förstagradsfaktorer saknas.

c) $x^3-1=x^3-1^3=\underline{(x-1)(x^2+x+1)}$. $\quad x^2+x+1=(x+\frac{1}{2})^2+\frac{3}{4}>0$.

d) $x^3+1=x^3+1^3=\underline{(x+1)(x^2-x+1)}$. $\quad x^2-x+1=(x-\frac{1}{2})^2+\frac{3}{4}>0$.

e) $x^4-1=(x^2)^2-1^2=(x^2-1)(x^2+1)=\underline{(x-1)(x+1)(x^2+1)}$ (Se a), b)).

f) $x^4+27x=x(x^3+27)=x(x^3+3^3)=\underline{x(x+3)(x^2-3x+9)}$.

g) $x^4-64=(x^2)^2-8^2=(x^2-8)(x^2+8)=\underline{(x-\sqrt{8})(x+\sqrt{8})(x^2+8)}$

Övning 1.19 (Sid. 18)

Lösning

$P(x)=x^5-10x^2+15x-6 \Rightarrow P(1)=0 \Leftrightarrow x-1$ faktor i $P(x)$

$$
\begin{array}{rl}
 & x^4+x^3+x^2-9x+6 \\
\hline
 & x^5+0x^4+0x^3-10x^2+15x-6 \quad \boxed{x-1} \\
(-) & \underline{x^5-x^4} \\
 & x^4+0x^3-10x^2+15x-6 \\
 & \underline{x^4-x^3} \\
 & x^3-10x^2+15x-6 \\
(-) & \underline{x^3-x^2} \\
 & -9x^2+15x-6 \\
(-) & \underline{-9x^2+9x} \\
 & 6x-6 \\
(-) & \underline{6x-6} \\
 & 0
\end{array}
$$

$P(x)=(x-1)\cdot Q(x)$; $\quad Q(x)=x^4+x^3+x^2-9x+6$;

$Q(1)=0 \Leftrightarrow x-1$ faktor i $Q(x)$ (enl. faktorsatsen).

$$
\begin{array}{rl}
 & x^3+2x^2+3x-6 \\
\hline
 & x^4+x^3+x^2-9x+6 \quad \boxed{x-1} \\
(-) & \underline{x^4-x^3} \\
 & 2x^3+x^2-9x+6 \\
(-) & \underline{2x^3-2x^2} \\
 & 3x^2-9x+6 \\
(-) & \underline{3x^2-3x} \\
 & -6x+6 \\
(-) & \underline{-6x+6} \\
 & 0
\end{array}
$$

$P(x)=(x-1)^2R(x)$; $\quad R(x)=x^3+2x^2+3x-6$;

$R(1)=0 \Leftrightarrow x-1$ faktor i $R(x)$ (enligt faktorsatsen).

$$
\begin{array}{rl}
 & x^2+3x+6 \\
\hline
 & x^3+2x^2+3x-6 \quad \boxed{x-1} \\
(-) & \underline{x^3-x^2} \\
 & 3x^2+3x-6 \\
(-) & \underline{3x^2-3x} \\
 & 6x-6 \\
(-) & \underline{6x-6} \\
 & 0
\end{array}
$$

$P(x)=(x-1)^3(x^2+3x+6)=(x-1)^3\cdot S(x)$; $\quad S(x)=x^2+3x+6=$

$=(x+\frac{3}{2})^2+\frac{15}{4}>0 \Rightarrow S(x)$ är irreduktibelt i $\mathbb{R}$.

Resultat: 3; $\quad P(x)=(x-1)^3(x^2+3x+6)$.

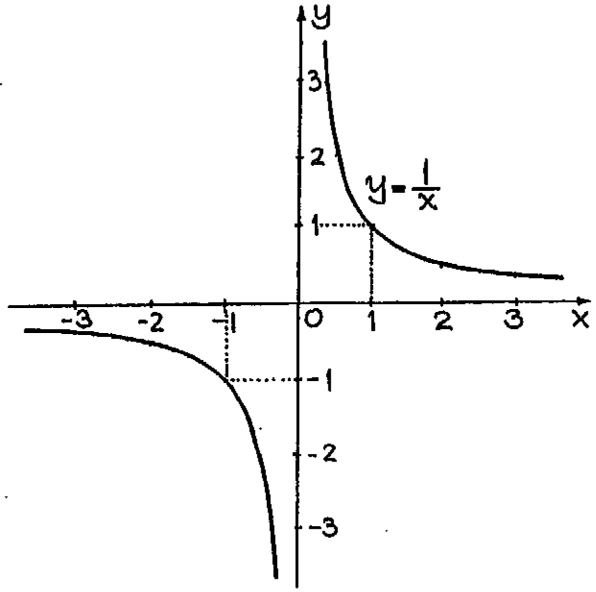
Övning 1.20 (Sid. 18)

Lösning

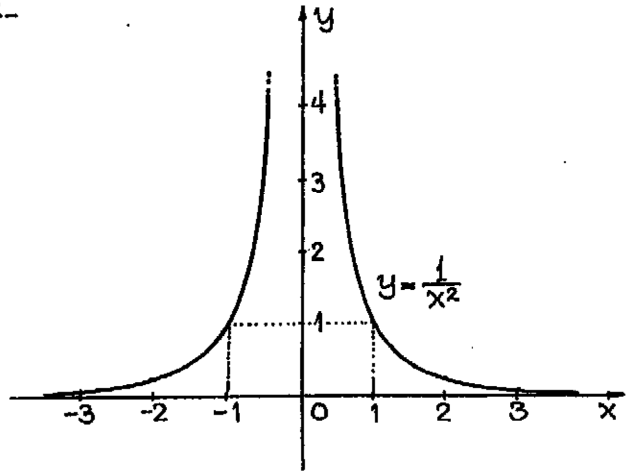
Se nästa sida.

a) $f(x) = \frac{1}{x}$.

$\begin{cases} x \to 0^+ \Rightarrow y \to \infty \\ x \to \infty \Rightarrow y \to 0^+ \end{cases}$; $\begin{cases} x \to 0^- \Rightarrow y \to -\infty \\ x \to -\infty \Rightarrow y \to 0^- \end{cases}$.



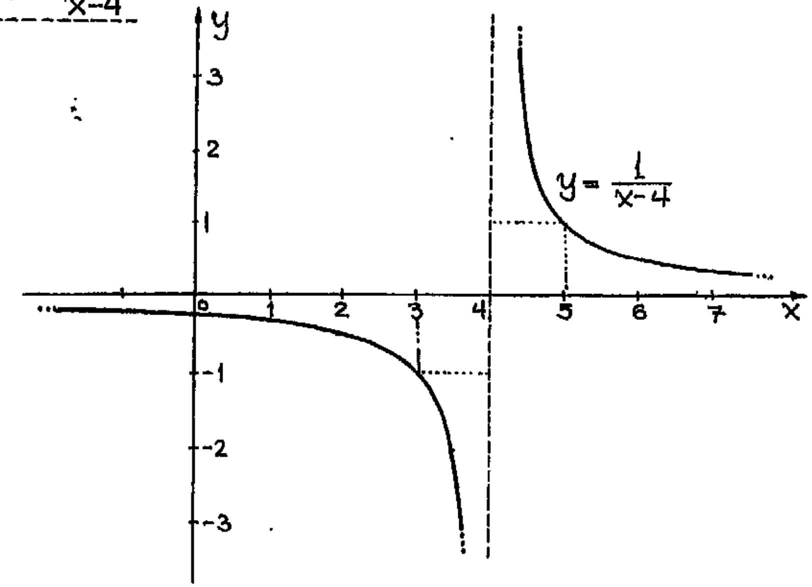
b) $f(x) = \frac{1}{x^2}$.



fortS.

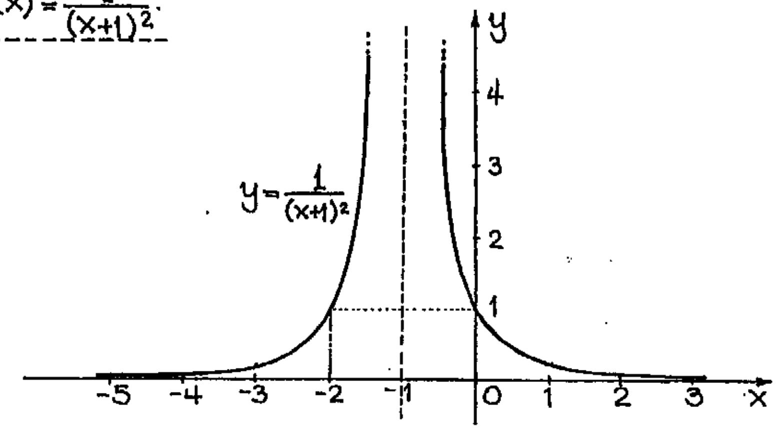
$\begin{cases} x \to 0^+ \Rightarrow y \to \infty \\ x \to \infty \Rightarrow y \to 0^+ \end{cases}$; $\begin{cases} x \to 0^- \Rightarrow y \to \infty \\ x \to -\infty \Rightarrow y \to 0^+ \end{cases}$.

c) $f(x) = \frac{1}{x-4}$.

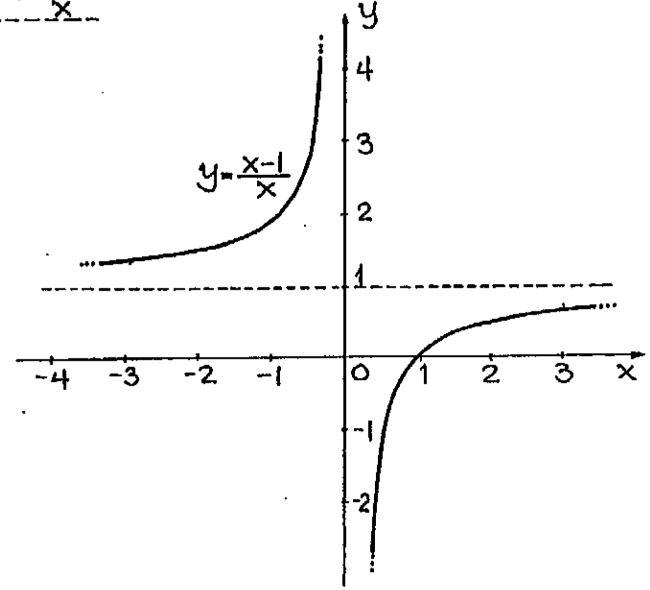


$y = \frac{1}{x-4}$ är $y = \frac{1}{x}$ förskjuten 4 enheter åt höger.

d) $f(x) = \frac{1}{(x+1)^2}$.

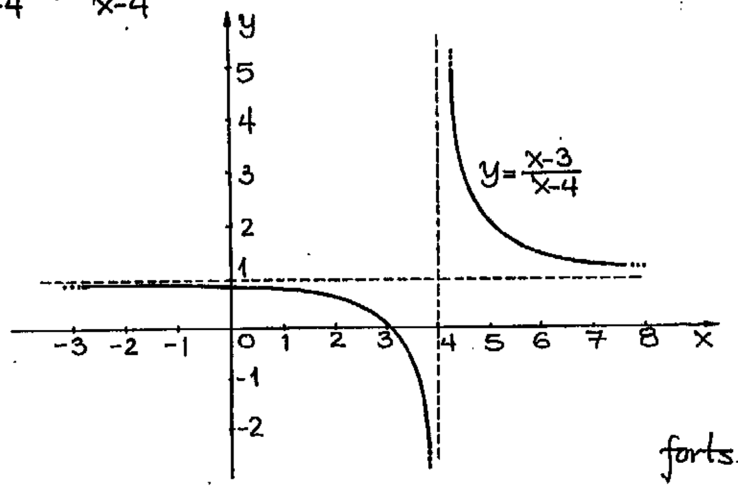


e) $\underline{f(x) = 1 - \frac{1}{x}}$.



Anm. $y = \frac{1}{x}$ speglas i x-axeln och förskjuts 1 enhet uppåt.

f) $f(x) = \frac{x-3}{x-4} = 1 + \frac{1}{x-4}$.



forts.

Anm. $y = \frac{1}{x}$ förskjuts 1 enhet uppåt och 4 enheter åt höger.

Övning 1.21 (Sid. 18)

Lösning

$$HL = \frac{(2^6 \cdot 5^5)^3}{(2^4 \cdot 5^2)^4} \cdot \frac{2^{18}}{5^7} = \frac{2^{18} \cdot 5^{15}}{2^{16} \cdot 5^8} \cdot \frac{2^{18}}{5^7} = 2^{18-16} \cdot 5^{15-8} \cdot 2^{18} \cdot 5^{-7} =$$
$$= 2^2 \cdot 5^7 \cdot 2^{18} \cdot 5^{-7} = 2^{20};$$

$$VL = (4^x)^5 = 2^{20} = HL \Leftrightarrow ((2^2)^x)^5 = 2^{20} \Leftrightarrow 2^{10x} = 2^{20} \Leftrightarrow$$

$$\Leftrightarrow 10x = 20 = 10 \cdot 2 \Leftrightarrow \underline{x = 2}.$$

Övning 1.22 (Sid. 18)

Lösning

a) $\frac{3^2 \cdot 2^4}{6^3} = \frac{3^2 \cdot 2^4}{(3 \cdot 2)^3} = \frac{3^2 \cdot 2^4}{3^3 \cdot 2^3} = 3^{2-3} \cdot 2^{4-3} = 3^{-1} \cdot 2^1 = \underline{\frac{2}{3}}$.

b) $\left(\frac{1}{4}\right)^{-1/2} = (4^{-1})^{-1/2} = (2^{-2})^{-1/2} = 2^{(-2)(-1/2)} = 2^1 = \underline{2}$.

c) $(\sqrt{64})^{2/3} = ((64)^{1/2})^{2/3} = (64)^{(1/2)(2/3)} = 64^{1/3} = (4^3)^{1/3} = 4^1 = \underline{4}$.

d) $\left(\frac{1}{3}\right)^{-1} = (3^{-1})^{-1} = 3^{(-1)^2} = 3^1 = \underline{3}$.

e) $2^{(2^3)} = 2^8 = \underline{256}$.

f) $(2^2)^3 = 2^{2 \cdot 3} = 2^6 = \underline{64}$.

Övning 1.23 (Sid. 18)

Lösning

Se nästa sida.

a) $\frac{a^{3,3}\cdot a^{-2,1}}{a^{0,8}} = a^{3,3-2,1-0,8} = \underline{a^{0,4}}$.

b) $\frac{a\sqrt{a}}{\sqrt[3]{a^2}} = \frac{a\cdot a^{1/2}}{(a^2)^{1/3}} = \frac{a\cdot a^{1/2}}{a^{2/3}} = a^{1+1/2-2/3} = a^{1/2+1/3} = \underline{a^{5/6}}$.

c) $\sqrt{\sqrt[3]{x}}\cdot\sqrt[6]{x^{-1}}\cdot(x^2)^{2/3} = (\sqrt[3]{x})^{1/2}\cdot(x^{-1})^{1/6}\cdot x^{4/3} = (x^{1/3})^{1/2}\cdot x^{-1/6}\cdot x^{4/3}$

$= x^{1/6}\cdot x^{-1/6}\cdot x^{4/3} = x^{1/6-1/6+4/3} = \underline{x^{4/3}}$

d) $(ab\sqrt[4]{a^3/\sqrt{b\sqrt{b}}})^2 = (ab\sqrt[4]{a^3/\sqrt{b\cdot b^{1/2}}})^2 = (ab\sqrt[4]{a^3/\sqrt{b^{3/2}}})^2 =$

$= (ab\sqrt[4]{a^3/(b^{3/2})^{1/2}})^2 = (ab\sqrt[4]{a^3/b^{3/4}})^2 = (ab(a^3b^{-3/4})^{1/4})^2$

$= (ab\cdot a^{3/4}\cdot b^{-3/16})^2 = (a^{7/4}\cdot b^{13/16})^2 = \underline{a^{7/2}\cdot b^{13/8}}$.

Jag har börjat förenklingen "inifrån"; det går bra att börja "utifrån". Det ändrar inget.

Övning 1.24 (Sid. 18)

Lösning

a) $3^x + 3^{x+1} = 3^x + 3^x\cdot 3 = \underline{4\cdot 3^x}$.

b) $e^x + e^{x+1} = e^x\cdot 1 + e^x\cdot e = (1+e)e^x$.

c) $(e^x+e^{-x})^2 - e^{2x} - e^{-2x} = (e^x)^2 + (e^{-x})^2 + 2\cdot e^x\cdot e^{-x} - e^{2x} - e^{-2x} =$

$= e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} = \underline{2}$.

d) $\frac{1}{e^x} + \frac{1}{e^{x+1}} = (e^x)^{-1} + (e^{x+1})^{-1} = e^{-x} + e^{-(x+1)} = e^{-x} + e^{-x-1} =$

$= e^{-x}\cdot 1 + e^{-x}\cdot e^{-1} = \underline{(1+e^{-1})e^{-x}}$.

Anm. $(1+e^{-1})e^{-x} = (e\cdot e^{-1} + e^{-1})e^{-x} = (e+1)e^{-1}\cdot e^{-x} = (e+1)e^{-x-1}$.

Övning 1.25 (Sid. 18)

Lösning

Logaritmlagarna

(1) ${}^a\log 1 = 0$.

(2) ${}^a\log(st) = {}^a\log s + {}^a\log t$.

(3) ${}^a\log(\frac{s}{t}) = {}^a\log s - {}^a\log t$.

(4) ${}^a\log(s^t) = t\cdot {}^a\log s$.

(5) ${}^b\log s = \frac{{}^a\log s}{{}^a\log b}$.

a) $\lg\frac{7}{4} + \lg\frac{8}{7} \overset{(2)}{=} \lg\frac{7}{4}\cdot\frac{8}{7} = \lg\frac{7\cdot 8}{4\cdot 7} = \underline{\lg 2}$.

b) $\frac{1}{2}\ln 100 - 2\ln 2 \overset{(4)}{=} \ln 100^{1/2} - \ln 2^2 = \ln 10 - \ln 4 \overset{(3)}{=}$

$= \ln\frac{10}{4} = \underline{\ln\frac{5}{2}}$.

c) $-\lg 6 = (-1)\lg 6 \overset{(4)}{=} \lg 6^{-1} = \underline{\lg\frac{1}{6}}$.

d) $\lg 36 - 3\lg 6 = \lg 6^2 - 3\lg 6 = 2\lg 6 - 3\lg 6 = -\lg 6 = \underline{\lg\frac{1}{6}}$.

e) ${}^3\log 27 = {}^3\log 3^3 = \underline{3}$. $\quad ({}^a\log a^b = b)$.

f) ${}^2\log 11 + {}^2\log\frac{1}{11} \overset{(2)}{=} {}^2\log 11\cdot\frac{1}{11} = {}^2\log 1 = \underline{0}$.

Övning 1.26 (Sid. 19)

Lösning

a) $\ln\frac{1}{x^2} + \ln x^3 = \ln\frac{x^3}{x^2} = \ln x^{3-2} = \underline{\ln x}$.

forts.

b) $\ln e^{2x} = 2x$. ($\ln$ och exp är varandras inverser).

c) $e^{\ln t} = t \quad (t>0)$.

d) $\ln e^{x} + \ln e^{-x} = x + (-x) = x - x = 0$.

Övning 1.27 (Sid. 19)

Lösning

$VL = \ln(a+b) - \ln a - \ln b = \ln(a+b) - (\ln a + \ln b) =$

$= \ln(a+b) - \ln(ab) = \ln \frac{a+b}{ab} = \ln\left(\frac{1}{b}+\frac{1}{a}\right) = \ln\left(\frac{1}{a}+\frac{1}{b}\right) = HL.$

Svar: Ja, det är det.

Övning 1.28 (Sid. 19)

Lösning

$1+e^{-x} = e^{-x}(1+e^{x}) \Leftrightarrow \sqrt{1+e^{-x}} = \sqrt{e^{-x}(1+e^{x})} = \sqrt{e^{-x}}\sqrt{1+e^{x}} =$

$= e^{-x/2}\sqrt{1+e^{x}} \Leftrightarrow \sqrt{1+e^{-x}}+1 = e^{-x/2}\sqrt{1+e^{x}}+1 \Leftrightarrow \sqrt{1+e^{-x}}+1 =$

$= e^{-x/2}(\sqrt{1+e^{x}}-\sqrt{e^{x}}) \Leftrightarrow \ln(\sqrt{1+e^{-x}}+1) = \ln e^{-x/2}(\sqrt{1+e^{x}}+\sqrt{e^{x}})$

$\Leftrightarrow \ln(\sqrt{1+e^{-x}}+1) = \ln e^{-x/2} + \ln(\sqrt{1+e^{x}}+\sqrt{e^{x}}) = -\frac{x}{2} +$

$+ \ln(\sqrt{1+e^{x}}+\sqrt{e^{x}}) \Leftrightarrow \underline{\ln(x(\sqrt{1+e^{x}}-\sqrt{e^{x}})) + \ln(\sqrt{1+e^{x}}+1)} =$

$= \ln(x(\sqrt{1+e^{x}}-\sqrt{e^{x}})) + \ln(\sqrt{1+e^{x}}+\sqrt{e^{x}}) - \frac{x}{2} = \ln x +$

$+ \ln(\sqrt{1+e^{x}}-\sqrt{e^{x}}) + \ln(\sqrt{1+e^{x}}+\sqrt{e^{x}}) - \frac{x}{2} = \ln x - \frac{x}{2} +$

$+ \ln(\sqrt{1+e^{x}}-\sqrt{e^{x}})(\sqrt{1+e^{x}}+\sqrt{e^{x}}) = \ln x - \frac{x}{2} + \ln(1+e^{x}-e^{x}) =$

$= \ln x - \frac{x}{2} + \ln 1 = \underline{\ln x - \frac{x}{2}}.$

Svar: Nej.

Kortare lösning

$\ln(x(\sqrt{1+e^{x}}-\sqrt{e^{x}})) + \ln(\sqrt{1+e^{-x}}+1) = \ln(x(\sqrt{1+e^{x}}-\sqrt{e^{x}})) +$

$+ \ln(e^{-x/2}(\sqrt{e^{x}}(\sqrt{1+e^{-x}}+1))) = \ln x + \ln(\sqrt{1+e^{x}}-\sqrt{e^{x}}) +$

$+ \ln(e^{-x/2}(\sqrt{1+e^{x}}+\sqrt{e^{x}})) = \ln x + \ln(\sqrt{1+e^{x}}-\sqrt{e^{x}}) +$

$+ \ln e^{-x/2} + \ln(\sqrt{1+e^{x}}+\sqrt{e^{x}}) = \ln x - \frac{x}{2} + \ln(\sqrt{1+e^{x}}-\sqrt{e^{x}}) +$

$+ \ln(\sqrt{1+e^{x}}+\sqrt{e^{x}}) = \ln x - \frac{x}{2} + \ln((\sqrt{1+e^{x}})^2 - (\sqrt{e^{x}})^2) =$

$= \ln x - \frac{x}{2} + \ln(1+e^{x}-e^{x}) = \ln x - \frac{x}{2} + \ln 1 = \ln x - \frac{x}{2}.$

Övning 1.29 (Sid. 19)

Lösning

a) $\lg x = {}^{10}\log x = \frac{\ln x}{\ln 10}$.

b) ${}^{2}\log x = \frac{\ln x}{\ln 2}$.

c) $({}^{3}\log 2)({}^{2}\log 3) = \frac{\ln 2}{\ln 3} \cdot \frac{\ln 3}{\ln 2} = 1$.

Övning 1.30 (Sid 19)

Lösning

Se författarnas förslag.

Övning 1.31 (Sid. 19)

Lösning

a) $D_{\ln} = \mathbb{R}_+ \Rightarrow x>0 \wedge x-2>0 \Rightarrow \underline{x>2}$ (villkor på x). (*)

$\ln x + \ln(x-2) = 2 \Leftrightarrow \ln x(x-2) = 2 \Leftrightarrow x^2-2x = e^2 \Leftrightarrow \underline{x = 1+\sqrt{e^2+1}}.$

b) $\ln(3^x + 3^{x+1}) = 1 \Leftrightarrow 3^x + 3\cdot 3^x = e \Leftrightarrow 4\cdot 3^x = e \Leftrightarrow 3^x = \frac{e}{4} \Leftrightarrow$

$\Leftrightarrow x = {}^3\log\frac{e}{4} = \frac{\ln(e/4)}{\ln 3} = \underline{\frac{1-2\ln 2}{\ln 3}}.$

Övning 1.32 (Sid. 19)

Lösning

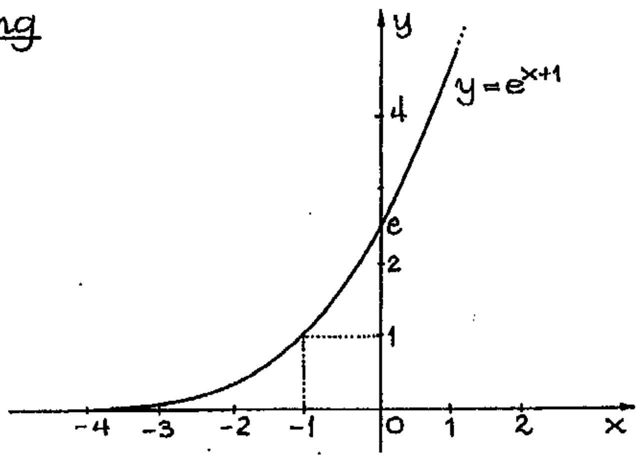
$m(T) = \frac{1}{2}\cdot m(0) \Leftrightarrow m(0)e^{-\lambda T} = \frac{1}{2}m(0) \Leftrightarrow e^{-\lambda T} = 2^{-1} \Leftrightarrow$

$\Leftrightarrow e^{\lambda T} = 2 \Leftrightarrow \lambda T = \ln 2 \Leftrightarrow T = \frac{\ln 2}{\lambda}.$

Övning 1.33 (Sid. 19)

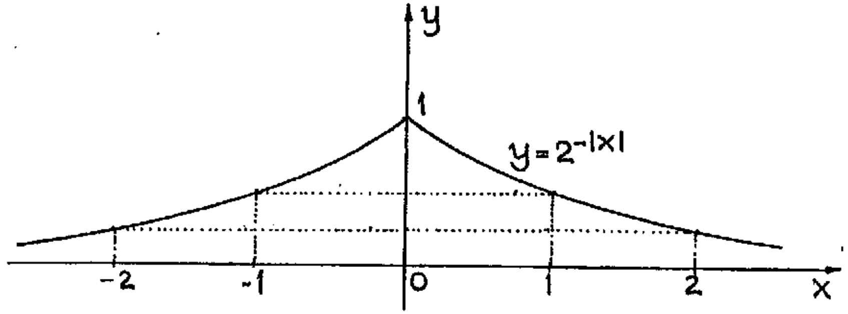
Lösning

a)



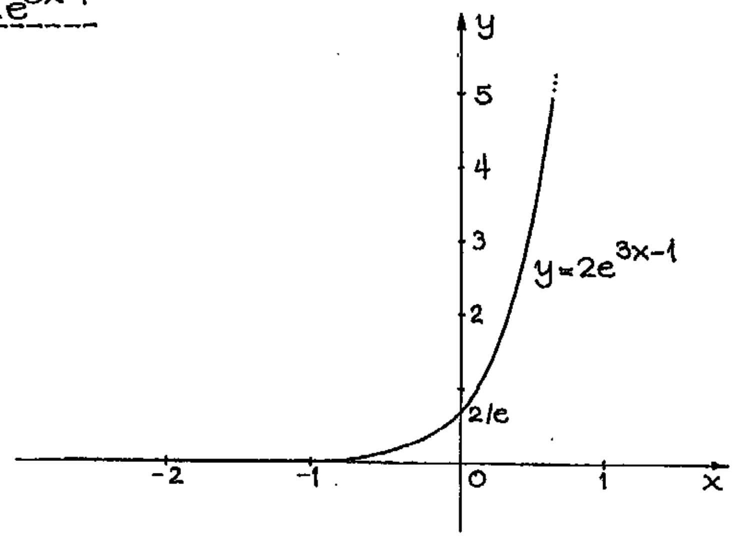
Kurvan $y = e^{x+1}$ är kurvan $y = e^x$ förskjuten 1 enhet åt vänster. (Värdetabell finns i ... miniräknarer.)

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x<0 \end{cases} \Leftrightarrow -|x| = \begin{cases} -x, & x\geq 0 \\ x, & x<0 \end{cases} \Leftrightarrow y = 2^{-|x|} = \begin{cases} 2^{-x}, & x\geq 0 \\ 2^{x}, & x<0 \end{cases}$



Grafen är spegelsymmetrisk m.a.p. y-axeln.

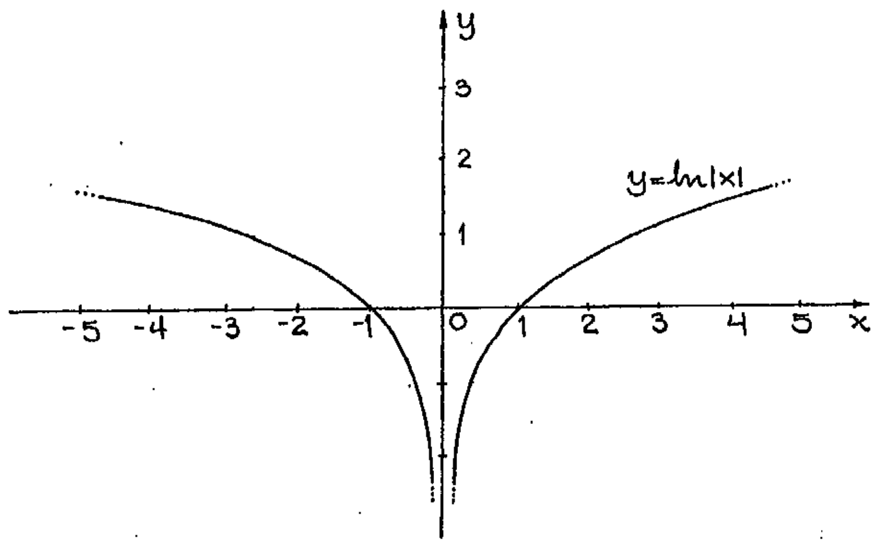
c) $\underline{y = 2e^{3x-1}}$



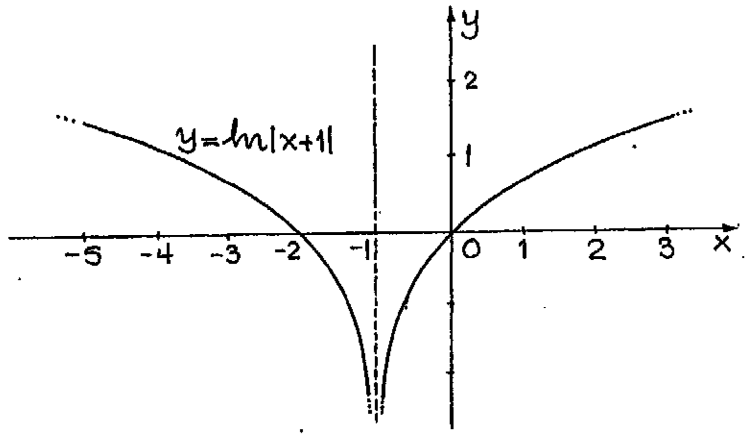
Övning 1.34 (Sid. 19)

Lösning

a) $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \Leftrightarrow y = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$

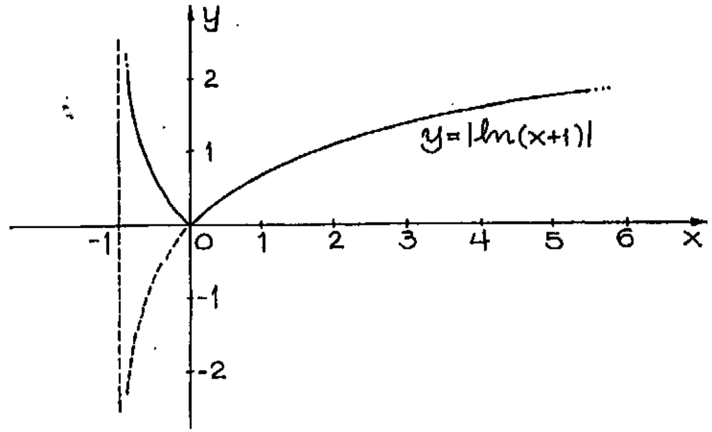


b)



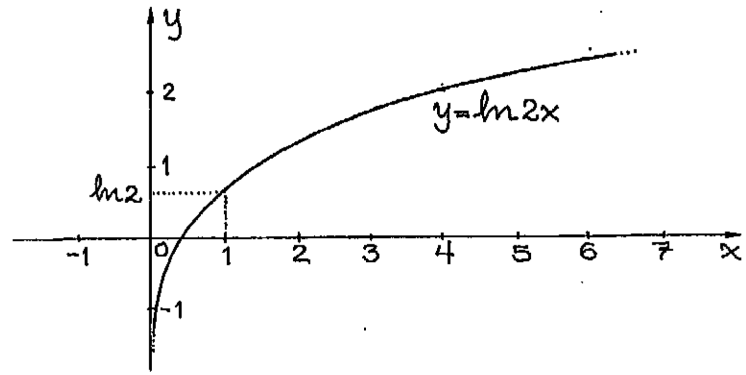
Kurvan $y = \ln|x+1|$ är kurvan $y = \ln|x|$ förskjuten 1 enhet åt vänster.

c) $y = |\ln(x+1)|$.



$y = \ln(x)$ förskjutes 1 enhet åt vänster varefter den del av grafen som ligger under x-axeln speglas i samma axel.

d) $y = \ln 2x$

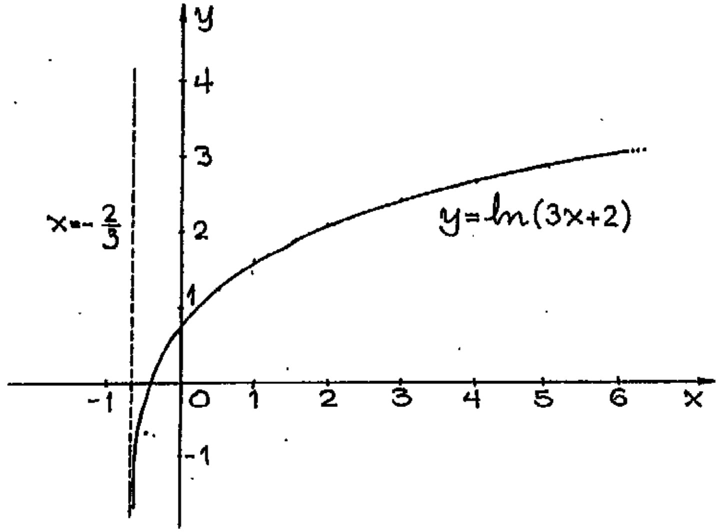


$y = \ln 2x = \ln 2 + \ln x$, så $y = \ln 2x$ är $y = \ln x$ förskjuten $\ln 2 \approx 0{,}69$ enheter uppåt.

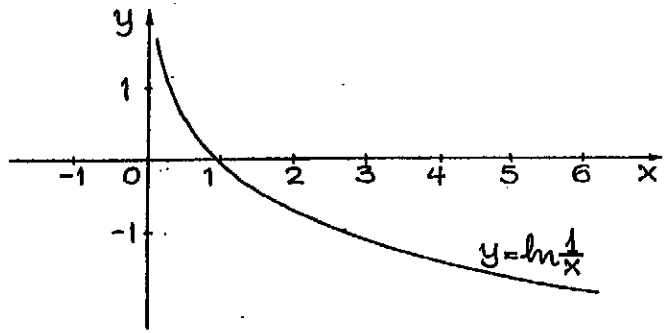
e) $y = \ln(2+3x)$

$D_{\ln} = \mathbb{R}_+ \Rightarrow 2+3x > 0 \Leftrightarrow 3x > -2 \Leftrightarrow x > -\frac{2}{3}$.

x	-0,5	0	0,5	1	2	3	4	5
y	-0,69	0,69	1,25	1,61	2,08	2,40	2,64	2,83



f) $y = \ln\frac{1}{x} = -\ln x$.

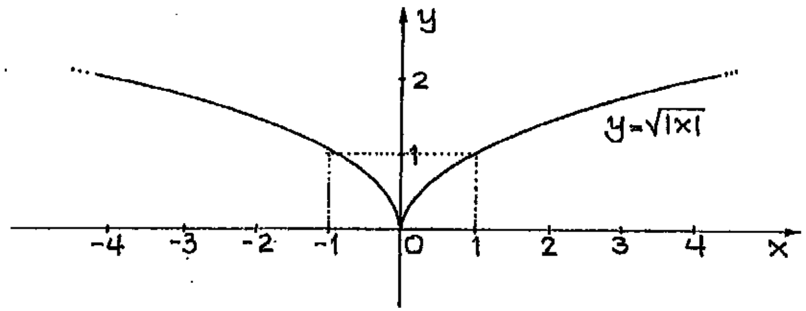


Övning 1.35 (Sid. 20)

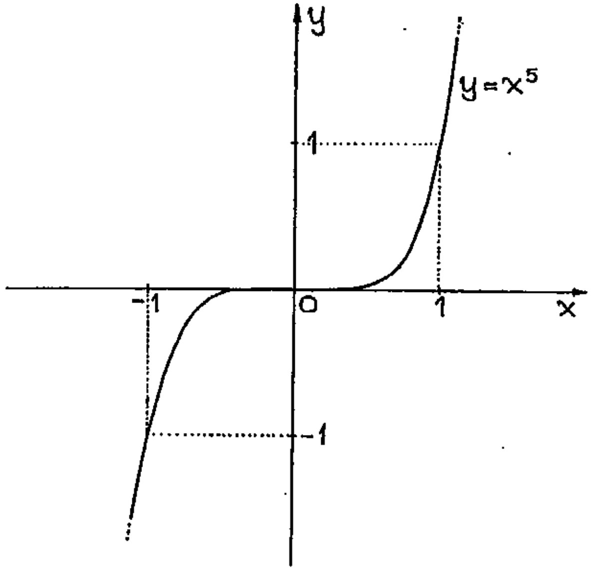
Lösning

a) $y = \sqrt{|x|}$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \Rightarrow y = \sqrt{|x|} = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases};$$



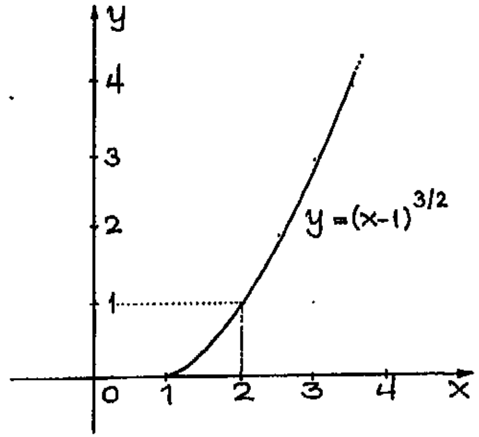
b) $y = x^5$



forts.

b) $y = (x-1)^{3/2}$

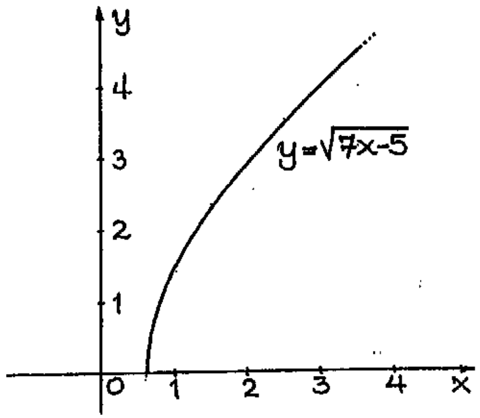
x	1,5	2	2,5	3	3,5	4
y	0,35	1	1,84	2,83	3,95	5,20



c) $y = \sqrt{7x-5}$

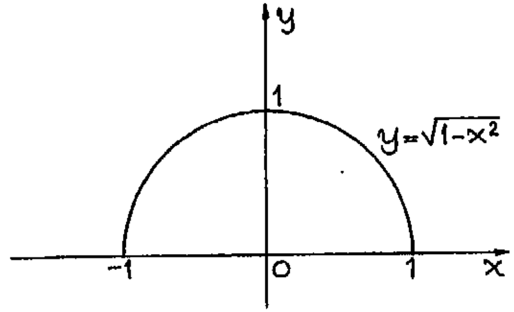
$D_{\sqrt{}} = \{x \in \mathbb{R} : x \geq 0\} \Rightarrow 7x-5 \geq 0 \Leftrightarrow 7x \geq 5 \Leftrightarrow x \geq \frac{5}{7}$.

x	1	1,5	2	2,5	3	3,5	4
y	1,41	2,35	3	3,54	4	4,42	4,80



e) $y = \sqrt{1-x^2}$

$y = \sqrt{1-x^2} \geq 0 \Leftrightarrow y^2 = 1-x^2 \wedge y \geq 0 \Leftrightarrow x^2+y^2 = 1,\ y \geq 0.$



Övning 1.36 (Sid. 20)

Lösning

$$\lim_{x\to\infty} \frac{5x^2+2x+1}{2x+x^2} = \lim_{x\to\infty} \frac{\cancel{x^2}(5+2x^{-1}+x^{-2})}{\cancel{x^2}(1+2x^{-1})} = \lim_{x\to\infty} \frac{5+2x^{-1}+x^{-2}}{1+2x^{-1}} = 5.$$

Övning 1.37 (Sid. 20)

Lösning

a) $\lim_{x\to\infty} \frac{x^2-10x+1}{2x^3+4x^2+1} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{x^2(1-10x^{-1}+x^{-2})}{x^3(2+4x^{-1}+x^{-3})} = \lim_{x\to\infty} \frac{1}{2x} = 0.$

b) $\lim_{x\to\infty} \frac{x^2-10x+1}{3x^2+x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\cancel{x^2}(1-10x^{-1}+x^{-2})}{\cancel{x^2}(3+x^{-1})} = \frac{1-0+0}{3+0} = \frac{1}{3}.$

c) $\lim_{x\to\infty} \frac{(x^2+1)^3}{(x^3+2)^2} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\cancel{x^6}(1+x^{-2})^3}{\cancel{x^6}(1+2x^{-3})^2} = \lim_{x\to\infty} \frac{(1+x^{-2})^3}{(1+2x^{-3})^2} = 1.$

d) $\lim_{x\to\infty} \frac{x^2+10x+1}{2x+1} = \lim_{x\to\infty} \frac{x^2}{2x} = \lim_{x\to\infty} \frac{x}{2} = \infty.$ (Oegentligt)

Övning 1.38 (Sid. 20)

Lösning

Se nästa sida.

$\alpha > 0 \wedge a > 1 \Rightarrow \lim_{x\to\infty} \frac{x^\alpha}{a^x} = 0 \Rightarrow \lim_{x\to\infty} \frac{x^{100}}{1{,}01^x} = 0.$

$x=1$: $\frac{1^{100}}{1{,}01^1} = 0{,}\overline{9900}$ (överstrykning anger period).

$x=10$: $\frac{10^{100}}{1{,}01^{10}} = 10^{99{,}95678626}$;

$x=100$: $\frac{100^{100}}{1{,}01^{100}} = 10^{199{,}5678626}$ osv.

För små x måste logaritmer användas; dämpningen "inträder" för mycket stora x.

Övning 1.39 (Sid. 20)

Lösning

a) $\lim_{x\to\infty} \frac{x^6+4x+2^x}{2^x+x^6} = \lim_{x\to\infty} \frac{\cancel{2^x}(\frac{x^6}{2^x}+4\frac{x}{2^x}+1)}{\cancel{2^x}(1+\frac{x^6}{2^x})} = \lim_{x\to\infty} \frac{\frac{x^6}{2^x}+4\frac{x}{2^x}+1}{1+\frac{x^6}{2^x}} = 1.$

b) $\lim_{x\to\infty} \frac{e^x+2{,}5^x+\ln x}{2e^x+x^{10}} = \lim_{x\to\infty} \frac{\cancel{e^x}(1+(\frac{2{,}5}{e})^x+\frac{\ln x}{e^x})}{\cancel{e^x}(2+x^{10}/e^x)} = \frac{1+0+0}{2+0} = \frac{1}{2}.$

Anm. ${}^a\log x < x^\alpha < b^x$, $a>1$, $\alpha>0$, $b>1$, tolkas så:

$$\lim_{x\to\infty} \frac{{}^a\log x}{x^\alpha} = \lim_{x\to\infty} \frac{{}^a\log x}{b^x} = \lim_{x\to\infty} \frac{x^\alpha}{b^x} = 0.$$

$$0<c<1 \Rightarrow \lim_{x\to\infty} c^x = 0.$$

c) För stora x är $x^4+x\ln x \approx x^4$ och $x+(\frac{2}{3})^x \approx x$, s.a.

$\lim_{x\to\infty} \frac{x^4+x\ln x}{x+(2/3)^x} = \lim_{x\to\infty} \frac{x^4}{x} = \lim_{x\to\infty} x^3 = \infty.$

Övning 1.40 (Sid. 20)

Lösning följer.

Lösning

a) $\lim_{x\to\infty} \frac{(\ln x)^{100}}{x}$ $[\xi = \ln x \Leftrightarrow x = e^\xi] = \lim_{\xi\to\infty} \frac{\xi^{100}}{e^\xi} = 0.$

Anm. ln och dess invers växer mot ∞.

b) $\lim_{x\to\infty} \frac{\ln 5x^2}{\ln 6x^3} = \lim_{x\to\infty} \frac{\ln 5+2\ln x}{\ln 6+3\ln x} = \lim_{x\to\infty} \frac{2\cancel{\ln x}}{3\cancel{\ln x}} = \frac{2}{3}.$

c) $\lim_{x\to\infty} \frac{x\ln x}{x+\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}+\frac{1}{\ln x}} = \frac{1}{0+0} = \infty.$

Övning 1.41 (Sid. 20)

Lösning

$f(x) = \frac{x+1}{2}$, $-3 \leq x \leq 1$.

$y = \frac{x+1}{2} \Leftrightarrow x+1 = 2y \Leftrightarrow x = 2y-1$;

$-3 \leq x \leq 1 \Leftrightarrow -3+1 \leq x+1 \leq 1+1 \Leftrightarrow -2 \leq x+1 \leq 2 \Leftrightarrow -1 \leq \frac{x+1}{2} \leq 1$;

Resultat: $f^{-1}(t) = 2t-1$, $-1 \leq t \leq 1$.

Övning 1.42 (Sid. 20)

Lösning

En funktion har invers om varje linje parallell med x-axeln skär dess graf högst en gång.

Funktionerna i a) och c) har en invers.

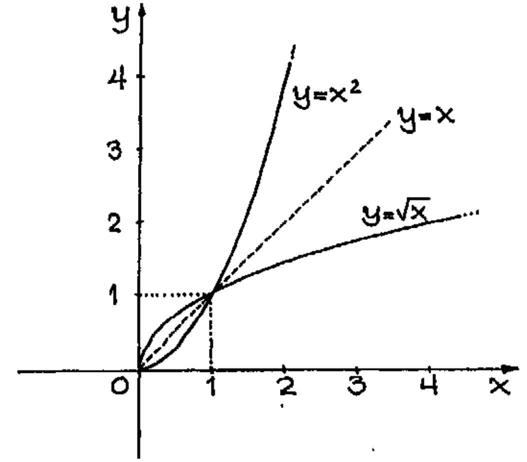
Anm. f inverterbar: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Övning 1.43 (Sid. 21)

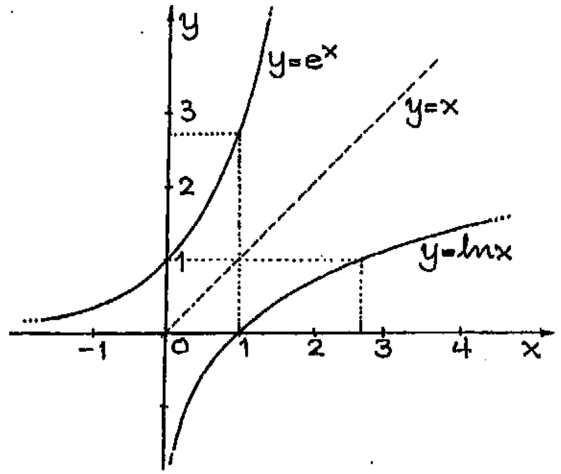
Lösning

a) $f(x)=x^2,\ x\geq 0.$

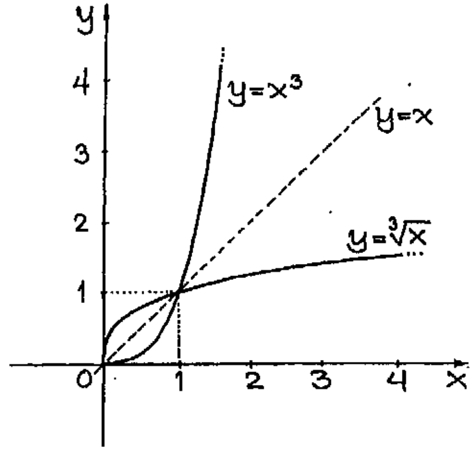
$x^2=y \Rightarrow x=\sqrt{y}=f^{-1}(y)$; $\quad f^{-1}(x)=\sqrt{x},\ x\geq 0$



b) $g(x)=e^x,\ x\in\mathbb{R}$; $\quad g^{-1}(x)=\ln x,\ x>0.$



c) $h(x)=x^3,\ x\geq 0.$ $\quad h^{-1}(x)=\sqrt[3]{x},\ x\geq 0.$



d) $f^{-1}\circ f(x)=g^{-1}\circ g(x)=h^{-1}\circ h(x)=x\ ,\ x>0.$

Övning 1.44 (Sid. 21)

Lösning

a) $f(x)=3x+4\ ,\ x\in\mathbb{R}$

$x_1<x_2 \Leftrightarrow 3x_1<3x_2 \Leftrightarrow 3x_1+4<3x_2+4 \Leftrightarrow f(x_1)<f(x_2) \Rightarrow$

$\Rightarrow f$ växande, dvs injektiv $\Rightarrow$ invers existerar.

$y=3x+4 \Leftrightarrow 3x=y-4 \Leftrightarrow x=\frac{y-4}{3}=f^{-1}(y)$; $\quad f^{-1}(x)=\frac{x-4}{3},\ x\in\mathbb{R}.$

b) $f(x)=|x|,\ x\in\mathbb{R}.$

$f(-1)=|-1|=1=f(1) \Rightarrow f$ icke-injektiv ; invers saknas.

c) $f(x)=\frac{1}{x+2},\ x>-2.$

fortş.

$-2<x_1<x_2 \Leftrightarrow 0<x_1+2<x_2+2 \Leftrightarrow \frac{1}{x_2+2}<\frac{1}{x_1+2} \Rightarrow f(x_2)<f(x_1)$

$\Rightarrow$ f avtagande, dvs. injektiv $\Rightarrow$ invers finns.

$\frac{1}{x+2}=y>0 \Leftrightarrow x+2=\frac{1}{y} \Leftrightarrow x=\frac{1}{y}-2=f^{-1}(y)$; $\underline{f^{-1}(x)=\frac{1}{x}-2,\ x>0.}$

d) $\underline{f(x)=x^2+4x+5,\ x\in\mathbb{R}}$

$f(-3)=2=f(-1) \Rightarrow f$ icke-injektiv; invers saknas.

e) $\underline{f(x)=x^2+4x+5,\ x\geq -2.}$

$x^2+4x+5=(x+2)^2+1$;

$-2<x_1<x_2 \Leftrightarrow 0<x_1+2<x_2+2 \Leftrightarrow (x_1+2)^2<(x_2+2)^2 \Leftrightarrow$

$\Leftrightarrow (x_1+2)^2+1<(x_2+2)^2+1 \Leftrightarrow f(x_1)<f(x_2) \Rightarrow f$ växande,

dvs. injektiv; invers existerar.

$(x+2)^2+1=y \Leftrightarrow (x+2)^2=y-1\geq 0 \Leftrightarrow x+2=\sqrt{y-1} \wedge y\geq 1 \Rightarrow$

$\Rightarrow \underline{f^{-1}(x)=-2+\sqrt{x-1},\ x\geq 1.}$

f) $\underline{f(x)=\sqrt{1+\frac{1}{x}},\ x>0.}$

$x_1\neq x_2 \Leftrightarrow \frac{1}{x_1}\neq\frac{1}{x_2} \Leftrightarrow \frac{1}{x_1}+1\neq\frac{1}{x_2}+1 \Leftrightarrow \sqrt{1+\frac{1}{x_1}}\neq\sqrt{1+\frac{1}{x_2}} \Leftrightarrow$

$\Leftrightarrow f(x_1)\neq f(x_2) \Rightarrow f$ injektiv $\Rightarrow f$ ~~inverterbar~~.

$y=\sqrt{1+\frac{1}{x}}>0 \Leftrightarrow 1+\frac{1}{x}=y^2 \wedge y>0 \Leftrightarrow \frac{1}{x}=y^2-1>0 \wedge y>0$

$\Leftrightarrow x=\frac{1}{y^2-1} \wedge y>1$; $\underline{f^{-1}(x)=\frac{1}{x^2-1},\ x>1.}$

Anm. $y^2-1>0 \Leftrightarrow y^2>1 \Leftrightarrow |y|>1 \overset{y>0}{\Leftrightarrow} y>1.$

Övning 1.45 (Sid. 21)

Lösning

$y=x^a \Leftrightarrow x=y^{1/a}$; $x^a=x^{1/a} \Leftrightarrow x^{a^2}=x \Leftrightarrow a^2=1 \Leftrightarrow a=\pm 1.$

Svar: $y=x$ och $y=x^{-1}$.

Övning 1.46 (Sid. 21)

Lösning

$\underline{f(x)=2x+1,\ x\in\mathbb{R}}$, $\underline{g(x)=x^2,\ x\in\mathbb{R}.}$

a) $y=2x+1 \Leftrightarrow 2x=y-1 \Leftrightarrow x=\frac{y-1}{2}$; $\underline{f^{-1}(u)=\frac{1}{2}u-\frac{1}{2},\ u\in\mathbb{R}.}$

b) $f\circ g(x)=f(g(x))=f(x^2)=\underline{2x^2+1.}$

c) $g\circ f(x)=g(f(x))=g(2x+1)=(2x+1)^2=\underline{4x^2+4x+1.}$

d) $f^{-1}\circ g(x)=f^{-1}(g(x))=f^{-1}(x^2)=\underline{\frac{1}{2}x^2-\frac{1}{2}.}$

Övning 1.47 (Sid. 21)

Lösning

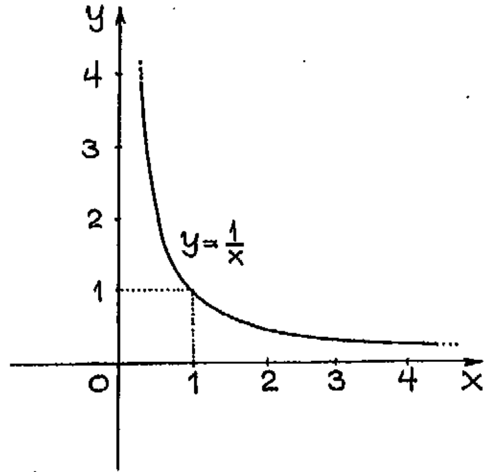
a) $\underline{y=\frac{1}{x},\ x\neq 0.}$

Grafen till $y=\frac{1}{x}$ är fallande för $x<0$ och $x>0$ för sig men inte för $x\neq 0$; $y=\frac{1}{x}$ är således inte monoton. $x_1\neq x_2 \Leftrightarrow y_1\neq y_2 \Rightarrow y=\frac{1}{x}$ injektiv.

$\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$ och $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$, så $y = \frac{1}{x}$ är obegränsad.

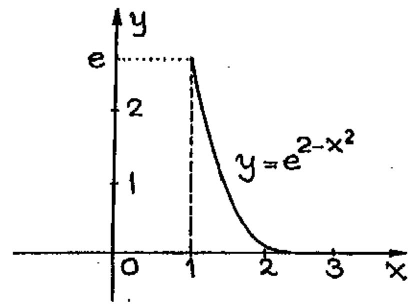
Anm. Grafen finns uppritad i Ö. 1.21.

b) $y = \frac{1}{x}$, $x > 0$.



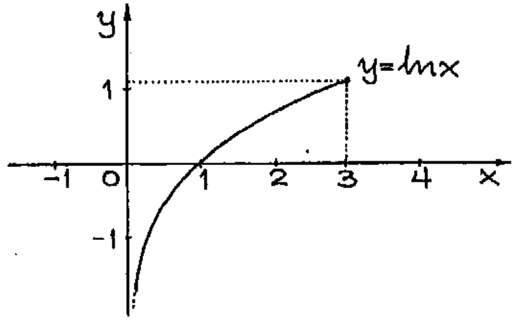
$0 < x_1 < x_2 \Leftrightarrow \frac{1}{x_2} < \frac{1}{x_1} \Rightarrow y = 1/x$ är strängt monotont avtagande och således injektiv. Den är begränsad nedåt men inte uppåt. $y = \frac{1}{x}$ är inte begränsad.

c) $y = e^{2-x^2}$, $x \geq 1$.



$y = e^{2-x^2}$ är monoton, injektiv och begränsad.

d) $y = \ln x$, $0 < x < 3$.



$y = \ln x$, $0 < x < 3$, är monoton (växande), injektiv och uppåt begränsad; den är inte nedåt begränsad.

Övning 1.47 (Sid. 21)

Lösning

a) $x_1 < x_2 \Rightarrow \begin{cases} f(x_1) \leq f(x_2) \\ g(x_1) \leq g(x_2) \end{cases} \Leftrightarrow \begin{cases} f(x_2) - f(x_1) \geq 0 \\ g(x_2) - g(x_1) \geq 0 \end{cases} \Rightarrow$

$\Rightarrow f(x_2) - f(x_1) + g(x_2) - g(x_1) = f(x_2) + g(x_2) - (f(x_1) + g(x_1)) =$

$= (f+g)(x_2) - (f+g)(x_1) \geq 0 \Leftrightarrow (f+g)(x_1) \leq (f+g)(x_2) \Rightarrow$

$\Rightarrow f+g$ växande.

b) Antag att f och g är växande och positiva.

$$x_1<x_2 \Rightarrow \begin{cases} 0<f(x_1)\leq f(x_2) \\ 0<g(x_1)\leq g(x_2) \end{cases} \Leftrightarrow \begin{cases} 1< f(x_2)/f(x_1) \\ 1< g(x_2)/g(x_1) \end{cases} \Rightarrow$$

$$\Rightarrow 1< \frac{f(x_2)}{f(x_1)}\cdot\frac{g(x_2)}{g(x_1)} \Leftrightarrow f(x_1)\cdot g(x_1) \leq f(x_2)\cdot g(x_2) \Leftrightarrow (f\cdot g)(x_1) \leq$$

$\leq (f\cdot g)(x_2) \Rightarrow f\cdot g$ växande.

Antag nu att både f och g är <u>växande men negativa</u>, dvs. $f(x)<0$ och $g(x)<0$.

$$x_1<x_2 \Rightarrow \begin{cases} f(x_1)\leq f(x_2)<0 \\ g(x_1)\leq g(x_2)<0 \end{cases} \Leftrightarrow \begin{cases} 0< f(x_2)/f(x_1)\leq 1 \\ 0< g(x_2)/g(x_1)\leq 1 \end{cases} \Rightarrow$$

$$\Rightarrow 0< \frac{f(x_2)}{f(x_1)}\cdot\frac{g(x_2)}{g(x_1)} \leq 1 \Leftrightarrow f(x_2)g(x_2) \leq f(x_1)g(x_1) \Leftrightarrow$$

$\Leftrightarrow (f\cdot g)(x_2) \leq (f\cdot g)(x_1) \Rightarrow f\cdot g$ avtagande.

<u>Resultat</u>: a) Se ovan! b) Produkten är i allmänhet inte växande.

<u>Övning 1.49 (Sid. 21)</u>

<u>Lösning</u>

a) $f_1(x)=x^2 \Rightarrow f_1(-x)=(-x)^2=x^2=f_1(x) \Rightarrow f_1(x)=x^2$ <u>jämn</u>.

b) $f_2(x)=x^3 \Rightarrow f_2(-x)=(-x)^3=-x^3=-f_2(x) \Rightarrow f_2(x)=x^3$ <u>udda</u>.

c) $f_3(x)=x^2+2x+1 \Rightarrow f_3(-x)=(-x)^2+2(-x)+1=x^2-2x+1 \neq$
$\neq \pm f(x) \Rightarrow f_3$ är <u>varken jämn eller udda</u>.

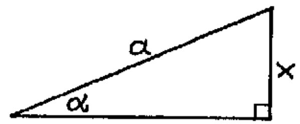
d) $f_4(x)=x(e^x-e^{-x}) \Rightarrow f_4(-x)=(-x)(e^{-x}-e^{-(-x)})=-x(e^{-x}+e^x)=$
$=-x(e^x+e^{-x})=-f_4(x) \Rightarrow$ <u>f_4 udda</u>.

e) $f_5(x)=x\ln x \Rightarrow D_{f_5}=]0,\infty[$; $x\in D_{f_5} \Rightarrow -x\notin D_{f_5}$; f_5 är <u>varken jämn eller udda</u>.

<u>Övning 1.50 (Sid. 21)</u>

<u>Lösning</u>

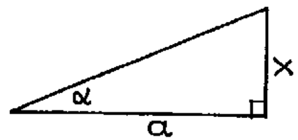
a)



(le)

$\sin\alpha=\frac{x}{a} \Leftrightarrow$ <u>$x=a\cdot\sin\alpha$</u>.

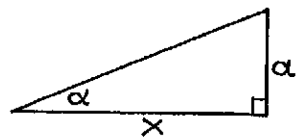
b)



(le)

$\tan\alpha=\frac{x}{a} \Leftrightarrow$ <u>$x=a\tan\alpha$</u>.

c)

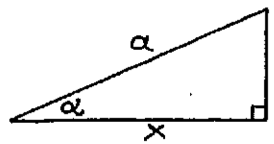


(le)

$\tan\alpha=\frac{a}{x} \Leftrightarrow$ <u>$x=a/\tan\alpha$</u>.

<u>Anm</u>. $\tan\alpha=\frac{\sin\alpha}{\cos\alpha} \Rightarrow \frac{1}{\tan\alpha}=\frac{\cos\alpha}{\sin\alpha}=\cot\alpha$.
Svaret kan alltså skrivas $x=a\cdot\cot\alpha$.

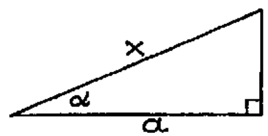
d)



(le)

$\cos\alpha = x/a \Leftrightarrow \underline{x = a\cdot\cos\alpha}.$

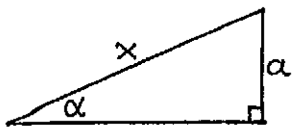
e)



(le)

$\cos\alpha = \frac{\alpha}{x} \Leftrightarrow \frac{x}{a} = \frac{1}{\cos\alpha} \Leftrightarrow \underline{x = \frac{a}{\cos\alpha}}.$

f)



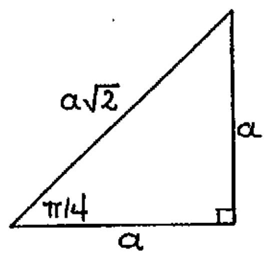
(le)

$\sin\alpha = \frac{a}{x} \Leftrightarrow \frac{x}{a} = \frac{1}{\sin\alpha} \Leftrightarrow \underline{x = \frac{a}{\sin\alpha}}.$

Övning 1.51 (Sid. 22)

Lösning

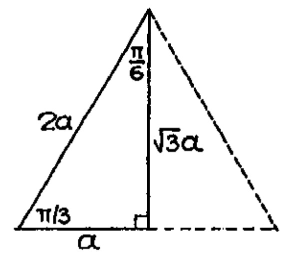
a)



(le)

$$\left.\begin{aligned} \sin\frac{\pi}{4} &= \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} \\ \cos\frac{\pi}{4} &= \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1.$$

b)



(le)

$$\left.\begin{aligned} \sin\frac{\pi}{3} &= \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2} \\ \cos\frac{\pi}{3} &= \frac{a}{2a} = \frac{1}{2} \end{aligned}\right\} \Rightarrow \tan\frac{\pi}{3} = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

$$\left.\begin{aligned} \sin\frac{\pi}{6} &= \frac{a}{2a} = \frac{1}{2} \\ \cos\frac{\pi}{6} &= \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2} \end{aligned}\right\} \Rightarrow \tan\frac{\pi}{6} = \frac{\sin(\pi/6)}{\cos(\pi/6)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}.$$

Övning 1.52 (Sid. 22)

Lösning

$(\cos\alpha)^2 + (\sin\alpha)^2 = 1 \Rightarrow (\cos\alpha)^2 + 0{,}36 = 1 \Leftrightarrow (\cos\alpha)^2 = 0{,}64$

$\Leftrightarrow \underline{\cos\alpha = -0{,}8} \lor \underline{\cos\alpha = 0{,}8}.$

Övning 1.53 (Sid. 22)

Lösning

a) $\sin x = \frac{1}{2} \Leftrightarrow \underline{x = \frac{\pi}{6} + m\cdot 2\pi} \lor \underline{x = \frac{5\pi}{6} + n\cdot 2\pi}, \quad m, n \in \mathbb{Z}.$

b) $\sin x = -\frac{\sqrt{3}}{2} \Leftrightarrow \sin(x+\pi) = \frac{\sqrt{3}}{2} \Leftrightarrow x + \pi = \frac{\pi}{3} + m\cdot 2\pi \lor$

$\lor\ x + \pi = \frac{2\pi}{3} + n\cdot 2\pi \Leftrightarrow \underline{x = -\frac{2\pi}{3} + m2\pi} \lor \underline{x = -\frac{\pi}{3} + n2\pi}.$

c) $\cos x = \frac{1}{\sqrt{2}} \Leftrightarrow \underline{x = \frac{\pi}{4} + m2\pi} \vee \underline{x = -\frac{\pi}{4} + n2\pi}$, $\underline{m, n \in \mathbb{Z}}$.

d) $\tan x = -1 \Leftrightarrow \underline{x = -\frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}}$.

e) $\tan x = \sqrt{3} \Leftrightarrow \underline{x = \frac{\pi}{3} + n\pi,\ n \in \mathbb{Z}}$.

f) $\cos 3x = \frac{1}{2} \Leftrightarrow 3x = \pm\frac{\pi}{3} + n \cdot 2\pi \Leftrightarrow \underline{x = \pm\frac{\pi}{9} + n \cdot \frac{2\pi}{3},\ n \in \mathbb{Z}}$.

Övning 1.54 (Sid. 22)

Lösning

$\sin 3x = \sin x \Leftrightarrow 3x = x + m2\pi \vee 3x = \pi - x + n2\pi \Leftrightarrow$

$\Leftrightarrow 2x = m2\pi \vee 4x = (2n+1)\pi \Leftrightarrow \underline{x = m\pi} \vee \underline{x = \frac{2n+1}{4}\pi}$.

Övning 1.55 (Sid. 22)

Lösning

a) $\cos 3x = \cos x \Leftrightarrow 3x = x + 2m\pi \vee 3x = -x + 2n'\pi \Leftrightarrow$

$\Leftrightarrow 2x = 2n\pi \vee 4x = 2n'\pi \Leftrightarrow x = n\pi \vee x = n' \cdot \frac{\pi}{2} \Leftrightarrow \underline{x = k \cdot \frac{\pi}{2}}$.

b) $\tan 3x = \tan x \Leftrightarrow 3x = x + n\pi \Leftrightarrow 2x = n\pi \Leftrightarrow \underline{x = n\frac{\pi}{2},\ n \in \mathbb{Z}}$.

Övning 1.56 (Sid. 22)

Lösning

a) $\cos 2x = \sin x = \cos(x - \frac{\pi}{2}) \Leftrightarrow 2x = x - \frac{\pi}{2} + m2\pi \vee 2x = \frac{\pi}{2} - x + 2n\pi$

$\Leftrightarrow x = (2m - \frac{1}{2})\pi \vee 3x = (2n + \frac{1}{2})\pi \Leftrightarrow \underline{x = \frac{4m-1}{2}\pi} \vee \underline{x = \frac{4n+1}{6}\pi}$.

Övning 1.57 (Sid. 22)

Lösning

$\cos 4x = \sin x = \cos(x - \frac{\pi}{2}) \Leftrightarrow 4x = x - \frac{\pi}{2} + m2\pi \vee 4x = \frac{\pi}{2} - x + n2\pi$

$\Leftrightarrow 3x = \frac{2m-1}{2}\pi \vee 5x = \frac{4n+1}{2}\pi \Leftrightarrow \underline{x = \frac{2m-1}{6}\pi} \vee \underline{x = \frac{4n+1}{10}\pi}$.

Övning 1.58 (Sid. 22)

Lösning

$A\sin(2x+\delta) = A\cos\delta \sin 2x + A\sin\delta \cos 2x = -4\sin 2x + 3\cos 2x$

$$\Leftrightarrow \begin{cases} A\cos\delta = -4 \\ A\sin\delta = 3 \end{cases} \Leftrightarrow \begin{cases} A^2 = 25 \\ A\cos\delta = -4 \\ A\sin\delta = 3 \end{cases} \Leftrightarrow \begin{cases} A = 5 \\ \cos\delta = -4/5 \\ \sin\delta = 3/5 \end{cases} \Leftrightarrow \begin{cases} A = 5 \\ \delta = 2{,}498 \end{cases}$$

Övning 1.59 (Sid. 22)

Lösning

a) $A\sin(x+\delta) = A\cos\delta \sin x + A\sin\delta \cos x = \sin x + \cos x \Leftrightarrow$

$$\Leftrightarrow \begin{cases} A\cos\delta = 1 \\ A\sin\delta = 1 \end{cases} \Leftrightarrow \begin{cases} A^2 = 2 \\ A\cos\delta = 1 \\ A\sin\delta = 1 \end{cases} \Leftrightarrow \begin{cases} A = \sqrt{2} \\ \cos\delta = 1/\sqrt{2} \\ \sin\delta = 1/\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} A = \sqrt{2} \\ \delta = \frac{\pi}{4} \end{cases}$$

b) $A\sin(2x+\delta) = A\cos\delta \sin 2x + A\sin\delta \cos 2x = -\sin 2x + \sqrt{3}\cos 2x$

$\Leftrightarrow A\cos\delta = -1 \wedge A\sin\delta = \sqrt{3} \Leftrightarrow A^2 = 4 \wedge \tan\delta = -\sqrt{3} \wedge \frac{\pi}{2} < \delta < \pi$

$\Leftrightarrow \underline{A = 2 \wedge \delta = \frac{2\pi}{3}}$.

c) $A\sin(3x+\delta) = A\cos\delta\sin 3x + A\sin\delta\cos 3x = -\cos 3x + \sqrt{3}\sin 3x$

$$\Leftrightarrow \begin{cases} A\cos\delta = \sqrt{3} \\ A\sin\delta = -1 \\ A>0 \end{cases} \Leftrightarrow \begin{cases} A^2 = 4 \\ A\cos\delta = \sqrt{3} \\ A\sin\delta = -1 \end{cases} \Leftrightarrow \begin{cases} A=2 \\ \cos\delta = \sqrt{3}/2 \\ \sin\delta = -1/2 \end{cases} \Leftrightarrow \begin{cases} A=2 \\ \delta = -\frac{\pi}{6} \end{cases}.$$

Övning 1.60 (Sid. 22)

Lösning

$A\sin(x+\delta) = A\cos\delta\sin x + A\sin\delta\cos x = \sin x + 2\cos x \Leftrightarrow$

$$\Leftrightarrow \begin{cases} A\cos\delta = 1 \\ A\sin\delta = 2 \\ A>0 \end{cases} \Leftrightarrow \begin{cases} A^2 = 5 \\ A\cos\delta = 1 \\ A\sin\delta = 2 \end{cases} \Leftrightarrow \begin{cases} A=\sqrt{5} \\ \tan\delta = 2 \end{cases} \Leftrightarrow \begin{cases} A=\sqrt{5} \\ \delta = 1{,}107 \end{cases};$$

$\Rightarrow \sin x + 2\cos x = \sqrt{5}\sin(x+1{,}107)$;

$-1 \leq \sin(x+1{,}107) \leq 1 \Leftrightarrow -\sqrt{5} \leq \sqrt{5}\sin(x+1{,}107) \leq \sqrt{2} \Leftrightarrow$

$\Leftrightarrow -\sqrt{5} \leq \sin x + 2\cos x \leq \sqrt{5} \Leftrightarrow |\sin x + 2\cos x| \leq \sqrt{5}$.

Övning 1.61 (Sid. 22)

Lösning

a) $\sin x\cos x = \frac{1}{4} \Leftrightarrow 2\sin x\cos x = \frac{1}{2} \Leftrightarrow \sin 2x = \frac{1}{2} = \sin\frac{\pi}{6} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 2x = \frac{\pi}{6} + 2m\pi = \frac{12m+1}{6}\pi \\ 2x = \frac{5\pi}{6} + 2n\pi = \frac{12n+5}{6}\pi \end{cases} \Leftrightarrow \begin{cases} x = \frac{12m+1}{12}\pi, \ m\in\mathbb{Z}. \\ x = \frac{12n+5}{12}\pi, \ n\in\mathbb{Z}. \end{cases}$$

b) $\cos^2 x - \sin^2 x = \frac{\sqrt{3}}{2} \Leftrightarrow \cos 2x = \frac{\sqrt{3}}{2} = \cos(\pm\frac{\pi}{6}) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 2x = \frac{\pi}{6} + 2m\pi = \frac{12m+1}{6}\pi \\ 2x = -\frac{\pi}{6} + 2n\pi = \frac{12n-1}{6}\pi \end{cases} \Leftrightarrow \begin{cases} x = \frac{12m+1}{12}\pi, \ m\in\mathbb{Z}. \\ x = \frac{12n-1}{12}\pi, \ n\in\mathbb{Z}. \end{cases}$$

c) $\sin x + \cos x = \frac{1}{\sqrt{2}} \overset{!}{\Leftrightarrow} \sqrt{2}\sin(x+\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \Leftrightarrow \sin(x+\frac{\pi}{4}) = \frac{1}{2}$

$$\Leftrightarrow \begin{cases} x+\frac{\pi}{4} = \frac{\pi}{6} + 2m\pi \\ x+\frac{\pi}{4} = \frac{5\pi}{6} + 2n\pi \end{cases} \Leftrightarrow \begin{cases} x = -\frac{\pi}{12} + m2\pi \\ x = \frac{7\pi}{12} + n\cdot 2\pi \end{cases} \Leftrightarrow \begin{cases} x = \frac{24m-1}{12}\pi. \\ x = \frac{24n+7}{12}\pi. \end{cases}$$

Anm. I $\overset{!}{\Leftrightarrow}$ har jag utnyttjat resultatet i Ö 1.59a.

d) $\cos 2x + 3\cos x - 1 = 0 \Leftrightarrow 2\cos^2 x - 1 + 3\cos x - 1 = 0 \Leftrightarrow 2\cos^2 x + 3\cos x - 2 = 0 \Leftrightarrow \cos^2 x + \frac{3}{2}\cos x - 1 = 0 \Leftrightarrow \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Leftrightarrow x = \frac{\pi}{3} + m2\pi \ \vee \ x = -\frac{\pi}{3} + n2\pi$, m, n heltal.

e) $\sin 4x = \cos 3x = \sin(\frac{\pi}{2} - 3x) \Leftrightarrow \begin{cases} 4x = \frac{\pi}{2} - 3x + m\cdot 2\pi \\ 4x = \frac{\pi}{2} + 3x + n\cdot 2\pi \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 7x = \frac{\pi}{2} + m\cdot 2\pi = \frac{4m+1}{2}\pi \\ x = \frac{\pi}{2} + n\cdot 2\pi = \frac{4n+1}{2}\pi \end{cases} \Leftrightarrow \begin{cases} x = \frac{4m+1}{14}\pi, \ m\in\mathbb{Z}. \\ x = \frac{4n+1}{2}\pi, \ n\in\mathbb{Z}. \end{cases}$$

f) $\cos 2x = 3\sin x + 2 \Leftrightarrow 1 - 2\sin^2 x = 3\sin x + 2 \Leftrightarrow 2\sin^2 x + 3\sin x + 1 = 0 \Leftrightarrow \sin^2 x + \frac{3}{2}\sin x + \frac{1}{2} = 0 \Leftrightarrow (t = \sin x) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \sin x = -1 \\ \sin x = -\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} x = -\frac{\pi}{2} + m2\pi, \ m\in\mathbb{Z}. \\ x = -\frac{\pi}{6} + n2\pi \ \vee \ x = \frac{7\pi}{6} + k\cdot 2\pi, \ n,k\in\mathbb{Z}. \end{cases}$$

Anm. $m\in\mathbb{Z}$ utläses "m är ett godtyckligt heltal".

Övning 1.62 (Sid. 23)

Lösning

$HL = (\sin x+\cos x)^2 = \cos^2 x+\sin^2 x+2\sin x\cos x = 1+\sin 2x = VL.$

Övning 1.63 (Sid. 23)

Lösning

$VL = \sin(x+\frac{\pi}{6})+\sin(x+\frac{\pi}{3}) =$

$= \sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}+\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3} =$

$= \sin x(\cos\frac{\pi}{6}+\cos\frac{\pi}{3})+\cos x(\sin\frac{\pi}{6}+\sin\frac{\pi}{3}) =$

$= \sin x\cdot(\frac{\sqrt{3}}{2}+\frac{1}{2})+\cos x(\frac{1}{2}+\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}+1}{2}(\cos x+\sin x) = HL.$

Övning 1.64 (Sid. 23)

Lösning

a) $-1\leq\cos x\leq 1 \Leftrightarrow |\cos x|\leq 1 \Leftrightarrow 0\leq\cos^2 x\leq 1 \Leftrightarrow -3\leq -3\cos^2 x\leq 0$

$\Leftrightarrow 1-3\leq 1-3\cos^2 x\leq 0+1 \Leftrightarrow -2\leq f(x)\leq 1 \Leftrightarrow \underline{V_f: -2\leq y\leq 1.}$

b) $g(x) = 2\cos 2x-\sin^2 x = 2\cos 2x-\frac{1-\cos 2x}{2} = \frac{5\cos 2x-1}{2};$

$-1\leq\cos 2x\leq 1 \Leftrightarrow -5\leq 5\cos 2x\leq 5 \Leftrightarrow -5-1\leq 5\cos 2x-1\leq 5-1 \Leftrightarrow$

$\Leftrightarrow -6\leq 5\cos 2x-1\leq 4 \Leftrightarrow -3\leq\frac{5\cos 2x-1}{2}\leq 2 \Rightarrow \underline{V_g: -3\leq y\leq 2.}$

Anm. Författarna använder bokstaven x i V_f.

Övning 1.65 (Sid. 23)

Lösning

$$\sin^2\frac{x}{2}+\frac{\frac{1}{\sin 2x}+\frac{1}{2\sin x}}{\tan x+\cot x} = \frac{1-\cos x}{2}+\frac{\frac{1}{2\sin x\cos x}+\frac{1}{2\sin x}}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}} =$$

$$= \frac{1-\cos x}{2}+\frac{\frac{1}{2\sin x\cdot\cos x}+\frac{\cos x}{2\sin x\cos x}}{\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}} = \frac{1-\cos x}{2}+$$

$$+\frac{\frac{1+\cos x}{2\sin x\cos x}}{\frac{2}{2\sin x\cos x}} = \frac{1-\cos x}{2}+\frac{1+\cos x}{2} = \frac{1-\cos x+1+\cos x}{2} = 1.$$

Övning 1.66 (Sid. 23)

Lösning

$$VL = \frac{1-\sin\theta}{1-\cos\theta} = \frac{1-\cos(\frac{\pi}{2}-\theta)}{1+\cos(\frac{\pi}{2}-\theta)} = \frac{2\sin^2\frac{1}{2}(\frac{\pi}{2}-\theta)}{2\cos^2\frac{1}{2}(\frac{\pi}{2}-\theta)} = \frac{\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}{\cos^2(\frac{\pi}{4}-\frac{\theta}{2})} =$$

$$= \left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{\cos(\frac{\pi}{4}-\frac{\theta}{2})}\right)^2 = (\tan(\frac{\pi}{4}-\frac{\theta}{2}))^2 = \tan^2(\frac{\pi}{4}-\frac{\theta}{2}) = HL.$$

Övning 1.67 (Sid. 23)

Lösning

$$\underline{\sin x} = \sin 2\cdot\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} =$$

$$= \frac{\cancel{\cos^2(x/2)}(2\cdot\sin(x/2)/\cos(x/2))}{\cancel{\cos^2(x/2)}(1+\sin^2(x/2)/\cos^2(x/2))} = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \underline{\frac{2t}{1+t^2}}.$$

$\underline{\cos x} = \cos 2\left(\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2(x/2) - \sin^2(x/2)}{\cos^2(x/2) + \sin^2(x/2)} =$

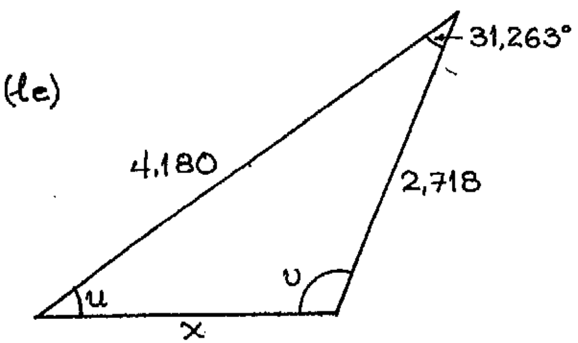
$= \frac{\cancel{\cos^2}(x/2)(1-\tan^2(x/2))}{\cancel{\cos^2}(x/2)(1+\tan^2(x/2))} = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \underline{\frac{1-t^2}{1+t^2}}.$

Övning 1.68 (Sid. 23)

Lösning

a)

(le)



Cos-satsen $\Rightarrow x^2 = 4{,}180^2 + 2{,}718^2 - 2\cdot 4{,}180\cdot 2{,}718\cos 31{,}263° =$

$= 5{,}4368812 \Leftrightarrow \underline{x = 2{,}332}.$

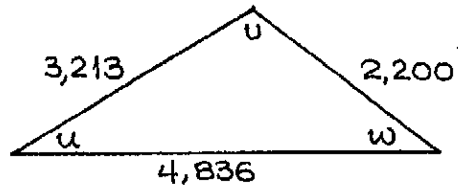
Sin-satsen $\Rightarrow \frac{\sin u}{2{,}718} = \frac{\sin 31{,}263°}{2{,}332} = \frac{\sin v}{4{,}180} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \frac{\sin u}{2{,}718} = \frac{\sin 31{,}263°}{2{,}332} \\ \frac{\sin v}{4{,}180} = \frac{\sin 31{,}263°}{2{,}332} \end{cases} \Leftrightarrow \begin{cases} \sin u = 0{,}6049 \\ \sin v = 0{,}9302 \end{cases} \Leftrightarrow \begin{cases} u = 37{,}219° \\ v = 111{,}518° \end{cases}.$

Triangelns övriga vinklar är 37,219° & 111,518°; den tredje sidan är 2,332 le.

Anm. $31{,}263° + 37{,}219° + 111{,}518° = 180°$.

b)



Cos-satsen $\Rightarrow \cos v = \frac{4{,}836^2 - 2{,}200^2 - 3{,}213^2}{-2\cdot 3{,}213\cdot 2{,}200} = -0{,}582$

$\Leftrightarrow \underline{v = 125{,}570°}$

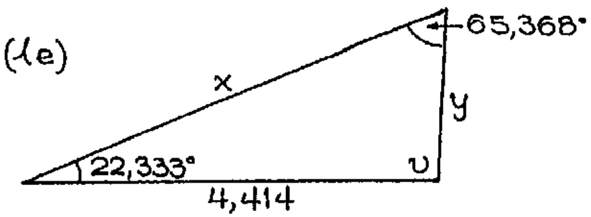
Sin-satsen $\Rightarrow \frac{\sin w}{3{,}213} = \frac{\sin 125{,}570°}{4{,}836} \Leftrightarrow \sin w = 0{,}5395$

$\Leftrightarrow w = 32{,}712° \Rightarrow u = 180° - v - w = 21{,}718°.$

Triangelns vinklar är 32,712°, 21,718° & 125,570°

c)

(le)



$v = 180° - 22{,}333° - 65{,}368° = \underline{92{,}299°}.$

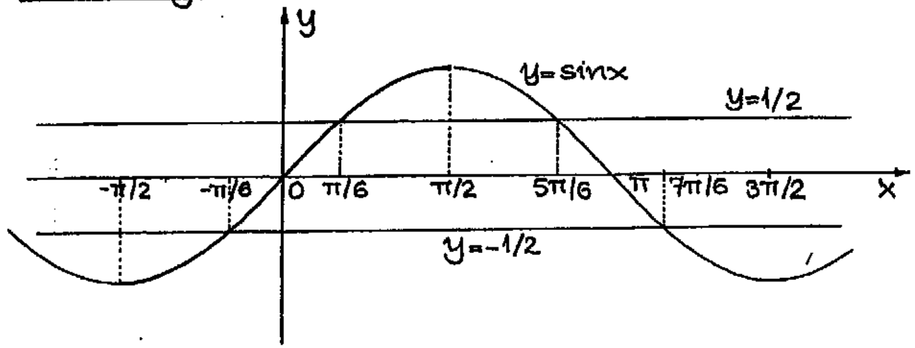
Sin-satsen $\Rightarrow \frac{x}{\sin 92{,}299°} = \frac{4{,}414}{\sin 65{,}368°} = \frac{y}{\sin 22{,}333°}$

$\Leftrightarrow \begin{cases} \frac{x}{\sin 92{,}299°} = \frac{4{,}414}{\sin 65{,}368°} \\ \frac{y}{\sin 22{,}333°} = \frac{4{,}414}{\sin 65{,}368°} \end{cases} \Leftrightarrow \begin{cases} x = 4{,}852 \text{ (le)}. \\ y = 1{,}845 \text{ (le)}. \end{cases}$

Triangeln tredje vinkel är 92,299°; de andra två sidorna är 4,852 och 1,845.

Övning 1.69 (Sid. 24)

Lösning



a) $\sin x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{6} + m\,2\pi \ \vee\ x = \frac{5\pi}{6} + n\,2\pi \quad (m, n \in \mathbb{Z})$.

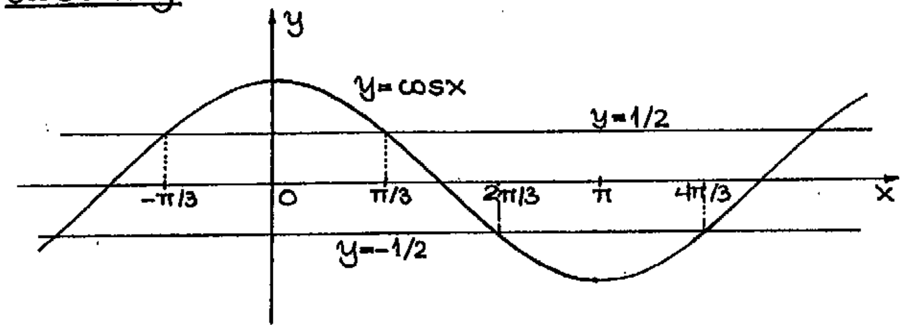
b) $x = \arcsin\frac{1}{2} \Leftrightarrow \sin x = \frac{1}{2} \wedge -\frac{\pi}{2} < x < \frac{\pi}{2} \Leftrightarrow x = \frac{\pi}{6}$.

c) $\sin x = -\frac{1}{2} \Leftrightarrow x = -\frac{\pi}{6} + m \cdot 2\pi \ \vee\ x = \frac{7\pi}{6} + n\,2\pi \quad (m, n \in \mathbb{Z})$.

d) $x = \arcsin(-\frac{1}{2}) \Leftrightarrow \sin x = -\frac{1}{2} \wedge -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Leftrightarrow x = -\frac{\pi}{6}$.

Övning 1.70 (Sid. 24)

Lösning



a) $\cos x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{3} + m\,2\pi \ \vee\ x = -\frac{\pi}{3} + n \cdot 2\pi \quad (m, n \in \mathbb{Z})$.

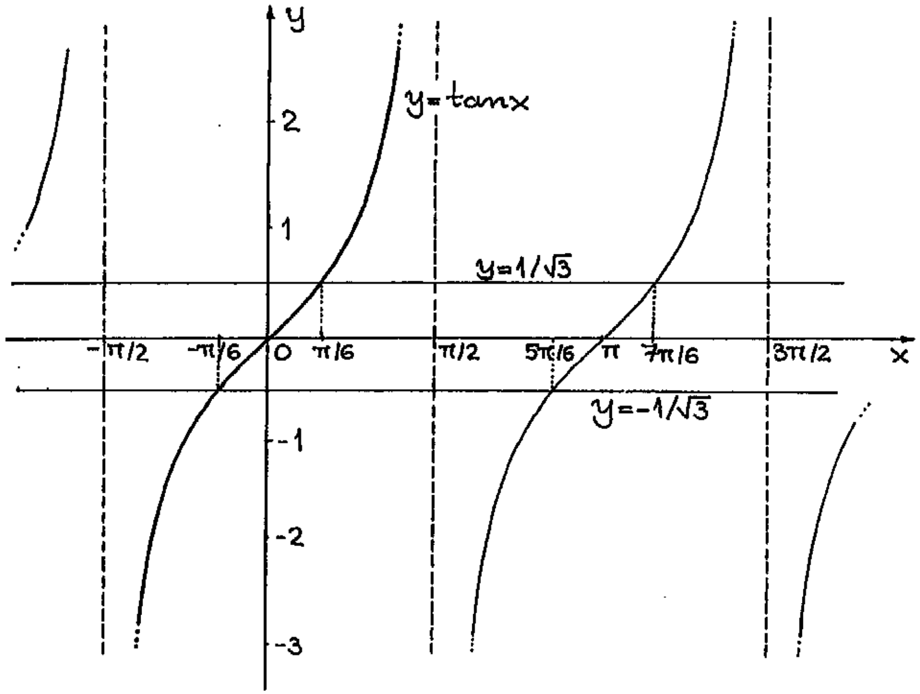
b) $x = \arccos\frac{1}{2} \Leftrightarrow \cos x = \frac{1}{2} \wedge 0 \leq x \leq \pi \Leftrightarrow x = \frac{\pi}{3}$.

c) $\cos x = -\frac{1}{2} \Leftrightarrow x = \frac{2\pi}{3} + m\,2\pi \ \vee\ x = \frac{4\pi}{3} + n\,2\pi \quad (m, n \in \mathbb{Z})$.

d) $x = \arccos(-\frac{1}{2}) \Leftrightarrow \cos x = -\frac{1}{2} \wedge 0 \leq x \leq \pi \Leftrightarrow x = \frac{2\pi}{3}$.

Övning 1.71 (Sid. 24)

Lösning



a) $\tan x = 1/\sqrt{3} \Leftrightarrow x = \frac{\pi}{6} + n\pi,\ n \in \mathbb{Z}$.

b) $x = \arctan\frac{1}{\sqrt{3}} \Leftrightarrow \tan x = \frac{1}{\sqrt{3}} \wedge -\frac{\pi}{2} < x < \frac{\pi}{2} \Leftrightarrow x = \frac{\pi}{6}$.

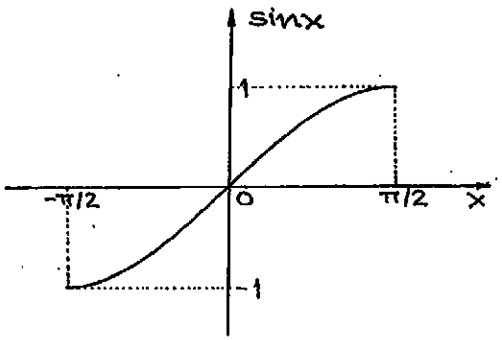
c) $\tan x = -\frac{1}{\sqrt{3}} \Leftrightarrow x = -\frac{\pi}{6} + n\pi,\ n \in \mathbb{Z}$.

d) $x = \arctan(-\frac{1}{\sqrt{3}}) \Leftrightarrow \tan x = -\frac{1}{\sqrt{3}} \wedge -\frac{\pi}{2} < x < \frac{\pi}{2} \Leftrightarrow x = -\frac{\pi}{6}$.

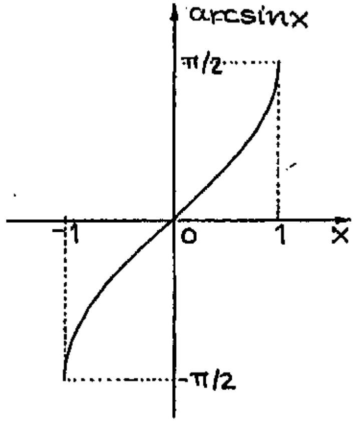
Övning 1.72 (Sid. 24)

Lösning

a)

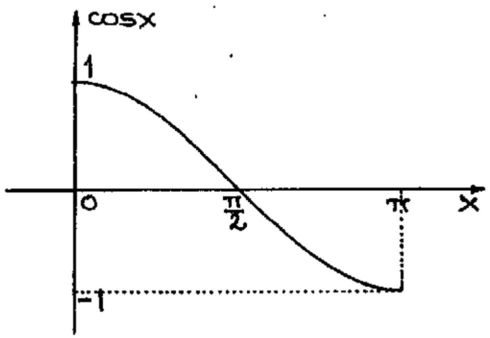


$\begin{cases} D: -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ V: -1 \le y \le 1 \end{cases}$

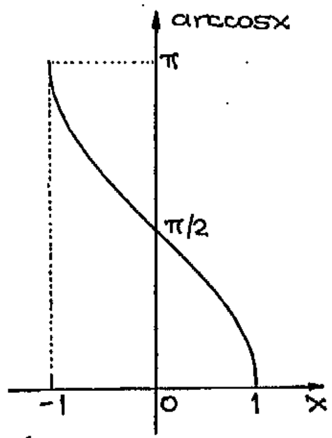


$\begin{cases} D: -1 \le x \le 1 \\ V: -\frac{\pi}{2} \le y \le \frac{\pi}{2} \end{cases}$

b)



$\begin{cases} D: 0 \le x \le \pi \\ V: -1 \le y \le 1 \end{cases}$

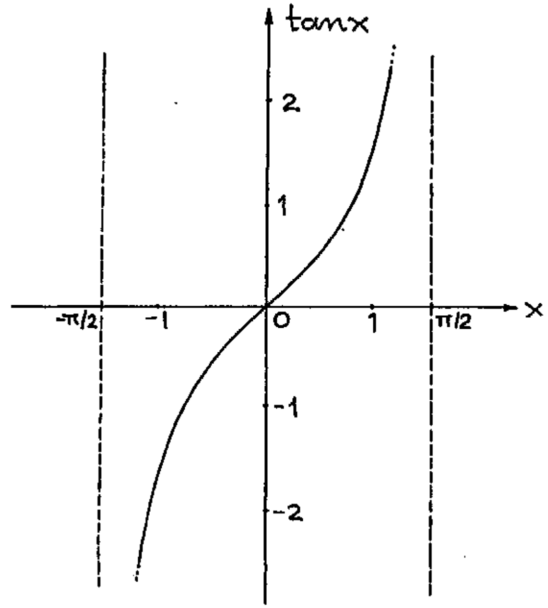


$\begin{cases} D: -1 \le x \le 1 \\ V: 0 \le y \le \pi. \end{cases}$

Övning 1.74 (Sid. 24)

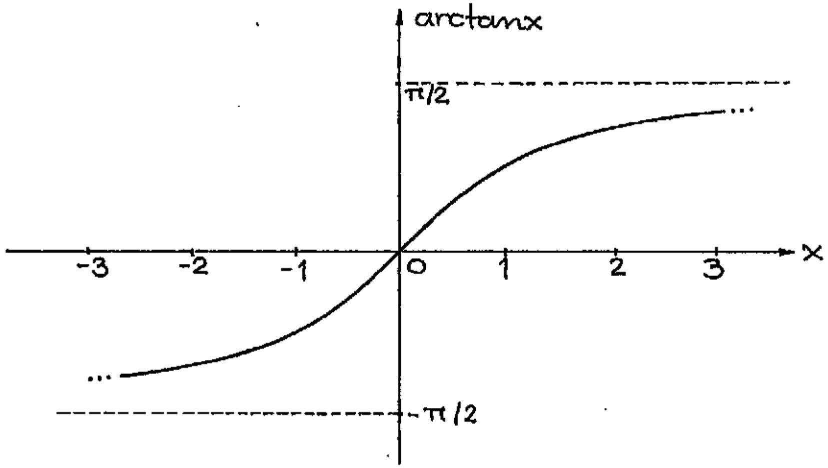
Lösning

a)



$D: -\frac{\pi}{2} < x < \frac{\pi}{2}$; $V: -\infty < y < \infty$.

b)



$D: -\infty < x < \infty$; $V: -\frac{\pi}{2} < y < \frac{\pi}{2}$.

Övning 1.76 (Sid. 24)

Lösning

a) $u=\arcsin 1 \Leftrightarrow \sin u=1 \wedge -\frac{\pi}{2}\leq u\leq\frac{\pi}{2} \Leftrightarrow u=\arcsin 1=\frac{\pi}{2}.$

$v=\arccos 1 \Leftrightarrow \cos v=1 \wedge 0\leq v\leq\pi \Leftrightarrow v=\arccos 1=0.$

b) $u=\arcsin(-1) \Leftrightarrow \sin u=-1 \wedge -\frac{\pi}{2}\leq u\leq\frac{\pi}{2} \Leftrightarrow u=\arcsin(-1)=-\frac{\pi}{2}$

$v=\arccos(-1) \Leftrightarrow \cos v=-1 \wedge 0\leq v\leq\pi \Leftrightarrow v=\arccos(-1)=\pi.$

c) $u=\arcsin(-\frac{\sqrt{3}}{2}) \Leftrightarrow \sin u=-\frac{\sqrt{3}}{2} \wedge |u|\leq\frac{\pi}{2} \Leftrightarrow u=\arcsin(-\frac{\sqrt{3}}{2})=-\frac{\pi}{3}.$

$v=\arccos(-\frac{\sqrt{3}}{2}) \Leftrightarrow \cos v=-\frac{\sqrt{3}}{2} \wedge 0\leq v\leq\pi \Leftrightarrow v=\arccos(-\frac{\sqrt{3}}{2})=\frac{5\pi}{6}.$

d) $\pi\notin D_{\arcsin}$ och $\pi\notin D_{\arccos}$; lösning saknas.

e) $u=\arcsin\frac{1}{2} \Leftrightarrow \sin u=\frac{1}{2} \wedge |u|\leq\frac{\pi}{2} \Leftrightarrow u=\arcsin\frac{1}{2}=\frac{\pi}{6}.$

$v=\arccos\frac{1}{2} \Leftrightarrow \cos v=\frac{1}{2} \wedge 0\leq v\leq\pi \Leftrightarrow v=\arccos\frac{1}{2}=\frac{\pi}{3}.$

f) $u=\arcsin 0 \Leftrightarrow \sin u=0 \wedge |u|\leq\frac{\pi}{2} \Leftrightarrow u=\arcsin 0=0.$

$v=\arccos 0 \Leftrightarrow \cos v=0 \wedge 0\leq v\leq\pi \Leftrightarrow v=\arccos 0=\pi/2.$

g) $u=\arcsin(-\frac{1}{\sqrt{2}}) \Leftrightarrow \sin u=-\frac{1}{\sqrt{2}} \wedge |u|\leq\frac{\pi}{2} \Leftrightarrow u=-\pi/4.$

$v=\arccos(-\frac{1}{\sqrt{2}}) \Leftrightarrow \cos v=-\frac{1}{\sqrt{2}} \wedge 0\leq v\leq\pi \Leftrightarrow v=3\pi/4.$

Övning 1.77 (Sid. 24)

Lösning

a) $\arctan 1=\operatorname{arccot} 1=\frac{\pi}{4}$, ty $\tan\frac{\pi}{4}=\cot\frac{\pi}{4}=1.$

b) $\begin{cases} u=\arctan(-1) \Leftrightarrow \tan u=-1 \wedge |u|<\frac{\pi}{2} \Leftrightarrow u=-\pi/4. \\ v=\operatorname{arccot}(-1) \Leftrightarrow \cot v=-1 \wedge 0<u<\pi \Leftrightarrow v=\frac{3\pi}{4}. \end{cases}$

c) $\begin{cases} u=\arctan(-\sqrt{3}) \Leftrightarrow \tan u=-\sqrt{3} \wedge |u|<\frac{\pi}{2} \Leftrightarrow u=-\pi/3. \\ v=\operatorname{arccot}(-\sqrt{3}) \Leftrightarrow \cot v=-\sqrt{3} \wedge 0<v<\pi \Leftrightarrow v=\frac{5\pi}{6}. \end{cases}$

d) $\begin{cases} u=\arctan 0 \Leftrightarrow \tan u=0 \wedge |u|<\frac{\pi}{2} \Leftrightarrow u=0. \\ v=\operatorname{arccot} 0 \Leftrightarrow \cot v=0 \wedge 0<v<\pi \Leftrightarrow v=\frac{\pi}{2}. \end{cases}$

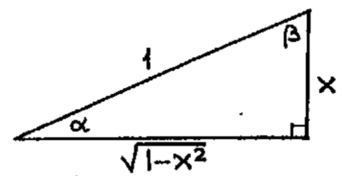
e) $\begin{cases} u=\arctan\frac{1}{\sqrt{3}} \Leftrightarrow \tan u=\frac{1}{\sqrt{3}} \wedge |u|<\frac{\pi}{2} \Leftrightarrow u=\frac{\pi}{6}. \\ v=\operatorname{arccot}\frac{1}{\sqrt{3}} \Leftrightarrow \cot v=\frac{1}{\sqrt{3}} \wedge 0<v<\pi \Leftrightarrow v=\frac{\pi}{3}. \end{cases}$

Övning 1.78 (Sid. 24)

Lösning

$\arcsin Q=\frac{\pi}{6} \Leftrightarrow Q=\sin\frac{\pi}{6}=\frac{1}{2}=\cos\frac{\pi}{3} \Leftrightarrow \underline{\arccos Q=\frac{\pi}{3}}.$

Anm. $x>0 \Rightarrow \arcsin x+\arccos x=\frac{\pi}{2}$ (se bevis).



$\frac{\pi}{2}=\alpha+\beta=\arcsin x+\arccos x$ (ty $\sin\alpha=x=\cos\beta$).

Övning 1.79 (Sid. 24)

Lösning

Se nästa sida.

a) $\underline{\arccos(\cos x) = x}$

$V_{\arccos} = [-1,1] \Rightarrow 0 \le x \le \pi.$

b) $\underline{\cos(\arccos x) = x}$

$D_{\arccos} = [-1,1] \Rightarrow -1 \le x \le 1.$

c) $\underline{\arctan(\tan x) = x} \Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}.$

Övning 1.80 (Sid. 25)

Lösning

$0 < \operatorname{arccot} x < \pi \Leftrightarrow -\pi < -\operatorname{arccot} x < 0 \Leftrightarrow \underbrace{-\frac{\pi}{2} < \frac{\pi}{2} - \operatorname{arccot} x < \frac{\pi}{2}}_{(*)};$

$\tan\left(\frac{\pi}{2} - \operatorname{arccot} x\right) = \cot(\operatorname{arccot} x) = x \overset{(*)}{\Leftrightarrow} \frac{\pi}{2} - \operatorname{arccot} x =$

$= \arctan x \Leftrightarrow \arctan x + \operatorname{arccot} x = \underline{\frac{\pi}{2} = f(x)}.$

Övning 1.81 (Sid. 25)

Lösning

$u = \frac{1}{2}\arctan\frac{12}{5} \Leftrightarrow 2u = \arctan\frac{12}{5} \Leftrightarrow \tan 2u = \frac{12}{5} \Leftrightarrow$

$\Leftrightarrow \frac{2\tan u}{1-\tan^2 u} = \frac{12}{5} \Leftrightarrow 5\tan u = 6 - 6\tan^2 u \Leftrightarrow \tan^2 u +$

$+ \frac{5}{6}\tan u = 1 \Leftrightarrow \tan u = -\frac{5}{12} + \frac{13}{12} = \frac{2}{3} \Leftrightarrow u = \arctan\frac{2}{3}.$

Övning 1.82 (Sid. 25)

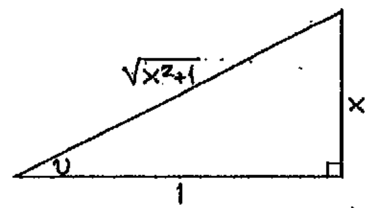
Lösning

$y = \arctan x \Rightarrow x = \tan y \Leftrightarrow x^2+1 = \tan^2 y + 1 = \frac{\sin^2 y}{\cos^2 y} + 1$

$= \frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y} \Leftrightarrow \cos^2 y = \frac{1}{x^2+1}.$

Övning 1.83 (Sid. 25)

Lösning



$$\left.\begin{array}{l} x>0 \Rightarrow v = \arctan x = \arccos\frac{1}{\sqrt{x^2+1}} \\ x<0 \Rightarrow -x>0 \Rightarrow \arctan(-x) = \arctan\frac{1}{\sqrt{(-x)^2+1}} \end{array}\right\} \Rightarrow$$

$\Rightarrow \arctan(\pm x) = \arccos\frac{1}{\sqrt{x^2+1}} \Leftrightarrow \underline{\arctan|x| = \arccos\frac{1}{\sqrt{x^2+1}}}.$

Anm. För $x=0$ är $\arctan 0 = 0 = \arccos 1$.

Övning 1.84 (Sid. 25)

Lösning

a) $\cosh u = \frac{e^u+e^{-u}}{2} \Rightarrow \cosh(-x) = \frac{e^{-x}+e^{-(-x)}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh x.$

b) $\sinh v = \frac{e^v-e^{-v}}{2} \Rightarrow \sinh(-x) = \frac{e^{-x}-e^{-(-x)}}{2} = \frac{e^{-x}-e^{x}}{2} = -\sinh x.$

c) $\cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x) =$

$= \left(\frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2}\right)\cdot\left(\frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2}\right) = e^{-x}\cdot e^x = e^{-x+x} = e^0 = 1.$

d) $\cosh^2 x + \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{1}{4}(e^x+e^{-x})^2 +$

$+\frac{1}{4}(e^x-e^{-x})^2=\frac{1}{4}(e^{2x}+e^{-2x}+2+e^{2x}+e^{-2x}-2)=\frac{1}{2}(e^{2x}-e^{-2x})=$

$=\cosh 2x.$

e) $2\sinh x\cosh x=2\frac{e^x-e^{-x}}{2}\cdot\frac{e^x+e^{-x}}{2}=\frac{e^{2x}-e^{-2x}}{2}=\sinh 2x.$

Övning 1.85 (Sid. 25)

Lösning

a) $\sum_{n=1}^{5} n^3=1^3+2^3+3^3+4^3+5^3=1+8+27+64+125=225.$

b) $\sum_{k=0}^{4}(k^2-3k)=\sum_{k=0}^{4}k(k-3)=0\cdot(-3)+1\cdot(-2)+2\cdot(-1)+3(0)+4\cdot 1=0.$

c) $\sum_{k=2}^{100}3=\underbrace{3+3+\cdots+3}_{99\text{ termer}}=99\cdot 3=297.$

d) $\sum_{k=m}^{n}3=\sum_{k=1}^{n}3-\sum_{k=1}^{m-1}3=n\cdot 3-(m-1)\cdot 3=\underline{(n-m+1)3}.$ (Jfr c)).

Övning 1.86 (Sid. 25)

Lösning

a) $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10}=\sum_{k=1}^{10}\frac{1}{k}.$

b) $2\cdot 3+3\cdot 4+4\cdot 5+\cdots+n(n+1)=\sum_{k=2}^{n}k(k+1)=\sum_{k=2}^{n}(k^2+k).$

c) $1+3+9+27+81+243=3^0+3^1+3^2+3^3+3^4+3^5=\sum_{k=0}^{5}3^k.$

Övning 1.87 (Sid. 25)

Lösning

a) $1+2+4+8+16+32=2^{1-1}+2^{2-1}+2^{3-1}+2^{4-1}+2^{5-1}+2^{6-1}=\sum_{k=1}^{6}2^{k-1}.$

b) $1-3+9-27+81-243=(-3)^{1-1}+(-3)^{2-1}+(-3)^{3-1}+(-3)^{4-1}+(-3)^{5-1}+$

$+(-3)^{6-1}=\sum_{k=1}^{6}(-3)^k.$

c) $2+1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{128}=\left(\frac{1}{2}\right)^{1-2}+\left(\frac{1}{2}\right)^{2-2}+\left(\frac{1}{2}\right)^{3-2}+\left(\frac{1}{2}\right)^{4-2}+\cdots+$

$+\left(\frac{1}{2}\right)^{9-2}=\sum_{k=1}^{9}\left(\frac{1}{2}\right)^{k-2}=2\cdot\frac{1-(1/2)^9}{1-1/2}=4\left(1-\frac{1}{2^9}\right)=4-\frac{1}{128}=\frac{511}{128}.$

d) $e+e^2+e^3+\cdots+e^{10}=\sum_{k=1}^{10}e^k=e\cdot\frac{e^{10}-1}{e-1}.$

e) $1-x+x^2+\cdots+(-x^9)=(-x)^0+(-x)^1+(-x)^2+\cdots+(-x)^9=\sum_{k=1}^{10}(-x)^k=$

$=1\cdot\frac{1-(-x)^{10}}{1-(-x)}=\frac{1-x^{10}}{1+x}.$

Övning 1.88 (Sid. 25)

Lösning

a) $\sum_{k=0}^{n}3\cdot 2^{-k}=3\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k=3\cdot 1\cdot\frac{1-(1/2)^{n+1}}{1-1/2}=6\left(1-\frac{1}{2^{n+1}}\right).$

b) $\sum_{k=1}^{10}3\cdot 2^k=3\sum_{k=1}^{10}2^k=3\left(2\cdot\frac{2^{10}-1}{2-1}\right)=3(2^{11}-2)=6138.$

c) $\sum_{k=3}^{10}3\cdot 2^k=3\sum_{k=3}^{10}2^k=3\left(\sum_{k=1}^{10}-\sum_{k=1}^{2}\right)2^k=3\left(2\frac{2^{10}-1}{2-1}-2-4\right)=$

$=3(2^{11}-2-2-4)=3(2^{11}-8)=6120.$

d) $\sum_{k=m}^{n}3\cdot 2^k=3\sum_{k=m}^{n}2^k=3\left(\sum_{k=1}^{n}-\sum_{k=1}^{m-1}\right)2^k=3\left(2\frac{2^n-1}{2-1}-2\frac{2^{m-1}-1}{2-1}\right)=$

$=3\cdot 2(2^n-1-2^{m-1}+1)=6(2^n-2^{m-1}).$

Övning 1.89 (Sid. 25)

Lösning

a) $\sum_{k=0}^{n} 3\cdot 2^{-k} = 3\sum_{k=0}^{n}(\frac{1}{2})^k = 3\cdot 1\cdot\frac{1-(1/2)^{n+1}}{1-1/2} = 6(1-\frac{1}{2^{n+1}})$.

b) $\sum_{k=1}^{n} e^{-k} = \sum_{k=1}^{n}(\frac{1}{e})^k = \frac{1}{e}\cdot\frac{1-(1/e)^n}{1-e^{-1}} = \frac{1-e^{-n}}{e-1}$.

c) $\sum_{n=0}^{100} 1000\cdot(1,05)^n = 1000\cdot\sum_{n=0}^{100}(1,05)^n = 1000\cdot 1\cdot\frac{1,05^{101}-1}{1,05-1} =$
$= 1000\cdot\frac{1,05^{101}-1}{0,05} = 20000(1,05^{101}-1) \approx 2,7415264\cdot 10^6$.

d) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots+\frac{1}{2^{2n}} = \sum_{k=0}^{2n}(\frac{1}{2})^k = 1\cdot\frac{1-(-1/2)^{2n+1}}{1+1/2} = \frac{2}{3}(1-\frac{1}{2^{2n+1}})$.

e) $\sum_{k=2}^{5}\frac{k\cdot(-1)^k}{2^k} = \frac{2\cdot 1}{4}+\frac{3\cdot(-1)}{8}+\frac{4\cdot 1}{16}+\frac{5\cdot(-1)}{32} = \frac{1}{2}-\frac{3}{8}+\frac{1}{4}-\frac{5}{32} = \frac{7}{32}$.

Övning 1.90 (Sid. 26)

Lösning

$P(x) = 2+2x+2x^2+2x^3+\dots+2x^7 = 2(1+x+x^2+x^3+\dots+x^7) =$

$= 2\cdot\frac{x^8-1}{x-1} \Rightarrow P(3) = 2\cdot\frac{3^8-1}{3-1} = 3^8-1 = 6560$.

Övning 1.91 (Sid. 26)

Lösning

Strax efter en studs har bollen rört sig 1 meter.

Efter två studsar har den rört sig $1+2\cdot 0,9$ meter.

Efter tre studsar $1+2\cdot 0,9+2\cdot 0,9^2$ och efter tio studsar

$1+2\cdot 0,9+2\cdot 0,9^2+\dots+2\cdot 0,9^9 = 1+2(0,9+0,9^2+\dots+0,9^9) =$

$= 1+2\cdot 0,9\frac{1-0,9^{10}}{1-0,9} = 1+2\cdot 9\cdot(1-0,9^{10}) \approx 12$ meter.

Övning 1.92 (Sid. 26)

Lösning

$S = 1+2+3+\dots+98+99+100$

$S = 100+99+98+\dots+3+2+1$ (+)

$2S = 101+101+101+\dots+101+101+101 = 100\cdot 101 \Leftrightarrow S = \underline{\underline{50\cdot 101}} = 5050$.

Allmänt fås $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ (med induktion).

Övning 1.93 (Sid. 26)

Lösning

$3+6+9+12+\dots+99 = 3(1+2+3+4+\dots+33) = 3\cdot\frac{33\cdot 34}{2} = 1683$.

Övning 1.94 (Sid. 26)

Lösning

a) $7! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 = 5040$.

b) $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{5040}{6\cdot 24} = 35$.

c) $\binom{1001}{999} = \frac{1001!}{999!2!} = \frac{\cancel{999!}\,1000\cdot 1001}{\cancel{999!}\,2!} = 500\cdot 1001 = 500500$.

Övning 1.95 (Sid. 26)

Lösning

Se nästa sida.

a) $(a+b)^2=\sum_{k=0}^{2}\binom{2}{k}a^{2-k}b^k=\binom{2}{0}a^2+\binom{2}{1}ab+\binom{2}{2}b^2=a^2+2ab+b^2.$

b) $(a+b)^3=\sum_{k=0}^{3}\binom{3}{k}a^{3-k}b^k=\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3=$

$=a^3+3a^2b+3ab^2+b^3.$

c) $(a+b)^4=\sum_{k=0}^{4}\binom{4}{k}a^{4-k}b^k=\binom{4}{0}a^4+\binom{4}{1}a^3b+\binom{4}{2}a^2b^2+\binom{4}{3}ab^3+\binom{4}{4}b^4$

$=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

d)

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \dots & & & & &
\end{array}$$

Övning 1.96 (Sid. 26)

Lösning

a) $(1+x)^3=\underline{1+3x+3x^2+x^3}.$ $\quad(a=1,\ b=x;\ n=3).$

b) $(3+(-2x))^3=3^3+3\cdot3^2(-2x)+3\cdot3(-2x)^2+(-2x)^3=$

$=\underline{27-54x+36x^2-8x^3}.$ $\quad(a=3,\ b=-2x).$

c) $(1+x)^4=1^4+4\cdot1^3\cdot x+6\cdot1^2\cdot x^2+4\cdot1\cdot x^3+x^4=$

$=\underline{1+4x+6x^2+4x^3+x^4}.$ $\quad(a=1,\ =x;\ n=4)$

Övning 1.97 (Sid. 26)

Lösning

Koefficienten för x^{13} kallas C_{13}.

$(1+x)^{15}=(x+1)^{15}=\sum_{k=0}^{15}\binom{15}{k}x^{15-k}=\binom{15}{0}x^{15}+\binom{15}{1}x^{14}+$

$+\binom{15}{2}x^{13}+\text{annat}\Rightarrow C_{13}=\binom{15}{2}=\frac{15!}{13!\,2!}=\frac{14\cdot15}{2}=105.$

Övning 1.98 (Sid. 26)

Lösning

Den sökta koefficienten kallas C_3.

$(3-x)^8=(x-3)^8=\sum_{k=0}^{8}\binom{8}{k}x^{8-k}\cdot(-3)^k=\sum_{k=0}^{8}\binom{8}{k}(-3)^k x^{8-k};$

C_3 fås för $k=5$, dvs. $C_3=\binom{8}{5}(-3)^5=\frac{8!}{3!\,5!}(-3)^5=$

$=\frac{\cancel{5!}6\cdot7\cdot8}{6\cdot\cancel{5!}}(-3^5)=-7\cdot8\cdot3^5=\underline{-13608}.$

Övning 1.99 (Sid. 26)

Lösning

Den konstanta termen kallas C_0.

$(x^2+x^{-3})^{15}=\sum_{k=0}^{15}\binom{15}{k}(x^2)^{15-k}\cdot(x^{-3})^k=\sum_{k=0}^{15}\binom{15}{k}x^{30-2k}\cdot x^{-3k}=$

$=\sum_{k=0}^{15}\binom{15}{k}x^{30-5k}\Rightarrow C_0=\binom{15}{6}=\frac{15!}{6!\,9!}=5005.$

Anm. Den konstanta termen har exponenten 0, dvs. $30-5k=0$, m.a.o. $k=6$.

Övning 1.100 (Sid. 26)

Lösning

$$(x^3-2)^{16} = \sum_{k=0}^{16} \binom{16}{k}(x^3)^{16-k}\cdot(-2)^k = \binom{16}{0}x^{48} + \binom{16}{1}x^{45}\cdot(-2) + \dots$$

$$= x^{48} - 32x^{45} + \text{annat}.$$

$$(x^4+3)^{12} = \sum_{k=0}^{12} \binom{12}{k}(x^4)^{12-k}\cdot 3^k = \binom{12}{0}x^{48} + \binom{12}{1}x^{44}\cdot 3 + \dots =$$

$$= x^{48} + 36x^{44} + \text{annat}.$$

$$(x^3-2)^{16} - (x^4+3)^{12} = x^{48} - 32x^{45} - (x^{48} + 36x^{44}) + \text{övrigt} =$$

$$= -32x^{45} - 36x^{44} + \dots$$

Resultat: Högstagradstermen är $-32x^{45}$.

Övning 1.101 (Sid. 26)

Lösning

$$P(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \Rightarrow P(1) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Övning 1.102 (Sid. 26)

Lösning

$$P(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \Rightarrow P(-1) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0.$$

Övning 1.103 (Sid. 26)

Lösning

Koefficienten för x^8 kallas C_8; $C_8 = 180$.

$$(a+2x)^{10} = \sum_{k=0}^{10} \binom{10}{k} a^{10-k}\cdot(2x)^k = \sum_{k=0}^{10} \binom{10}{k} a^{10-k}\cdot 2^k\cdot x^k \Rightarrow$$

$$\Rightarrow C_8 = \binom{10}{8} a^2\cdot 2^8 = 45\cdot 256 a^2 = 180 \Leftrightarrow 64a^2 = 1 \Leftrightarrow \underline{a = \pm\tfrac{1}{8}}.$$

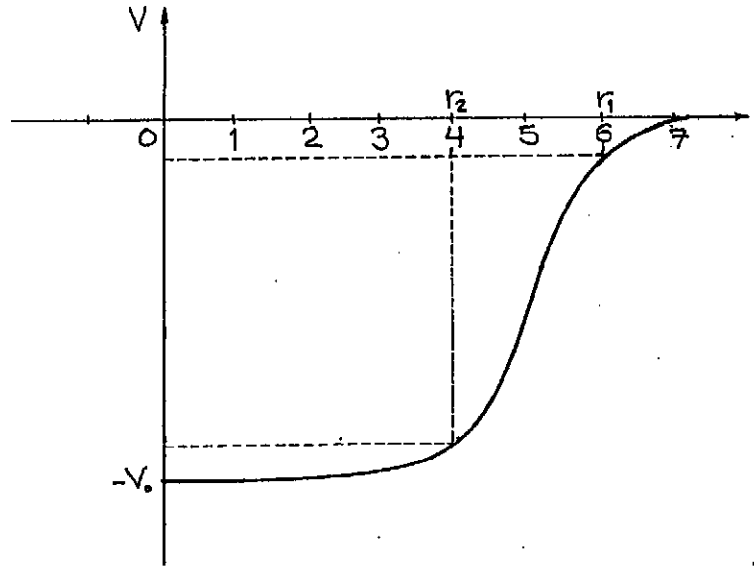
Övning 1.104 (Sid. 26)

Lösning

a) $V(r) = -\dfrac{V_0}{1+e^{2(r-5)}} \Rightarrow V'(r) = \dfrac{2V_0 e^{2(r-5)}}{(1+e^{2(r-5)})^2} > 0 \Rightarrow V$ växande.

$\lim_{r\to\infty} V(r) = 0^-$ (konvergensen är... meningslös, ty kärnkraften har ändlig räckvidd).

r	0	1	2	3	4	5	6
V/V_0	-1	-1	-0,99	-0,98	-0,88	-0,5	-0,12



forts.

b) $V(r_1) = -0{,}1V_0 \Leftrightarrow 0{,}1 = \dfrac{1}{1+e^{(r_1-R)/a}} \Leftrightarrow 1+e^{(r_1-R)/a} = 10$

$\Leftrightarrow e^{(r_1-R)/a} = 9 \Leftrightarrow \dfrac{r_1-R}{a} = \ln 9 = 2\ln 3 \Leftrightarrow \underline{r_1 = R+2a\ln 3};$

$V(r_2) = -0{,}9V_0 \Leftrightarrow 0{,}9 = \dfrac{1}{1+e^{(r_2-R)/a}} \Leftrightarrow 1+e^{(r_2-R)/a} = \dfrac{10}{9}$

$\Leftrightarrow e^{(r_2-R)/a} = \dfrac{1}{9} = 3^{-2} \Leftrightarrow (r_2-R)/a = -2\ln 3 \Leftrightarrow \underline{r_2 = R-2a\ln 3};$

$r_1 - r_2 = R+2a\ln 3 - (R-2a\ln 3) = \underline{4a\ln 3}.$

Övning 1.105 (Sid. 27)

Lösning

$$\sum_{j=1}^{n}\left(1+\frac{k}{100}\right)^j = \left(1+\frac{k}{100}\right)\cdot\frac{\left(1+\frac{k}{100}\right)^n-1}{\frac{k}{100}} = \left(\frac{100}{k}+1\right)\left(\left(1+\frac{k}{100}\right)^n-1\right).$$

Formeln gäller endast om $k>0$.

Övning 1.106 (Sid. 27)

Lösning

a) Koefficienten för x^4 kallas C_4.

$(x+2^{-1})^8 = \sum_{k=0}^{8}\binom{8}{k}x^{8-k}\cdot 2^{-k} \Rightarrow C_4 = \binom{8}{4}\cdot 2^{-4} = 35/8.$

b) $x>0 \Rightarrow VL = \ln(x+2) = \ln x(x+1) = HL \Leftrightarrow x+2 = x^2+x$

$\Leftrightarrow x^2 = 2 \Leftrightarrow \underline{x = \sqrt{2}}.$

Övning 1.107 (Sid. 27)

Lösning

$x_1 < x_2 \Leftrightarrow -4x_2 < -4x_1 \Leftrightarrow e^{-4x_2} < e^{-4x_1} \Leftrightarrow 3e^{-4x_2} < 3e^{-4x_1}$

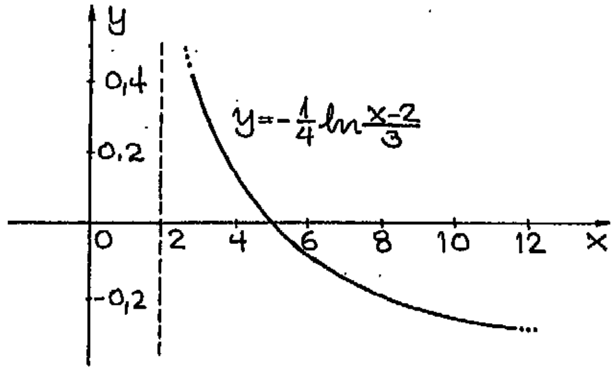
$\Leftrightarrow 2+3e^{-4x_2} < 2+3e^{-4x_1} \Leftrightarrow f(x_2) < f(x_1) \Rightarrow f$ strängt monotont avtagande $\Rightarrow f$ injektiv $\Rightarrow$ inversen finns och är även den avtagande.

$2+3e^{-4x} = y \Leftrightarrow 3e^{-4x} = y-2 > 0 \Leftrightarrow e^{-4x} = \dfrac{y-2}{3} \wedge y>2$

$\Leftrightarrow -4x = \ln\dfrac{y-2}{3} \Leftrightarrow x = -\dfrac{1}{4}\ln\dfrac{y-2}{3},\ y>2.$

Den inversa till f är $\underline{f^{-1}(x) = -\frac{1}{4}\ln\frac{x-2}{3},\ x>2}.$

x	2,5	3	4	5	6	7	8
$f^{-1}(x)$	0,44	0,27	0,10	0	-0,07	-0,13	-0,17



Anm. f^{-1} är uppenbarligen obegränsad.

Övning 1.108 (Sid. 27)

Lösning

a) $VL = \sum_{k=0}^{4}(1+x)^k = 1\cdot\frac{(1+x)^5-1}{1+x-1} = \frac{(1+x)^5-1}{x}$; (geom. serie).

$HL = \sum_{k=1}^{5}\binom{5}{k}x^{k-1} = \frac{1}{x}\sum_{k=1}^{5}\binom{5}{k}x^k = \frac{1}{x}\left(\sum_{k=0}^{5}\binom{5}{k}x^k - 1\right) =$

$= \frac{1}{x}((1+x)^5-1) = VL.$

b) $VL = \sum_{k=0}^{n-1}(1+x)^k = 1\cdot\frac{(1+x)^n-1}{1+x-1} = \frac{(1+x)^n-1}{x}$; (geom. serie).

$HL = \sum_{k=1}^{n}\binom{n}{k}x^{k-1} = \frac{1}{x}\sum_{k=1}^{n}\binom{n}{k}x^k = \frac{1}{x}\left(\sum_{k=0}^{n}\binom{n}{k}x^k - 1\right) =$

$= \frac{1}{x}((1+x)^n-1) = VL.$

Övning 1.109 (Sid. 27)

Lösning

a) $i = \binom{10}{5} = \frac{10!}{5!5!} = \frac{\cancel{5!}\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{\cancel{5!}\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{\cancel{6}\cdot 7\cdot \overset{2}{\cancel{8}}\cdot 9\cdot \overset{2}{\cancel{10}}}{\cancel{6}\cdot\cancel{4}\cdot\cancel{5}} = 36\cdot 7 = 6^2\cdot 7$;

$j = \binom{8}{4} = \frac{8!}{4!4!} = \frac{\cancel{4!}\cdot 5\cdot \cancel{6}\cdot 7\cdot 8}{\cancel{4!}\cdot 1\cdot\cancel{2}\cdot\cancel{3}\cdot 4} = \frac{5\cdot 7\cdot 8}{4} = 5\cdot 7\cdot 2$,

$\frac{i}{j} = \frac{6^2\cdot\cancel{7}}{2\cdot 5\cdot\cancel{7}} = \frac{\cancel{2}\cdot 3\cdot 6}{\cancel{2}\cdot 5} = \frac{18}{5}.$

b) $i = \binom{2n}{n} = \frac{(2n)!}{n!\cdot n!} = \frac{(2n)!}{(n!)^2}.$

$j = \binom{2n-2}{n-1} = \frac{(2n-2)!}{((n-1)!)^2} = \frac{n^2}{((n-1)!)^2\cdot n^2}\cdot\frac{(2n)!}{(2n-1)(2n)} =$

$= \frac{n^2}{(n!)^2}\cdot\frac{(2n)!}{(2n-1)\cdot 2n} = \frac{(2n)!}{(n!)^2}\cdot\frac{n}{2(2n-1)} = \binom{2n}{n}\cdot\frac{n}{2(2n-1)}$;

$\frac{i}{j} = \frac{\binom{2n}{n}}{\binom{2n-2}{n-1}} = \frac{1}{\frac{n}{2(2n-1)}} = \frac{4n-2}{n}.$ $\qquad \underbrace{\left(n=5 \Rightarrow \frac{i}{j} = \frac{4\cdot 5-2}{5} = \frac{18}{5}\right)}_{\text{specialfall}}.$

Övning 1.110 (Sid. 28)

Lösning

a) $e^x - e^{-x} = 6 \Leftrightarrow e^x(e^x - e^{-x}) = 6e^x \Leftrightarrow (e^x)^2 - 6e^x - 1 \Leftrightarrow$

$\Leftrightarrow e^x = 3+\sqrt{10} \Leftrightarrow \underline{x = \ln(3+\sqrt{10})}.$

b) $\ln\frac{1}{x} + \frac{1}{\ln x} = 2 \Leftrightarrow -\ln x + \frac{1}{\ln x} = 2 \Leftrightarrow -\ln^2 x + 1 = 2\ln x \Leftrightarrow$

$\Leftrightarrow (\ln x)^2 + 2\ln x = 1 \Leftrightarrow \ln x = -1+\sqrt{2} \vee \ln x = -1-\sqrt{2} \Leftrightarrow$

$\Leftrightarrow \underline{x = e^{-1+\sqrt{2}}} \vee \underline{x = e^{-1-\sqrt{2}}}.$

c) $(\ln x)\cdot(\ln x^2) = 3 \Leftrightarrow (\ln x)\cdot(2\ln x) = 3 \Leftrightarrow 2(\ln x)^2 = 3$

$\Leftrightarrow (\ln x)^2 = \frac{3}{2} \Leftrightarrow \ln x = \pm\sqrt{\frac{3}{2}} \Leftrightarrow \underline{x = e^{-\sqrt{3/2}}} \vee \underline{x = e^{\sqrt{3/2}}}.$

Övning 1.111 (Sid. 28)

Lösning

$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos 2x}{1+\cos 2x} \Rightarrow \tan^2\frac{\pi}{8} = \frac{1-\cos(\pi/4)}{1+\cos(\pi/4)} =$

$= \frac{1-1/\sqrt{2}}{1+1/\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{(\sqrt{2}-1)^2}{1} \Leftrightarrow \underline{\tan\frac{\pi}{8} = \sqrt{2}-1}.$

Övning 1.112 (Sid. 28)

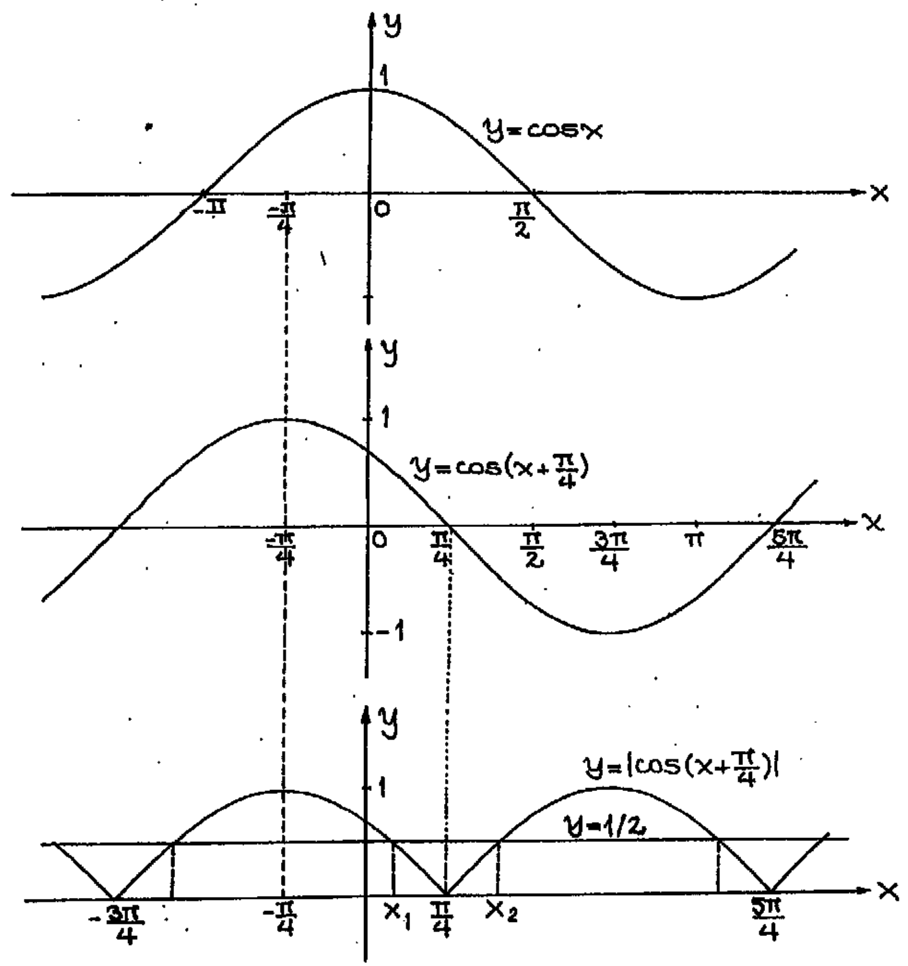
Lösning

$\cot(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} =$

$= \frac{\sin\alpha\sin\beta(\cot\alpha\cot\beta - 1)}{\sin\alpha\sin\beta(\cot\alpha + \cot\beta)} = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}.$

Övning 1.113 (Sid. 28)

Lösning



$y=|\cos(x+\frac{\pi}{4})|$ har perioden π.

$\cos(x+\frac{\pi}{4})=\frac{1}{2} \Rightarrow x+\frac{\pi}{4}=\frac{\pi}{3} \Rightarrow x_1=\frac{\pi}{12} \Rightarrow x_2=\frac{\pi}{12}+\frac{\pi}{3}=\frac{5\pi}{12}$.

$\therefore\ |\cos(x+\frac{\pi}{4})|<\frac{1}{2} \Leftrightarrow \frac{\pi}{12}+n\pi<x<\frac{5\pi}{12}+n\pi$, n heltaligt.

Övning 1.114 (Sid. 28)

Lösning

a) $\ln 2x+\ln 3x=\ln 4x \Leftrightarrow \ln(2x\cdot 3x)=\ln 4x \wedge \underline{x>0} \Leftrightarrow$

$\Leftrightarrow 2x\cdot 3x=4x \Leftrightarrow 3x=2 \Leftrightarrow \underline{x=\frac{2}{3}}$.

b) $\ln 2x\cdot\ln 3x=\ln 4 \Leftrightarrow (\ln 2+\ln x)(\ln 3+\ln x)=\ln 4x$

$\Leftrightarrow (\ln 2)\ln 3+(\ln 2+\ln 3)\ln x+(\ln x)^2=\ln 4+\ln x$

$\Leftrightarrow (\ln 2)\ln 3-2\ln 2+(\ln 2+\ln 3-1)\ln x+\ln^2 x=0 \Leftrightarrow$

$\Leftrightarrow (\ln x)^2+(\ln 6-1)\ln x=(2-\ln 3)\ln 2 \Leftrightarrow \ln x=\frac{1-\ln 6}{2}\pm$

$\pm\sqrt{\frac{1}{4}(1-\ln 6)^2+(2-\ln 3)\ln 2}$;

Svar: $\begin{cases} x_1=\exp\{\frac{1}{2}((1-\ln 6)+\sqrt{(2\ln 2)(2-\ln 3)+(\ln 6-1)^2})\} \\ x_2=\exp\{\frac{1}{2}((1-\ln 6)-\sqrt{(2\ln 2)(2-\ln 3)+(\ln 6-1)^2})\} \end{cases}$

c) ${}^2\log x+{}^3\log x={}^4\log x \overset{!}{\Leftrightarrow} \frac{\ln x}{\ln 2}+\frac{\ln x}{\ln 3}=\frac{\ln x}{\ln 4} \Leftrightarrow x=1$.

d) ${}^2\log x\cdot{}^3\log x={}^4\log x \overset{!}{\Leftrightarrow} \frac{\ln x}{\ln 2}\cdot\frac{\ln x}{\ln 3}=\frac{\ln x}{\ln 4} \Leftrightarrow \frac{\ln x}{2\ln 2}=$

$=\frac{(\ln x)^2}{\ln 2\cdot\ln 3} \Leftrightarrow \ln x=0 \vee \ln x=\frac{\cancel{\ln 2}\cdot\ln 3}{2\cancel{\ln 2}} \Leftrightarrow x=1 \vee$

$\vee \ln x=\ln\sqrt{3} \Leftrightarrow \underline{x=0} \vee \underline{x=\sqrt{3}}$.

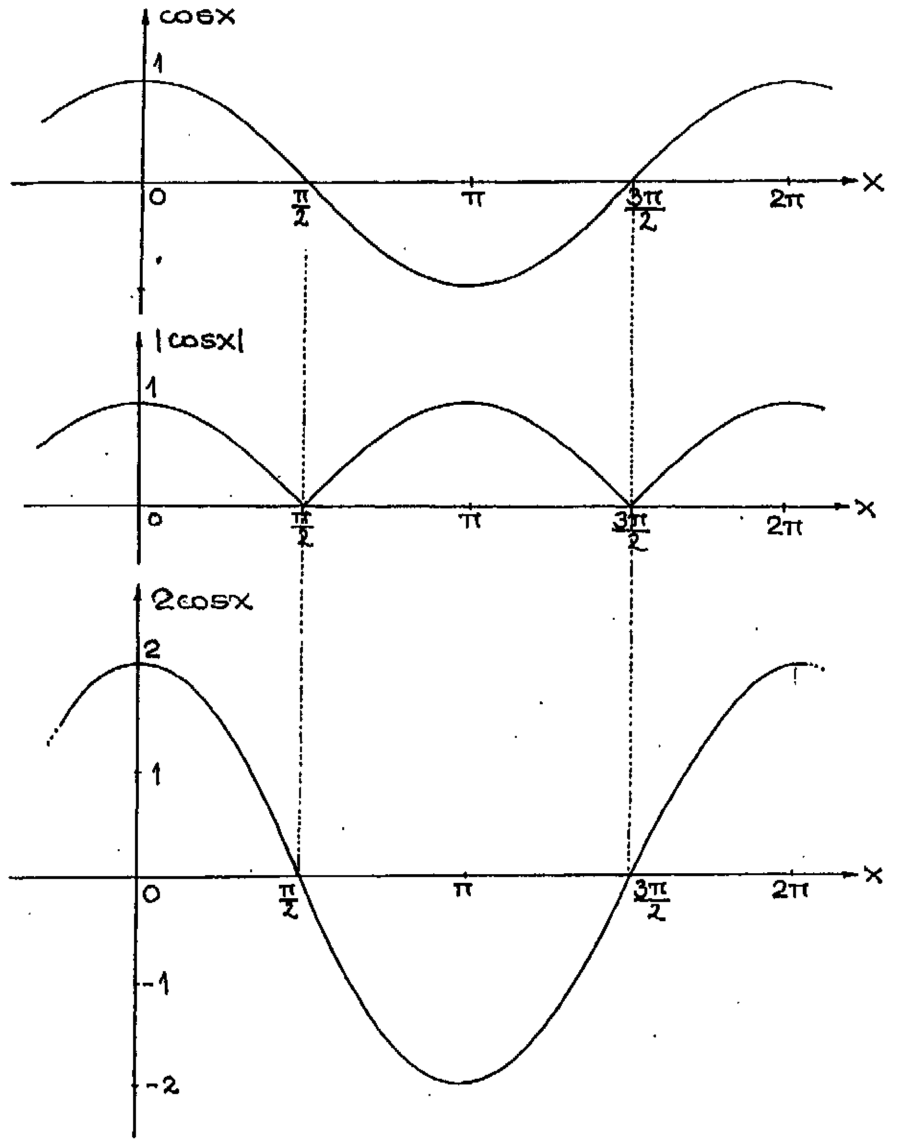
Anm. I $\overset{!}{\Leftrightarrow}$ har jag tillgripit basbyte.

Övning 1.115 (Sid. 28)

Lösning

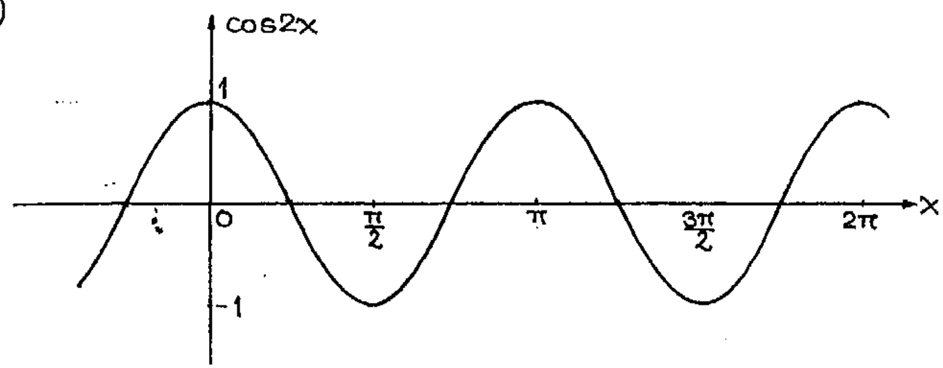
Se nästa sida.

a) b) c)

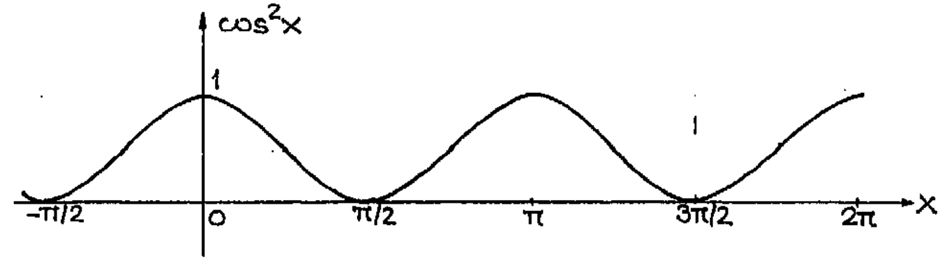


<u>Anm</u> $y = |\cos x|$ fås ur $y = \cos x$ genom spegling av den del av grafen som ligger under x-axeln i samma axel.

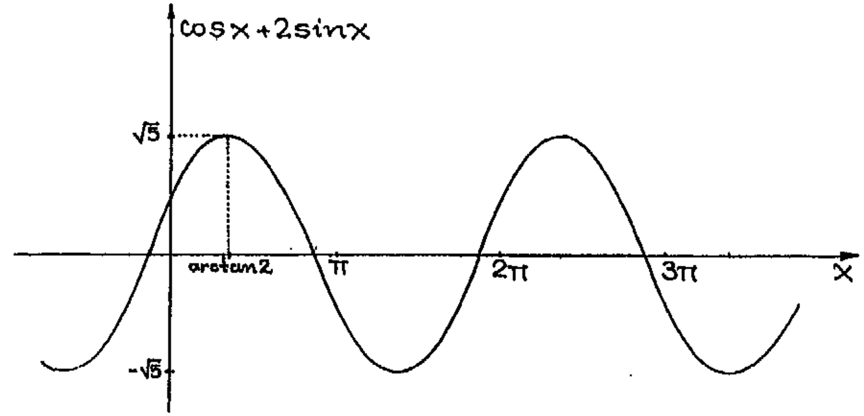
d)



e)



e) $y = \cos x + 2\sin x = \sqrt{5}\cos(x - \arctan 2)$



Inga tabeller här; dessa kurvor är ... kurs C.

Övning 1.116 (Sid. 28)

Lösning

a) $\sin 3x = \frac{\sqrt{3}}{2} \Leftrightarrow \begin{cases} 3x = \frac{\pi}{3} + m2\pi \\ 3x = \frac{2\pi}{3} + n2\pi \end{cases} \Leftrightarrow \begin{cases} x = \frac{\pi}{9} + m\cdot\frac{2\pi}{3}, & m \in \mathbb{Z} \\ x = \frac{2\pi}{9} + n\cdot\frac{2\pi}{3}, & n \in \mathbb{Z} \end{cases}$

b) $3\sin x = \frac{\sqrt{3}}{2} \Leftrightarrow \sin x = \frac{\sqrt{3}}{6} \Leftrightarrow \begin{cases} x = \arcsin\frac{\sqrt{3}}{6} + m2\pi, & m \in \mathbb{Z}. \\ x = -\arcsin\frac{\sqrt{3}}{6} + (2n+1)\pi, & n \in \mathbb{Z}. \end{cases}$

c) $\sin^3 x = 3\sin x \Leftrightarrow \sin x(\underbrace{\sin^2 x - 3}_{\neq 0}) = 0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = n\pi.$

d) $\cos^2 x - \sin^2 x = \frac{1}{2} \Leftrightarrow \cos 2x = \frac{1}{2} \Leftrightarrow 2x = \pm\frac{\pi}{3} + 2n\pi \quad (n \in \mathbb{Z}) \Leftrightarrow$

$\Leftrightarrow x = \pm\frac{\pi}{6} + n\pi, \ n \in \mathbb{Z}.$

e) $\cos^4 x - \sin^4 x = (\underbrace{\cos^2 x + \sin^2 x}_{=1})(\cos^2 x - \sin^2 x) = \cos 2x = \frac{1}{2}$ (Se d).

f) $\cos^4 x + \sin^4 x \overset{!}{=} (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x = 1 - \frac{1}{2}\sin^2 2x =$

$= 1 - \frac{1}{4}(1 - \cos 4x) = \frac{1}{4}(3 + \cos 4x) = \frac{1}{2} \Leftrightarrow \cos 4x = -1 \Leftrightarrow 4x = (2n+1)\pi$

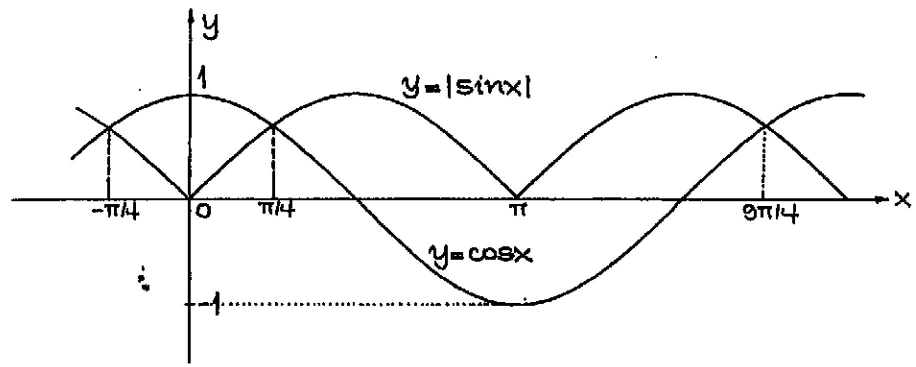
$\Leftrightarrow \underline{x = (2n+1)\frac{\pi}{4}, \ n \in \mathbb{Z}.}$

Anm. I ! har jag tillgripit kvadratkomplettering.

Övning 1.117 (Sid. 28)

Lösning

a) Jag löser ekvationen grafiskt (se nästa sida).



$|\sin x| = \cos x \Leftrightarrow x = -\frac{\pi}{4} + m2\pi \ \vee \ x = \frac{\pi}{4} + n2\pi, \ m, n \in \mathbb{Z}.$

b) $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = 7 \Leftrightarrow$

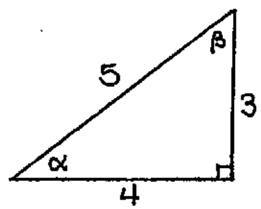
$\Leftrightarrow \sin x \cos x = \frac{1}{7} \Leftrightarrow \frac{1}{2}\sin 2x = \frac{1}{7} \Leftrightarrow \sin 2x = \frac{2}{7} \Leftrightarrow 2x =$

$= \arcsin\frac{2}{7} + n2\pi \ \vee \ 2x = \pi - \arcsin\frac{2}{7} + n'2\pi \quad (n, n' \in \mathbb{Z}) \Leftrightarrow$

$\Leftrightarrow x = \frac{1}{2}\arcsin\frac{2}{7} + n\pi \ \vee \ x = -\frac{1}{2}\arcsin\frac{2}{7} + (n' + \frac{1}{2})\pi, \ n, n' \in \mathbb{Z}.$

Övning 1.118 (Sid. 28)

Lösning



(le)

$\tan\alpha = \frac{3}{4} \Leftrightarrow \alpha = \arctan\frac{3}{4} \Rightarrow \beta = \frac{\pi}{2} - \arctan\frac{3}{4} = \arctan\frac{4}{3}.$

Övning 1.119 (Sid. 28)

$\sin x \cos x = a \Leftrightarrow \sin 2x = 2a$; skiljs. mellan $a > \frac{1}{2}$ & $a < \frac{1}{2}$.

2. Gränsvärden

Övning 2.1 (Sid. 48)

Lösning

a) $-1 \leq \cos x \leq 1 \Leftrightarrow \forall x > 1: -\frac{1}{\ln x} \leq \frac{\cos x}{\ln x} \leq \frac{1}{\ln x} \Leftrightarrow \left|\frac{\cos x}{\ln x}\right| \leq \frac{1}{\ln x}$;

$\lim_{x\to\infty} \frac{1}{\ln x} = 0 \Rightarrow \lim_{x\to\infty} \frac{\cos x}{\ln x} = 0$, enligt instängningsregeln.

b) Betrakta talföljderna

$a_n = (2n+\frac{1}{2})\pi, \quad b_n = n\pi, \quad c_n = (2n-\frac{1}{2})\pi; \quad n \geq 1.$

$$f(x) = x\sin x \Rightarrow \left\{\begin{array}{l} \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} (2n+\frac{1}{2})\pi = \infty \\ \lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} (n\pi).0 = 0 \\ \lim_{n\to\infty} f(c_n) = \lim_{n\to\infty} (2n-\frac{1}{2})\pi.(-1) = -\infty \end{array}\right\} \Rightarrow$$

$\Rightarrow \lim_{x\to\infty} x\sin x$ existerar inte.

c) $\arctan x < \frac{\pi}{2} \Leftrightarrow \frac{2}{\pi} < \frac{1}{\arctan x} \Leftrightarrow \frac{2}{\pi}\ln x < \frac{\ln x}{\arctan x}, x > 1.$

$\lim_{x\to\infty} \ln x = \infty \Rightarrow \lim_{x\to\infty} \frac{\ln x}{\arctan x} = \infty.$

Övning 2.2 (Sid. 48)

Lösning

a) $\lim_{x\to 1} \frac{x+2}{x-3} = \frac{\lim_{x\to 1}(x+2)}{\lim_{x\to 1}(x-3)} = \frac{1+2}{1-3} = \frac{3}{-2} = -\frac{3}{2}.$

b) $\lim_{x\to\infty} \frac{e^{1/x}}{x} \; [u = \frac{1}{x}] = \lim_{u\to 0} ue^u = 0\cdot e^0 = 0\cdot 1 = 0.$

c) $-1 \leq \sin x \leq 1 \Leftrightarrow -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \Leftrightarrow \left|\frac{\sin x}{x}\right| \leq \frac{1}{x}$, för stora x;

$\lim_{x\to\infty} \frac{1}{x} = 0 \Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$, enl. instängningsregeln.

d) Betrakta talföljderna $a_n = 2n\pi$ och $b_n = (2n+1)\pi$.

$$f(x) = x\cos x \Rightarrow \left\{\begin{array}{l} \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 2\pi n = \infty \\ \lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} (2n+1)\pi\cdot(-1) = -\infty \end{array}\right\} \Rightarrow$$

$\Rightarrow \lim_{x\to\infty} x\cos x$ existerar inte.

Övning 2.3 (Sid. 48)

Lösning

a) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ kan tas som standardgränsvärde.

b) $\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{2\sin x\cos x}{x} = 2\lim_{x\to 0} \frac{\sin x}{x} . \lim_{x\to 0} \cos x = 2.$

c) $\lim_{x\to 0} \frac{\sin 3x}{x} \; [u = 3x] = \lim_{u\to 0} \frac{\sin u}{u/3} = 3\lim_{u\to 0} \frac{\sin u}{u} = 3\cdot 1 = 3.$

d) $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 2x}{x} \cdot \frac{x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 2x}{x} \left(\frac{\sin 3x}{x}\right)^{-1}$

$= \lim_{x\to 0} \frac{\sin 2x}{x} \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)^{-1} = 2\cdot(3)^{-1} = \frac{2}{3}.$

e) $\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} = \lim_{u\to 0} \frac{\sin u}{u} = 1.$

Övning 2.4 (Sid. 48)

Lösning

Se nästföljande sida.

a) $\lim\limits_{x\to 0}\frac{\sin 2x}{5x}=\frac{1}{5}\lim\limits_{x\to 0}\frac{\sin 2x}{x}=\frac{1}{5}\cdot 2=\frac{2}{5}$ (Se ö. 2.3 b)).

b) $\lim\limits_{x\to 0}\frac{x}{2\sin 2x}=\frac{1}{2}\lim\limits_{x\to 0}\frac{x}{\sin 2x}=\frac{1}{2}\left(\lim\limits_{x\to 0}\frac{\sin 2x}{x}\right)^{-1}=\frac{1}{2}(2)^{-1}=\frac{1}{4}$.

c) $\lim\limits_{x\to 0}\frac{\tan 3x}{\sin 4x}=\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 4x}\cdot\frac{1}{\cos 3x}\overset{I}{=}\lim\limits_{x\to 0}\frac{\sin 3x}{x}\cdot\frac{x}{\sin 4x}=$

$=\lim\limits_{x\to 0}\frac{\sin 3x}{x}\cdot\left(\lim\limits_{x\to 0}\frac{\sin 4x}{x}\right)^{-1}=3\cdot 4^{-1}=\frac{3}{4}$.

I $\overset{I}{=}$ antar jag $\lim\limits_{x\to 0}\frac{1}{\cos 3x}=1$.

Anm. $\lim\limits_{x\to 0}\frac{\sin 4x}{x}=\lim\limits_{x\to 0}2\,\frac{\sin 2x}{x}\cos 2x=2\cdot 2\cdot 1=4$.

Övning 2.5 (Sid. 48)

Lösning

a) $\lim\limits_{x\to 0}\frac{e^x-1}{x}\ [u=e^x-1]=\lim\limits_{u\to 0}\frac{u}{\ln(1+u)}=\left(\lim\limits_{u\to 0}\frac{\ln(1+u)}{u}\right)^{-1}=1$.

b) $\lim\limits_{x\to 0}\frac{e^{2x}-1}{x}=\lim\limits_{x\to 0}\frac{e^x-1}{x}\cdot(e^x+1)=\lim\limits_{x\to 0}\frac{e^x-1}{x}\cdot\lim\limits_{x\to 0}(e^x+1)=2$.

c) $\lim\limits_{x\to 0}\frac{e^{3x}-1}{x}\ [u=3x]=\lim\limits_{u\to 0}\frac{e^u-1}{u/3}=3\lim\limits_{u\to 0}\frac{e^u-1}{u}=3$.

d) $\lim\limits_{x\to 0}\frac{e^{2x}-1}{e^{3x}-1}=\lim\limits_{x\to 0}\frac{\cancel{(e^x-1)}(e^x+1)}{\cancel{(e^x-1)}(e^{2x}+e^x+1)}=\lim\limits_{x\to 0}\frac{e^x+1}{e^{2x}+e^x+1}=\frac{2}{3}$.

Anm. $a^3-b^3=(a-b)(a^2+ab+b^2)$.

e) $\lim\limits_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\ [u=\sin x]=\lim\limits_{u\to 0}\frac{e^u-1}{u}=1$.

f) $\lim\limits_{x\to 0}\frac{e^{\sin x}-1}{x}\ [u=\sin x]=\lim\limits_{u\to 0}\frac{e^u-1}{\arcsin u}=\lim\limits_{u\to 0}\frac{e^u-1}{u}\cdot\frac{u}{\sin^{-1}u}$

$=\lim\limits_{u\to 0}\frac{e^u-1}{u}\cdot\left(\lim\limits_{u\to 0}\frac{\arcsin u}{u}\right)=1\cdot\lim\limits_{z\to 0}\frac{z}{\sin z}=1$.

Övning 2.6 (Sid. 48)

Lösning

a) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$ tas som standardgränsvärde.

Lägg märke till att för små $|x|$ är $\ln(1+x)\approx x$.

b) $\lim\limits_{x\to 0}\frac{\ln(1+2x)}{x}\ [u=2x]=\lim\limits_{u\to 0}\frac{\ln(1+u)}{u/2}=2\lim\limits_{u\to 0}\frac{\ln(1+u)}{u}=2$.

c) $\lim\limits_{x\to 0}\frac{\ln(1+3x)}{x}\ [u=3x]=\lim\limits_{u\to 0}\frac{\ln(1+u)}{u/3}=3\lim\limits_{u\to 0}\frac{\ln(1+u)}{u}=3$.

d) $\lim\limits_{x\to 0}\frac{\ln(1+2x)}{\ln(1+3x)}=\lim\limits_{x\to 0}\frac{\ln(1+2x)}{x}\cdot\left(\frac{\ln(1+3x)}{x}\right)^{-1}=2\cdot 3^{-1}=\frac{2}{3}$.

e) $\lim\limits_{x\to 0}\frac{\ln(1+\sin x)}{\sin x}\ [u=\sin x]=\lim\limits_{u\to 0}\frac{\ln(1+u)}{u}=1$.

Övning 2.7 (Sid. 48)

Lösning

a) $\lim\limits_{x\to 0^+}x\ln x=0^-$ tas som standardgränsvärde.

b) $\lim\limits_{x\to 0^+}x\ln 2x=\lim\limits_{x\to 0}\frac{1}{2}(2x)\ln 2x=\frac{1}{2}\lim\limits_{u\to 0}u\ln u=0$.

c) $\lim\limits_{x\to 0^+}x\ln 3x=\lim\limits_{x\to 0^+}x(\ln x+\ln 3)=\lim\limits_{x\to 0^+}x\ln x+\lim\limits_{x\to 0}x\ln 3=0$.

d) $\lim\limits_{x\to 0^+}\frac{\ln 3x}{\ln 2x}=\lim\limits_{x\to 0^+}\frac{\ln 3+\ln x}{\ln 3+\ln x}=\lim\limits_{x\to 0^+}\frac{\ln x}{\ln x}=1$.

Anm. För små x ar $\ln 2x\approx\ln x\approx\ln 3x$.

e) $\lim\limits_{x\to 0^+}\sin x\cdot\ln(\sin x)=[u=\sin x]=\lim\limits_{u\to 0^+}u\ln u=0$.

f) $\lim\limits_{x\to 0}x\cdot\ln(\sin x)=\lim\limits_{x\to 0}\frac{x}{\sin x}\cdot\sin x\cdot\ln(\sin x)=1\cdot 0=0$.

Övning 2.8 (Sid. 48)

Lösning

a) $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ tas som <u>standardgränsvärde.</u>

b) $\lim_{n\to\infty}(1+\frac{1}{n})^{2n} = \lim_{n\to\infty}((1+\frac{1}{n})^n)^2 = (\lim_{n\to\infty}(1+\frac{1}{n})^n)^2 = e^2.$

c) $\lim_{n\to\infty}(1+\frac{1}{2n})^n = [\nu = 2n] = \lim_{\nu\to\infty}(1+\frac{1}{\nu})^{\nu/2} = \lim_{\nu\to\infty}((1+\frac{1}{\nu})^\nu)^{1/2} =$
$= (\lim_{\nu\to\infty}(1+\frac{1}{\nu})^\nu)^{1/2} = e^{1/2} = \sqrt{e}.$

d) $\lim_{n\to\infty}(2-\frac{1}{n})^n = \infty$, ty $2-\frac{1}{n} > 1$ för stora n.

Övning 2.9 (Sid. 48)

Lösning

a) $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = \exp\{\lim_{x\to 0^+} x\ln x\} = e^{0^-} = 1^-.$

b) $\lim_{x\to 0}(\sin x)^x = \lim_{x\to 0^+} e^{x\ln(\sin x)} = \exp\{\lim_{x\to 0^+} x\ln(\sin x)\} =$
$= \exp\{\lim_{x\to 0^+}\frac{x}{\sin x}\cdot\sin x\ln(\sin x)\} = \exp\{\lim_{x\to 0^+}\sin x\ln(\sin x)\}$
$\overset{!}{=} \exp\{\lim_{u\to 0^+} u\ln u\} = e^{0^-} = 1^-.$

<u>Anm.</u> I $\overset{!}{=}$ underförstås substitutionen $u = \sin x$.

Övning 2.10 (Sid. 49)

Lösning

a) $\lim_{n\to\infty}\sqrt[n]{n^2} = \lim_{n\to\infty}\sqrt[n]{n\cdot n} = \lim_{n\to\infty}\sqrt[n]{n}\cdot\sqrt[n]{n} = (\lim_{n\to\infty}\sqrt[n]{n})^2 = 1^2 = 1.$

b) $\lim_{n\to\infty}\frac{1}{\sqrt[n]{n^3}} = (\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}})^3 = (\frac{1}{1})^3 = 1.$

c) $\lim_{n\to\infty}\sqrt[n]{\frac{n^4}{2^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n^4}}{\sqrt[n]{2^n}} = \lim_{n\to\infty}\frac{(\sqrt[n]{n})^4}{2} = \frac{1}{2}.$

Övning 2.11 (Sid. 49)

Lösning

a) $n>1 \Rightarrow \frac{1}{3n}+\frac{1}{3n} < 1+\frac{1}{3n} < 1+1 \Leftrightarrow \frac{2}{3n} < 1+\frac{1}{3n} < 2 \Leftrightarrow \sqrt[n]{\frac{2}{3n}} < \sqrt[n]{1+\frac{1}{3n}} < \sqrt[n]{2}$;
$1 \xleftarrow[n\to\infty]{} \sqrt[n]{\frac{2}{3n}} < \sqrt[n]{1+\frac{1}{3n}} < \sqrt[n]{2} \xrightarrow[n\to\infty]{} 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{1+\frac{1}{3n}} = 1.$

b) $n>1 \Rightarrow n^2 > n \Leftrightarrow 3n^2 > 3n \Leftrightarrow 3n+n < 3n^2+n < 3n^2+n^2 \Leftrightarrow$
$\Leftrightarrow 4n < 3n^2+n < 4n^2 \Leftrightarrow \sqrt[n]{4n} < \sqrt[n]{3n^2+n} < \sqrt[n]{4n^2}$;
$1 \xleftarrow[n\to\infty]{} \sqrt[n]{4n} < \sqrt[n]{3n^2+n} < \sqrt[n]{4n^2} \xrightarrow[n\to\infty]{} 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{3n^2+n} = 1.$

Övning 2.12 (Sid. 49)

Lösning

a) $n>1 \Rightarrow \frac{1}{n} < 1 \Leftrightarrow \frac{1}{n}+\frac{1}{n} < 1+\frac{1}{n} < 1+1 \Leftrightarrow \frac{2}{n} < 1+\frac{1}{n} < 2$;
$1 \xleftarrow[n\to\infty]{} \sqrt[n]{\frac{2}{n}} < \sqrt[n]{1+\frac{1}{n}} < \sqrt[n]{2} \xrightarrow[n\to\infty]{} 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{1+\frac{1}{n}} = 1.$

b) $n>1 \Rightarrow n < n^2 \Rightarrow n+n < n^2+n < n^2+n^2 \Leftrightarrow 2n < n^2+n < 2n^2$;
$1 \xleftarrow[n\to\infty]{} \sqrt[n]{2n} < \sqrt[n]{1+\frac{1}{n}} < \sqrt[n]{2} \xrightarrow[n\to\infty]{} 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{1+\frac{1}{n}} = 1.$

Övning 2.13 (Sid. 49)

Lösning

a) $n>1 \Rightarrow \frac{1}{n^2} < 2 \Rightarrow \frac{1}{n^2}+\frac{1}{n^2} < 2+\frac{1}{n^2} < 2+2 \Leftrightarrow \frac{2}{n^2} < 2+\frac{1}{n^2} < 4$;
$1 \xleftarrow[n\to\infty]{} \sqrt[n]{\frac{2}{n^2}} < \sqrt[n]{2+\frac{1}{n^2}} < \sqrt[n]{4} \xrightarrow[n\to\infty]{} 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{2+\frac{1}{n^2}} = 1.$

b) $n>1 \Rightarrow n<n^3 \Leftrightarrow 2n<2n^3 \Leftrightarrow 2n+n<2n^3+n<2n^3+n^3 \Leftrightarrow$

$\Leftrightarrow 3n<2n^3+n<3n^3 \Leftrightarrow \sqrt[n]{3n}<\sqrt[n]{2n^3+n}<\sqrt[n]{3n^3}$;

$1 \xleftarrow[n\to\infty]{} \sqrt[n]{3n}<\sqrt[n]{2n^3+n}<\sqrt[n]{3n^3} \xrightarrow[n\to\infty]{} 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{2n^3+n}=1.$

Övning 2.14 (Sid. 49)

Lösning

a) $\lim_{x\to 0}\frac{x}{\arctan x}\,[x=\tan u]=\lim_{u\to 0}\frac{\tan u}{u}=\lim_{u\to 0}\frac{\sin u}{u}\cdot\frac{1}{\cos u}=$

$=\lim_{u\to 0}\frac{\sin u}{u}\cdot\lim_{u\to 0}\frac{1}{\cos u}=1\cdot 1=1.$

b) $\lim_{x\to 1}\frac{\ln x}{x^2-1}=\lim_{x\to 1}\frac{\ln x}{x-1}\,\frac{1}{x+1}=\lim_{x\to 1}\frac{\ln x}{x-1}\cdot\frac{1}{2}\,[u=x-1]=$

$=\lim_{u\to 0}\frac{\ln(1+u)}{u}\cdot\frac{1}{2}=1\cdot\frac{1}{2}=\frac{1}{2}.$

Övning 2.15 (Sid. 49)

Lösning

a) $\lim_{x\to 0}\frac{\arctan 2x}{3x}\,[2x=\tan u]=\lim_{u\to 0}\frac{2}{3}\left(\frac{\tan u}{u}\right)^{-1}\overset{!}{=}\frac{2}{3}\cdot 1=\frac{2}{3}.$

I $\overset{!}{=}$ underförstås resultatet i Ö. 2.14 a).

b) $\lim_{x\to\pi/2}\frac{\cot x}{2x-\pi}=\left(\frac{0}{0}\right)=\lim_{x\to\pi/2}\frac{\tan(\pi/2-x)}{-2(\pi/2-x)}\overset{!}{=}-\frac{1}{2}\lim_{u\to 0}\frac{\tan u}{u}\overset{!!}{=}-\frac{1}{2}.$

I $\overset{!}{=}$ underförstås substitutionen $u=\frac{\pi}{2}-x$; i $\overset{!!}{=}$ å sin sida underförstås resultatet i Ö. 2.14 a.

c) $\lim_{x\to 0^+}x^3e^{1/x}=(0\cdot\infty)=\lim_{u\to\infty}\frac{e^u}{u^3}=\infty,$ (med $u=\frac{1}{x}$).

Övning 2.16 (Sid. 49)

Lösning

$\lim_{x\to\infty}(\sqrt{x^2+x+1}-x)=(\infty-\infty)=\lim_{x\to\infty}\frac{(\sqrt{x^2+x+1}-x)(\sqrt{x^2+x+1}+x)}{\sqrt{x^2+x+1}+x}=$

$=\lim_{x\to\infty}\frac{(\sqrt{x^2+x+1})^2-x^2}{\sqrt{x^2+x+1}+x}=\lim_{x\to\infty}\frac{\cancel{x^2}+x+1-\cancel{x^2}}{\sqrt{x^2+x+1}+x}=\lim_{x\to\infty}\frac{\cancel{x}(1+x^{-1})}{\cancel{x}(\sqrt{1+x^{-1}+x^{-2}}+1)}$

$=\lim_{x\to\infty}\frac{1+1/x}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}=\frac{1}{1+1}=\frac{1}{2}.$

Övning 2.17 (Sid. 49)

Lösning

a) $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x})=(\infty-\infty)=\lim_{x\to\infty}\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}}=$

$=\lim_{x\to\infty}\frac{(\sqrt{x+1})^2-(\sqrt{x})^2}{\sqrt{x+1}+\sqrt{x}}=\lim_{x\to\infty}\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}}=0.$

b) $\lim_{x\to\infty}(\sqrt{x^2+3x}-\sqrt{x^2+1})=(\infty-\infty)=\lim_{x\to\infty}\frac{(\sqrt{x^2+3x}-\sqrt{x^2+1})(\sqrt{x^2+3x}+\sqrt{x^2+1})}{\sqrt{x^2+3x}+\sqrt{x^2+1}}$

$=\lim_{x\to\infty}\frac{(\sqrt{x^2+3x})^2-(\sqrt{x^2+1})^2}{\sqrt{x^2+3x}+\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{\cancel{x^2}+3x-\cancel{x^2}-1}{\sqrt{x^2+3x}+\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{3x}{x+x}=\frac{3}{2}.$

Anm För stora x har vi $3x-1\approx 3x$, $\sqrt{x^2+1}\approx x\approx\sqrt{x^2+3x}$.

c) För stora $|x|$ har vi $3x+1\approx 3x$, $\sqrt{x^2+1}\approx|x|\approx\sqrt{x^2+3x}$; (*)

$\lim_{x\to-\infty}(\sqrt{x^2+3x}-\sqrt{x^2+1})=\lim_{x\to-\infty}\frac{3x-1}{\sqrt{x^2+3x}+\sqrt{x^2+1}}=$ (Se (*)) $=$

$\lim_{x\to-\infty}\frac{x(3-1/x)}{|x|(\sqrt{1+3/x}+\sqrt{1+1/x^2})}=\lim_{x\to-\infty}\frac{\cancel{x}(3-1/x)}{-\cancel{x}(\sqrt{1+3/x}+\sqrt{1+1/x^2})}=\frac{3}{-2}=-\frac{3}{2}.$

Övning 2.18 (Sid. 49)

Lösning

Se nästföljande sida.

a) $\underline{a_0 = 1,\ a_k = 2a_{k-1},\ k \geq 1.}$

$a_0 = 1 \Rightarrow a_1 = 2a_0 = 2 \Rightarrow a_2 = 2a_1 = 4 \Rightarrow a_3 = 2a_2 = 8.$

b) $\underline{a_0 = 1,\ a_k = (a_{k-1})^2 - 1,\ k \geq 1.}$

$a_0 = 1 \Rightarrow a_1 = a_0^2 - 1 = 0 \Rightarrow a_2 = a_1^2 - 1 = -1 \Rightarrow a_3 = a_2^2 - 1 = 0.$

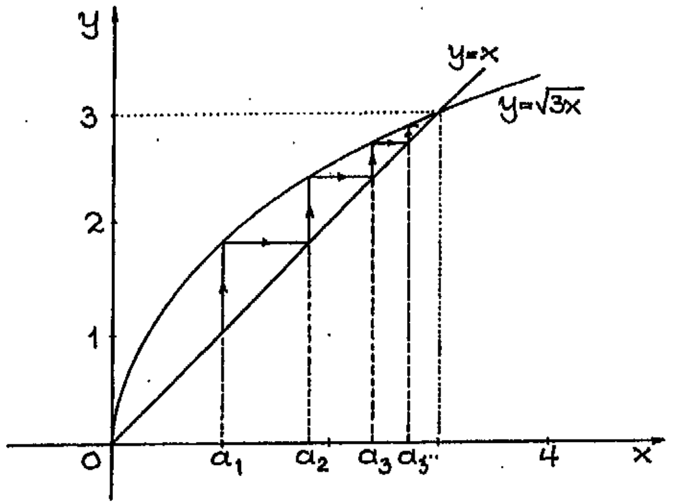
Resultat: a) 2, 4, 8; b) 0, −1, 0.

Övning 2.19 (Sid. 49)

Lösning

a) $\underline{a_1 = 1,\ a_{n+1} = \sqrt{3a_n},\ n \geq 1.}$

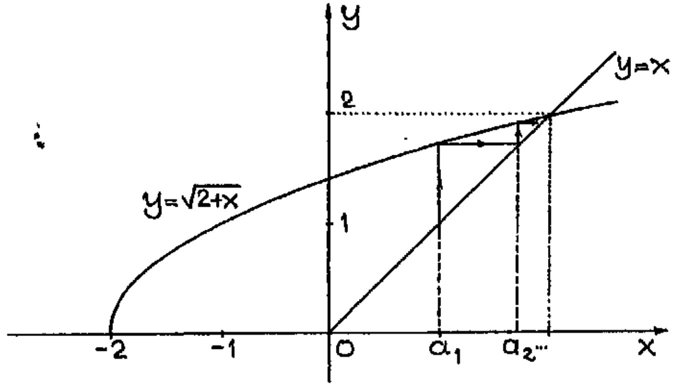
$a_1 = 1 \Rightarrow \underbrace{a_2 = \sqrt{3}}_{1,732} \Rightarrow \underbrace{a_3 = \sqrt{3\sqrt{3}}}_{2,280} \Rightarrow \underbrace{a_4 = \sqrt{3\sqrt{3\sqrt{3}}}}_{2,615} \Rightarrow \dots$ osv.



a_n tycks konvergera mot 3; $\lim_{n\to\infty} a_n = 3.$

b) $\underline{a_1 = 1,\ a_{n+1} = \sqrt{2 + a_n},\ n \geq 1.}$

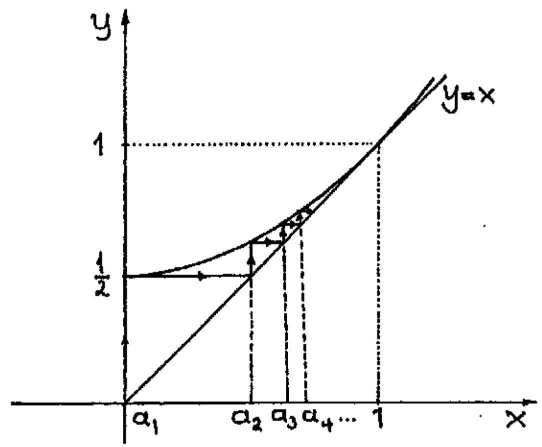
$a_1 = 1 \Rightarrow \underbrace{a_2 = \sqrt{3}}_{1,732} \Rightarrow \underbrace{a_3 = \sqrt{2+\sqrt{3}}}_{1,932} \Rightarrow \underbrace{a_4 = \sqrt{2+\sqrt{2+\sqrt{3}}}}_{1,983} \Rightarrow$ osv.



a_n tycks konvergera mot 2; $\lim_{n\to\infty} a_n = 2.$

c) $\underline{a_1 = 0,\ a_{n+1} = \frac{1 + a_n^2}{2},\ n \geq 1.}$

$a_1 = 0 \Rightarrow a_2 = \frac{1}{2} \Rightarrow a_3 = \frac{5}{8} \Rightarrow a_4 = \frac{89}{128} \Rightarrow \dots$ osv.



a_n tycks konvergera mot 1; $\lim_{n\to\infty} a_n = 1.$

Övning 2.20 (Sid. 50)

Lösning

a) $f(x)=\begin{cases} x^2, & x>0 \\ -x, & x<0 \end{cases}$; $g(x)=\begin{cases} -x^2, & x\geq 1 \\ 1+x, & 0\leq x<1 \\ 1, & x<0 \end{cases}$

f är kontinuerlig men inte g. Se figurerna i Ö 1.8.

b) $\lim_{x\to 1} h(x)=\lim_{x\to 1}\frac{2x^2-x-1}{x-1}=\lim_{x\to 1}\frac{(2x+1)(x-1)}{x-1}=\lim_{x\to 1}(2x+1)=3=$ $=h(1) \Rightarrow h$ är en kontinuerlig funktion.

Övning 2.21 (Sid. 50)

Lösning

$VL=f(x)=x^3+3x^2+4x-5 \Rightarrow f(0)\cdot f(1)=(-5)\cdot 3=-15<0 \Rightarrow$ $\Rightarrow$ f byter tecken; f är kontinuerlig, ty alla polynomfunktioner är kontinuerliga; enligt satsen om mellanliggande värden finns $\xi\in]0,1[$ s.a. $f(\xi)=0$. (Roten kan bestämmas med t. ex en grafritande miniräknare). $f(0{,}74)\approx 0{,}008$.

Övning 2.22 (Sid. 50)

Lösning

$VL=f(x)=8x^3-36x^2+46x-15$.

f är en polynomfunktion och således kontinuerlig.

x	0	1	2	3
$f(x)$	−15	3	−3	15

f uppvisar teckenväxling i vart och ett av de intervallen som nämns i texten. Enligt satsen om mellanliggande värden har ekvationen exakt en rot i vart och ett av intervallen.

Övning 2.23 (Sid. 50)

Lösning

Nej, ty f är inte kontinuerlig i intervallet $]-1,1[$.

Övning 2.24 (Sid. 50)

Lösning

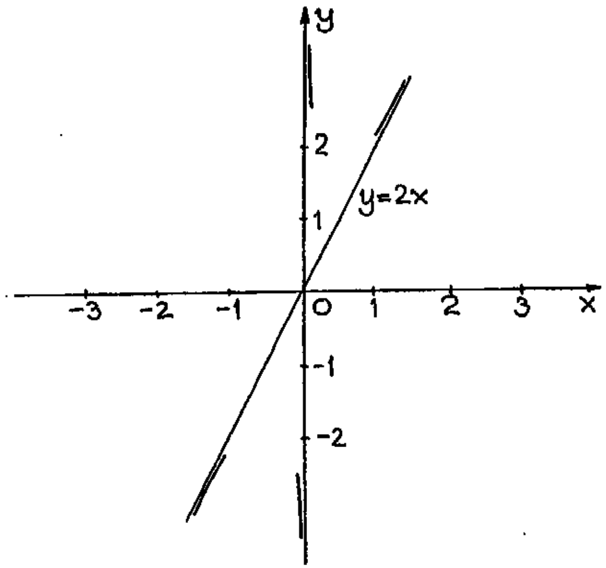
a) $f(x)=2x+\frac{1}{x}$.

$\lim_{x\to 0^-} f(x)=-\infty \wedge \lim_{x\to 0^+} f(x)=+\infty \Rightarrow$ y-axeln asymptot.

$\lim_{|x|\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)=2 \Rightarrow \lim_{|x|\to\infty}(f(x)-2x)=0 \Rightarrow$

$\Rightarrow y=2x$ (sned) asymptot.

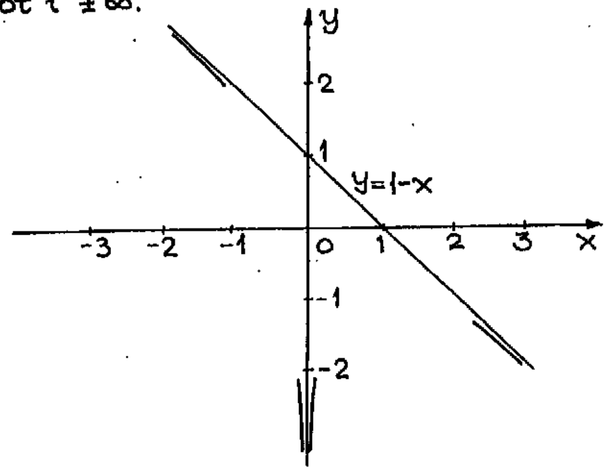
På nästa sidan "syns" ovanstående asymptoter.



b) $f(x) = 1 - x - \frac{1}{x^2}$

$\lim_{x\to 0^+} f(x) = -\infty = \lim_{x\to 0^+} f(x) \Rightarrow$ y-axeln asymptot i $-\infty$.

$\lim_{|x|\to\infty} (f(x) - (1-x)) = -\lim_{|x|\to\infty} \frac{1}{x^2} = 0 \Leftrightarrow y = 1 - x$ (sned) asymptot i $\pm\infty$.



c) $f(x) = \frac{2x^2+1}{x} = 2x + \frac{1}{x}$ (Se under a))

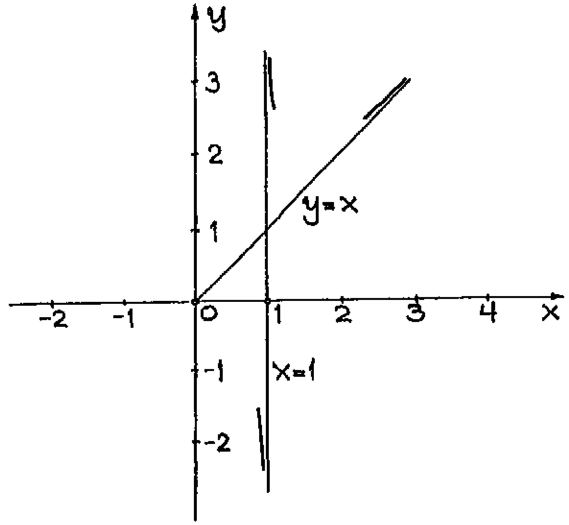
d) $f(x) = \frac{-x^3+x^2-1}{x^2} = 1 - x - \frac{1}{x^2}$ (Se under b)).

e) $f(x) = x + \frac{1}{\ln x}$, $x > 0$, $x \neq 1$

$\lim_{x\to 1^-} f(x) = -\infty \wedge \lim_{x\to 1^+} f(x) = \infty \Rightarrow x = 1$ asymptot i $\pm\infty$.

$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \left(1 + \frac{1}{x\ln x}\right) = 0 \Rightarrow \lim_{x\to\infty} (f(x) - x) =$

$= \lim_{x\to\infty} \frac{1}{\ln x} = 0 \Rightarrow y = x$ (sned) asymptot.



f) $f(x) = 3x + 2 + e^{-x^2}$.

$\lim_{|x|\to\infty} (f(x) - (3x+2)) = \lim_{|x|} e^{-x^2} = 0^+ \Rightarrow y = 3x + 2$ (sned) asymptot. Några andra asymptoter finns inte.

Anm. $y = kx + m$ asymptot till $y = f(x)$; $k = \lim_{|x|\to\infty} \frac{f(x)}{x}$ och $m = \lim_{|x|\to\infty} (f(x) - kx)$; tecknet på x väljs lämplit.

Övning 2.25 (Sid. 50)

Lösning

Fullständig lösning finns på sid. 58-59.

Övning 2.26 (Sid. 50)

Lösning

Fullständig lösning finns på sid. 59-60.

Övning 2.27 (Sid. 51)

Lösning

a) $f(x)=\frac{x^3-4x^2-3x}{x^2-1}=\frac{x(x^2-4x+3)}{(x-1)(x+1)}=\frac{x\cancel{(x-1)}(x-3)}{\cancel{(x-1)}(x+1)}=\frac{x(x-3)}{x+1}=$

$=\frac{x^2-3x}{x+1}=x-4+\frac{4}{x+1}$;

$\lim_{x\to -1^+} f(x)=\infty \wedge \lim_{x\to -1^-} f(x)=-\infty \Rightarrow x=-1$ asymptot i $\pm\infty$

$$\left.\begin{array}{l}\lim_{x\to\infty}(f(x)-(x-4))=0^+\\ \lim_{x\to-\infty}(f(x)-(x-4))=0^-\end{array}\right\}\Rightarrow y=x-4 \text{ asymptot i } \pm\infty.$$

Anm $\lim_{x\to\infty}(f(x)-(x-4))=0^+$ tolkas så att $y=f(x)$ närmar asymptoten $y=x-4$ uppifrån. Detta

b) $f(x)=\arctan\frac{x^2}{x-1}$.

$\lim_{x\to\infty}\arctan\frac{x^2}{x-1}=\lim_{x\to\infty}\arctan x=\frac{\pi}{2}^- \Rightarrow y=\frac{\pi}{2}$ asymptot.

$\lim_{x\to-\infty}\arctan\frac{x^2}{x-1}=\lim_{x\to-\infty}\arctan x=-\frac{\pi}{2}^+\Rightarrow y=-\frac{\pi}{2}$ asymptot.

c) $f(x)=\frac{x^3}{(x+1)^2}=x-2+\frac{3x+2}{(x+1)^2}$

$\lim_{x\to-1^+} f(x)=-\infty=\lim_{x\to-1^-} f(x)\Rightarrow x=-1$ asymptot i $-\infty$.

$$\left.\begin{array}{l}\lim_{x\to\infty}(f(x)-(x-2))=0^+\\ \lim_{x\to-\infty}(f(x)-(x-2))=0^-\end{array}\right\}\Rightarrow y=x-2 \text{ asymptot i } \pm\infty.$$

Övning 2.28 (Sid. 51)

Lösning

a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}=\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+\dots+(\frac{1}{2})^n=\frac{1}{2}\cdot\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}=1-\frac{1}{2^n}$.

b) $\sum_{k=1}^{n}\frac{1}{2^k}=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^n}=\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+\dots+(\frac{1}{2})^n=1-\frac{1}{2^n}$.

c) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^k}=\lim_{n\to\infty}(1-\frac{1}{2^n})=1$.

d) $\sum_{k=1}^{\infty}\frac{1}{2^k}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^k}=\lim_{n\to\infty}(1-2^{-n})=1$.

Övning 2.29 (Sid. 51)

Lösning

a) $\sum_{k=1}^{\infty}(\frac{1}{3})^k=\lim_{n\to\infty}\sum_{k=1}^{n}(\frac{1}{3})^k=\lim_{n\to\infty}\frac{1}{3}\cdot\frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}}=\frac{1}{3}\cdot\frac{1}{2/3}=\frac{1}{2}$.

b) $\sum_{k=1}^{\infty}(-\frac{1}{3})^k=\lim_{n\to\infty}\sum_{k=1}^{n}(-\frac{1}{3})^k=\lim_{n\to\infty}(-\frac{1}{3})\cdot\frac{1-(-1/3)^n}{1-(-1/3)}=(-\frac{1}{3})\cdot\frac{1}{4/3}=-\frac{1}{4}$.

c) $\sum_{k=0}^{\infty}\frac{2^k}{3^k}=\lim_{n\to\infty}\sum_{k=0}^{n-1}(\frac{2}{3})^k=\lim_{n\to\infty}\frac{1-(2/3)^n}{1-2/3}=\frac{1}{1/3}=3$.

d) $\sum_{k=1}^{\infty}2^k=\lim_{n\to\infty}\sum_{k=1}^{n}2^k=\lim_{n\to\infty}2\cdot\frac{2^n-1}{2-1}=2\lim_{n\to\infty}(2^n-1)=\infty$.

Övning 2.30 (Sid. 51)

Lösning

a) $\sum_{k=2}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=2}^{n+1} x^k = \lim_{n\to\infty} x^2 \frac{x^n-1}{x-1} = \frac{x^2}{1-x} \Leftrightarrow |x|<1.$

b) $\sum_{k=1}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=1}^{n} x^k = \lim_{n\to\infty} x\cdot \frac{1-x^n}{1-x} = \frac{x}{1-x} \Leftrightarrow |x|<1.$

c) $\sum_{n=1}^{\infty} x^n = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x} \Leftrightarrow |x|<1.$

d) $\sum_{n=1}^{\infty} (2x)^n = \lim_{N\to\infty} \sum_{n=1}^{N} (2x)^n = \lim_{N\to\infty} 2x\cdot \frac{1-(2x)^N}{1-2x} = \frac{2x}{1-2x} \Leftrightarrow |x|<\frac{1}{2}.$

e) $\sum_{j=0}^{\infty} x^{-j} = \lim_{n\to\infty} \sum_{j=0}^{n-1} x^{-j} = \lim_{n\to\infty} \frac{1-x^{-n}}{1-x^{-1}} = \frac{1}{1-x^{-1}} = \frac{x}{x-1} \Leftrightarrow |x|>1.$

f) $\sum_{j=0}^{\infty} (x+1)^{-j} \overset{!}{=} \frac{x+1}{(x+1)-1} = \frac{x+1}{x} \Leftrightarrow |x+1|>1 \Leftrightarrow \underline{x\le -2} \vee \underline{x>0}.$

I $\overset{!}{=}$ har jag utnyttjat resultatet i e).

$|x+1|>1 \Leftrightarrow x+1>1 \vee -(x+1)>1 \Leftrightarrow x>0 \vee x<-2.$

Övning 2.31 (Sid. 51)

Lösning

$\lim_{L\to\infty} \frac{q}{(1+\frac{z}{B})(1+\frac{z}{L})} = \frac{q}{(1+\frac{z}{B})} \lim_{L\to\infty} \frac{1}{1+z/L} = \frac{q}{1+z/B}.$

Övning 2.32 (Sid. 51)

Lösning

a) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0.$ (Standard med $a=e$ och $\alpha=\frac{1}{2}$).

b) $\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x} = \lim_{x\to\infty} \frac{\ln x^{1/2}}{x} = \lim_{x\to\infty} \frac{1}{2}\cdot\frac{\ln x}{x} = \frac{1}{2}\lim_{x\to\infty} \frac{\ln x}{x} = 0.$

c) $\lim_{x\to\infty} \frac{\sqrt{\ln x}}{x} = \lim_{x\to\infty} \sqrt{\frac{\ln x}{x^2}} = \sqrt{\lim_{x\to\infty} \frac{\ln x}{x^2}} = \sqrt{0} = 0.$

d) $\lim_{x\to\infty} \frac{2^{\ln x}}{x^{\ln 2}} = \lim_{x\to\infty} \frac{(e^{\ln 2})^{\ln x}}{(e^{\ln x})^{\ln 2}} = \lim_{x\to\infty} \frac{e^{(\ln 2)\ln x}}{e^{(\ln 2)(\ln x)}} = \lim_{x\to\infty} 1 = 1.$

e) $\lim_{x\to\infty} \frac{x^3 2^x + x + 1}{x^2 3^x + \sqrt{2x}} = \lim_{x\to\infty} \frac{x^3 2^x}{x^2 3^x} = \lim_{x\to\infty} \frac{x}{1{,}5^x} = 0.$

Anm. För stora x är $x^3 2^x + x + 1 \approx x^3 2^x$ och $x^2 3^x + \sqrt{2x} \approx x^2 3^x$. Sådana resonemang är tillåtna i t ex en tentamen.

f) $\lim_{x\to\infty} (\ln 2^x - \ln 3^x) = \lim_{x\to\infty} \ln\left(\frac{2}{3}\right)^x = \ln 0^+ = -\infty.$

g) $\lim_{x\to\infty} \frac{\ln 2^x}{\ln 3^x} = \lim_{x\to\infty} \frac{x\ln 2}{x\ln 3} = \lim_{x\to\infty} \frac{\ln 2}{\ln 3} = \frac{\ln 2}{\ln 3}.$

h) $\lim_{x\to\infty} (\ln 2x - \ln x) = \lim_{x\to\infty} \ln\frac{2x}{x} = \lim_{x\to\infty} \ln 2 = \ln 2.$

i) $\lim_{x\to\infty} (\ln(2+x) - \ln x) = \lim_{x\to\infty} \ln\frac{2+x}{x} = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right) = \ln 1 = 0.$

j) $\lim_{x\to\infty} (\ln x^2 - \ln x) = \lim_{x\to\infty} \ln\frac{x^2}{x} = \lim_{x\to\infty} \ln x = \infty.$

k) $\lim_{x\to\infty} x(x-\sqrt{x^2-1}) = \lim_{x\to\infty} \frac{x(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}{x+\sqrt{x^2-1}} =$

$= \lim_{x\to\infty} \frac{x(x^2-(x^2-1))}{x+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{x}{x+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{1+\sqrt{1-1/x^2}} = \frac{1}{2}.$

Övning 2.33 (Sid. 52)

Lösning Se nästa sida.

$(1+x)^n \cdot (1+x)^n = (1+x)^{2n}$; $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ (*) ;

När de två faktorerna i VL hopmultipliceras på formen (*), fås en x^n-term om $\binom{n}{n-r}x^{n-r}$ tas från den första faktor och $\binom{n}{r}x^r$ tas från den andra faktorn. Koefficienter för x^n blir

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\dots+\binom{n}{n-1}\binom{n}{1}+\binom{n}{n}\binom{n}{0}.$$

Men $\binom{n}{r}=\binom{n}{n-r}$, så koefficienten reduceras till

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\dots+\binom{n}{n-1}^2+\binom{n}{n}^2.$$

På HL blir x^n:s koefficient $\binom{2n}{n}$, varav följer att

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2.$$

a) $\binom{n}{k} \geq 1 \Rightarrow \binom{n}{k}^2 \geq \binom{n}{k} \Rightarrow \binom{2n}{n} \geq \sum_{k=0}^{n}\binom{n}{k} = 2^n \xrightarrow[n\to\infty]{} \infty \Rightarrow$

$\Rightarrow \lim_{n\to\infty}\binom{2n}{n} = \infty.$ (Se Ö. 1.101).

Övning 2.34 (Sid. 52)

Lösning

a) $\lim_{x\to 0^+}\frac{1}{x} = \infty \wedge \lim_{x\to 0^-}\frac{1}{x} = -\infty \Rightarrow \lim_{x\to 0}\frac{1}{x}$ existerar inte.

b) $\lim_{x\to 0^+}\frac{1}{x^2} = \infty = \lim_{x\to 0^-}\frac{1}{x^2} \Rightarrow \lim_{x\to 0}\frac{1}{x^2} = \infty$ (oegentligt).

c) $\begin{cases} \lim_{x\to 0^+}\frac{\ln x^2}{\sin x} = \frac{-\infty}{0^+} = -\infty \\ \lim_{x\to 0^-}\frac{\ln x^2}{\sin x} = \frac{-\infty}{0^-} = \infty \end{cases} \Rightarrow \lim_{x\to 0}\frac{\ln x^2}{\sin x}$ existerar inte.

Övning 2.35 (Sid. 52)

Lösning

a) $\lim_{x\to\pi}\frac{\sin x}{\pi - x}\,[u=\pi-x] = \lim_{u\to 0}\frac{\sin(\pi-u)}{u} = \lim_{u\to 0}\frac{\sin u}{u} = 1.$

b) $\lim_{x\to\pi}\frac{\cos x}{\pi - x} = \frac{-1}{0}$, gränsvärdet existerar inte.

c) $\lim_{x\to\pi}\frac{\tan x}{\pi - x}\,[u=\pi-x] = \lim_{u\to 0}\frac{\tan(\pi-u)}{u} = \lim_{u\to 0}\left(-\frac{\tan u}{u}\right) =$

$= -\lim_{u\to 0}\frac{\sin u}{u}\frac{1}{\cos u} = -\lim_{u\to 0}\frac{\sin u}{u}\cdot\lim_{u\to 0}\frac{1}{\cos u} = -1\cdot 1\cdot 1 = -1.$

d) $\lim_{x\to\pi}\frac{\cot x}{\pi - x}\,[u=\pi-x] = \lim_{u\to 0}\frac{\cot(\pi-u)}{u} = \lim_{u\to 0}\frac{-\cot u}{u} =$

$= -\lim_{u\to 0}\cos u\cdot\frac{1}{u\sin u} \overset{!}{=} -\infty$; gränsvärdet är oegentligt.

e) $\lim_{x\to\pi}\frac{\sin x}{\pi} = \frac{\sin\pi}{\pi} = 0.$

Övning 2.36 (Sid. 52)

Lösning

$\lim_{x\to 1^-}\frac{\arccos x}{\sqrt{1-x}}\,[u=\arccos x] = \lim_{u\to 0^+}\frac{u}{\sqrt{1-\cos u}} = \lim_{u\to 0^+}\frac{u}{\sqrt{2\sin^2\frac{u}{2}}} =$

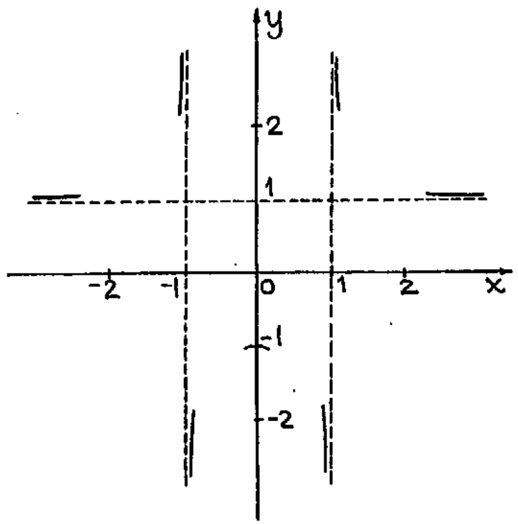
$= \lim_{u\to 0^+}\frac{1}{\sqrt{2}}\frac{u}{\sin(u/2)}\,[u=2v] = \lim_{v\to 0^+}\frac{2}{\sqrt{2}}\cdot\left(\frac{\sin v}{v}\right)^{-1} = \sqrt{2}\cdot 1 = \sqrt{2}.$

Övning 2.37 (Sid. 52)

Lösning

a) $f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2-1+2}{x^2-1} = 1 + \frac{2}{x^2-1} = 1 + \frac{2}{(x-1)(x+1)}$

x	$-\infty$		-1		1		∞
$\mathrm{sgn}(f(x))$	1^+	$+$	$\infty \mid -\infty$	$-$	$-\infty \mid \infty$	$+$	1^+



$x=\pm 1$ är lodräta asymptoter

$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty}(1+\frac{2}{x^2-1}) = 1^+ \Rightarrow y=1$ vågrät asymptot.

b) $f(x) = \frac{\ln x}{x-2},\ x>0.$

$\lim_{x\to 0^+} f(x) = \frac{-\infty}{-2} = \infty \Rightarrow$ y-axeln asymptot i ∞.

$\left.\begin{array}{l}\lim_{x\to 2^-} f(x) = \frac{\ln 2}{0^-} = -\infty \\ \lim_{x\to 2^+} f(x) = \frac{\ln 2}{0^+} = +\infty\end{array}\right\} \Rightarrow x=2$ asymptot i $\pm\infty$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\ln x}{x-2} = 0^+ \Rightarrow$ x-axeln vågrät asymptot.

c) $f(x) = \frac{x\cdot\ln x}{x-1},\ x>0$

$\lim_{x\to 0} f(x) = \frac{0}{-1} = 0;\ \lim_{x\to 1}\frac{x\ln x}{x-1}\ [u=x-1] = \lim_{\to 0}\frac{(1+u)\ln(1+u)}{u} = 1.$

$\lim_{x\to\infty} f(x) = \infty;$ f saknar asymptoter.

Övning 2.38 (Sid. 52)

Lösning

a) $\lim_{x\to\infty}\frac{\ln x+\ln 2x}{\ln x^2} = \lim_{x\to\infty}\frac{\ln x+\ln 2+\ln x}{2\ln x} = \lim_{x\to\infty}\frac{2\ln x+\ln 2}{2\ln x} =$

$= \lim_{x\to\infty}(1+\frac{\ln 2}{2}\cdot\frac{1}{\ln x}) = 1.$

b) $\lim_{x\to 0}\frac{e^{2x}-1}{\sin 3x} = \lim_{x\to 0}\frac{(e^x-1)(e^x+1)}{\sin 3x} = \lim_{x\to 0}\frac{e^x-1}{x}\cdot\frac{x}{\sin 3x}\cdot(1+e^x) =$

$= \lim_{x\to 0}\frac{e^x-1}{x}\cdot\underbrace{\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)^{-1}}_{\text{Se Ö. 2.3 c)}}\cdot\lim_{x\to 0}(e^x+1) = 1\cdot 3^{-1}\cdot 2 = \frac{2}{3}.$

c) $\lim_{x\to 0^+} x^2 e^{1/x}\ [u=\frac{1}{x}] = \lim_{u\to\infty}\frac{e^u}{u^2} = \infty$ (oegentligt).

d) $\lim_{n\to\infty}\sum_{k=0}^{n}(\frac{1}{3})^k = \lim_{n\to\infty}\frac{1-(1/3)^{n+1}}{1-1/3} = \frac{3}{2}\lim_{n\to\infty}(1-\frac{1}{3^{n+1}}) = \frac{3}{2}.$

Övning 2.39 (Sid. 52)

Lösning

a) $\lim_{x\to\infty}\frac{2x^3+x^2}{4x^3+x} = (\frac{\infty}{\infty}) = \lim_{x\to\infty}\frac{\cancel{x^3}(2+x^{-1})}{\cancel{x^3}(4+x^{-2})} = \lim_{x\to\infty}\frac{2+x^{-1}}{4+x^{-2}} = \frac{2}{4} = \frac{1}{2}.$

b) $\lim_{x\to\infty}\frac{xe^{x^2+2x}}{e^{x^2+3x}} = \lim_{x\to\infty} xe^{x^2+2x-(x^2+3x)} = \lim_{x\to\infty} xe^{-x} = 0.$

c) $\lim_{x\to\infty}\frac{\cos x}{2x-\pi} = \lim_{x\to\infty}\frac{\cos x}{2x} = 0.$

3. Derivator

Övning 3.1 (Sid. 62)

Lösning

a) $(x+h)^4 = x^4+4x^3h+6x^2h^2+4xh^3+h^4 \Rightarrow (x+h)^4-x^4 =$
$= h(4x^3+6x^2h+4xh^2+h^3) \Rightarrow \frac{(x+h)^4-x^4}{h} = 4x^3+6x^2h+$
$+4xh^2+h^3 \Rightarrow \lim_{h\to 0}\frac{(x+h)^4-x^4}{h} = 4x^3.$

b) $\sin(x+h)-\sin x = \sin x\cosh h+\cos x\sin h-\sin x =$
$= \sin x(\cos h-1)+\cos x\sin h \Rightarrow \frac{\sin(x+h)-\sin x}{h} =$
$= \cos x\cdot\frac{\sin h}{h}-\sin x\cdot\frac{1-\cos h}{h} \Rightarrow \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} =$
$= \cos x\cdot\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1}+\sin x\cdot\underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{\sin 0=0} = \cos x.$

c) $\lim_{x\to x_0}\frac{e^x-e^{x_0}}{x-x_0} = \lim_{x\to x_0} e^{x_0}\cdot\frac{e^{x-x_0}-1}{x-x_0}\,[u=x-x_0] = e^{x_0}\lim_{u\to 0}\frac{e^u-1}{u} =$
$= e^{x_0}\cdot 1 = e^{x_0}.$

d) $\lim_{x\to x_0}\frac{\ln x-\ln x_0}{x-x_0}\left[\begin{array}{l} u=\ln x \Leftrightarrow x=e^u \\ u_0=\ln x_0 \Leftrightarrow x_0=e^{u_0}\end{array}\right] = \lim_{u\to u_0}\frac{u-u_0}{e^u-e^{u_0}} =$
$= \left(\lim_{u\to u_0}\frac{e^u-e^{u_0}}{u-u_0}\right)^{-1} = (e^{u_0})^{-1} = x_0^{-1} = \frac{1}{x_0}.$

Övning 3.2 (Sid. 62)

Lösning

Antag att vid tiden t finns $n(t)$ kärnor av preparatet. Δt tidsenheter senare finns $n(t+\Delta t)$ kärnor kvar; $n(t+\Delta t)<n(t)$, p.g.a. sönderfallet.

$-\frac{n(t+\Delta t)-n(t)}{\Delta t} \propto n(t) \Leftrightarrow -\frac{\Delta n}{\Delta t} = kn \Rightarrow \lim_{\Delta t\to 0}\frac{\Delta n}{\Delta t} = -kn$
$\Leftrightarrow \frac{dn}{dt} = -kn,\ k>0,\ n=n(t).$

Övning 3.3 (Sid. 62)

Lösning

$n=n(t) \Rightarrow \frac{dn}{dt} = kn,\ k>0.$

Övning 3.4 (Sid. 62)

Lösning

a) s är sträckan; $v=\frac{ds}{dt}$ är hastigheten; $s=s(t)$;
$v\propto\frac{1}{t} \Rightarrow \frac{ds}{dt} = \frac{k}{t},\ k>0.$

b) $\vartheta=\vartheta(t)$ är temperaturen vid tiden t, ϑ_0 är det omgivande mediets (kvasi)konstanta temperatur. Modellen i fråga är

$$\vartheta'(t) = k(\vartheta(t)-\vartheta_0),\ k>0.$$

c) $a(t)=v'(t)=\frac{dv}{dt}$ kallas som bekant acceleration.
$a\propto F \Rightarrow \frac{dv}{dt} = kF,\ k>0;\ v=v(t),\ F=F(t),\ t=$ tiden.

d) Laddningen är $q = q(t)$ och strömstyrkan $i = i(t)$.

$$\frac{dq}{dt} \propto i(t) \Leftrightarrow \frac{dq}{dt} = k \cdot i(t),\ k>0.$$

Anm. Symbolen $\propto$ utläses "är proportionellt mot".

Övning 3.5 (Sid. 62)

Lösning

$f(x) = x^4+2,\ x_0 = 2.$ $\qquad f(2) = 18\ ;\ f'(2) = 32.$

Tangent: $y = f(x_0) + f'(x_0)(x-x_0) \Rightarrow y = 32x+46.$

Normal: $y = f(x_0) - \frac{1}{f'(x_0)}(x-x_0) \Rightarrow y = -\frac{1}{32}x + \frac{289}{16}.$

Anm. $k_t \cdot k_n = -1$ med $k_t = f'(x_0)$.

Övning 3.6 (Sid. 62)

Lösning

a) $f(x) = \cos 2x,\ x_0 = \frac{\pi}{6}.$

$f'(x) = -2\sin 2x\ ;\quad f(\frac{\pi}{6}) = \cos\frac{\pi}{3} = \frac{1}{2}\ ;\ f'(\frac{\pi}{6}) = -2\sin\frac{\pi}{3} = -\sqrt{3}.$

Tangent: $y = f(\frac{\pi}{6}) + f'(\frac{\pi}{6})(x-\frac{\pi}{6}) \Rightarrow y = -\sqrt{3}x + \frac{1}{2} + \frac{\pi\sqrt{3}}{6}.$

Normal: $y = f(\frac{\pi}{6}) - \frac{1}{f'(\frac{\pi}{6})}(x-\frac{\pi}{6}) \Rightarrow y = \frac{\sqrt{3}}{3}x + \frac{1}{2} - \frac{\pi\sqrt{3}}{6}.$

b) $f(x) = \ln x,\ x_0 = 2.$

$f'(x) = \frac{1}{x}\ ;\quad f(2) = \ln 2\ ;\ f'(2) = \frac{1}{2}.$

Tangent: $y = f(2) + f'(2)(x-2) \Rightarrow y = \frac{1}{2}x + \ln 2 - 1.$

Normal: $y = f(2) - \frac{1}{f'(2)}(x-2) \Rightarrow y = -2x+4+\ln 2.$

Anm. En del mellanräkningar har jag inte tagit med; uppgiften är rent gymnasial.

Övning 3.7 (Sid. 63)

Lösning

a) $y = e^{2x} - \sin 3x \Rightarrow (y' = e^{2x}\cdot 2 - \cos 3x \cdot 3) \Rightarrow y' = 2e^{2x} - 3\cos 3x.$

b) $y = \ln x + \arctan x \Rightarrow y' = \frac{1}{x} + \frac{1}{x^2+1}.$

c) $y = \arcsin 2x + (2x+1)^7 \Rightarrow (y' = \frac{1}{\sqrt{1-(2x)^2}}\cdot(2x)' + 7(2x+1)^6\cdot 2)$

$\Rightarrow y' = \frac{2}{\sqrt{1-4x^2}} + 14(2x+1)^6.$

d) $y = \tan(\pi x) + \cos(\frac{\pi x}{2}) \Rightarrow (y' = \frac{1}{\cos^2(\pi x)}(\pi x)' - \sin(\frac{\pi x}{2})\cdot\frac{\pi}{2})$

$\Rightarrow y' = \frac{\pi}{\cos^2 \pi x} - \frac{\pi}{2}\sin\frac{\pi x}{2} \Leftrightarrow y' = \pi(\tan^2 \pi x + 1 - \frac{1}{2}\sin\frac{\pi x}{2}).$

e) $y = \sqrt{x} = x^{1/2} \Rightarrow y' = \frac{1}{2}x^{-1/2} \Leftrightarrow y' = \frac{1}{2\sqrt{x}}.$

f) $y = \frac{1}{x} = x^{-1} \Rightarrow y' = -x^{-2} \Leftrightarrow y' = -\frac{1}{x^2}.$

j) $y = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow y' = -\frac{1}{2}x^{-3/2} \Leftrightarrow y' = -\frac{1}{2x\sqrt{x}}.$

Övning 3.8 (Sid. 63)

Lösning

Se nästa sida.

a) $D e^{2x}\sin 3x = 2e^{2x}\sin 3x + e^{2x}3\cos 3x = \underline{e^{2x}(2\sin 3x + 3\cos 3x)}$.

b) $D e^{-x}(x^2+x) = -e^{-x}(x^2+x) + e^{-x}(2x+1) = \underline{e^{-x}(1+x-x^2)}$.

c) $D\dfrac{x}{x+1} = \dfrac{1\cdot(x+1) - x\cdot 1}{(x+1)^2} = \dfrac{x+1-x}{(x+1)^2} = \underline{\dfrac{1}{(x+1)^2}}$.

d) $D\dfrac{2x+1}{(x+1)^2} = \dfrac{2(x+1)^2 - (2x+1)2(x+1)}{(x+1)^4} = \dfrac{(x+1)\cdot 2(x+1-2x-1)}{(x+1)^4} = \underline{-\dfrac{2x}{(x+1)^3}}$

e) $D e^{-x}\ln x = -e^{-x}\ln x + e^{-x}\cdot\dfrac{1}{x} = \underline{\dfrac{1-x\ln x}{xe^x}}$.

f) $D x\ln|x| = 1\cdot\ln|x| + x\cdot\dfrac{1}{x} = \underline{\ln|x| + 1}$.

Övning 3.9 (Sid. 63)

Lösning

a) $D\ln(x^2+1) = \dfrac{1}{x^2+1}D(x^2+1) = \underline{\dfrac{2x}{x^2+1}}$.

b) $D(\ln x)^2 = 2(\ln x)D\ln x = 2\ln x\cdot\dfrac{1}{x} = \underline{\dfrac{2\ln x}{x}}$.

c) $D e^{-x^2} = e^{-x^2}D(-x^2) = \underline{-2xe^{-x^2}}$.

d) $D e^{-1/x} = e^{-1/x}D(-x^{-1}) = e^{-1/x}(x^{-2}) = \underline{\dfrac{e^{-1/x}}{x^2}}$.

e) $D\sin\sqrt{x} = \cos\sqrt{x}\cdot D\sqrt{x} = \underline{\dfrac{\cos\sqrt{x}}{2\sqrt{x}}}$.

f) $D\sin^2 x = 2\sin x\cdot D\sin x = 2\sin x\cos x = \underline{\sin 2x}$.

g) $D\tan^2 x = 2\tan x\cdot D\tan x = 2\tan x\cdot\dfrac{1}{\cos^2 x} = \underline{\dfrac{2\sin x}{\cos^3 x}}$.

h) $D\arctan\dfrac{1}{x} = D(\dfrac{\pi}{2} - \arctan x) = -D\arctan x = \underline{-\dfrac{1}{x^2+1}}$.

i) $D\arcsin\sqrt{x} = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}}D\sqrt{x} = \dfrac{1}{\sqrt{1-x}}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x-x^2}}$.

Anm. $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}$.

Övning 3.10 (Sid. 63)

Lösning

a) $D\arctan(e^x) = \dfrac{1}{1+(e^x)^2}De^x = \underline{\dfrac{e^x}{e^{2x}+1}}$.

b) $De^{\arcsin x} = e^{\arcsin x}D\arcsin x = \underline{\dfrac{e^{\arcsin x}}{\sqrt{1-x^2}}}$.

c) $D\sqrt{1-x^2} = D(1-x^2)^{1/2} = \dfrac{1}{2}(1-x^2)^{-1/2}D(1-x^2) = \underline{-\dfrac{x}{\sqrt{1-x^2}}}$.

d) $D\dfrac{1}{\sqrt{1-x^2}} = D(1-x^2)^{-1/2} = -\dfrac{1}{2}(1-x^2)^{-3/2}D(1-x^2) = \underline{\dfrac{x}{(1-x^2)^{3/2}}}$.

e) $D(1+x^2)^{3/2} = \dfrac{3}{2}(1+x^2)^{1/2}D(1+x^2) = \underline{x\sqrt{1+x^2}}$.

f) $D\arcsin\dfrac{1}{x} = \dfrac{1}{\sqrt{1-(1/x)^2}}D\dfrac{1}{x} = \dfrac{1}{\sqrt{\dfrac{x^2-1}{x^2}}}(-\dfrac{1}{x^2}) = -\dfrac{1}{\dfrac{\sqrt{x^2-1}}{\sqrt{x^2}}}\cdot\dfrac{1}{x^2} =$

$= -\dfrac{1}{\sqrt{x^2-1}}\,\dfrac{\sqrt{x^2}}{x^2} = -\dfrac{1}{\sqrt{x^2-1}}\cdot\dfrac{1}{\sqrt{x^2}} = \underline{-\dfrac{1}{|x|\sqrt{x^2-1}}}$.

Övning 3.11 (Sid. 63)

Lösning

a) $D\ln(x+\sqrt{x^2+1}) = \dfrac{1}{x+\sqrt{x^2+1}}D(x+\sqrt{x^2+1}) = \dfrac{1+\dfrac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \underline{\dfrac{1}{\sqrt{x^2+1}}}$.

b) $\ln\dfrac{|x|}{\sqrt{x^2+1}} = \ln|x| - \ln\sqrt{x^2+1} = \ln|x| - \dfrac{1}{2}\ln(x^2+1) \Rightarrow$

$\Rightarrow D\ln\dfrac{|x|}{\sqrt{x^2+1}} = D(\ln|x| - \dfrac{1}{2}\ln(x^2+1)) = \underline{\dfrac{1}{x} - \dfrac{x}{x^2+1}}$.

c) $D\ln|\ln|x|| = \dfrac{1}{\ln|x|}D\ln|x| = \underline{\dfrac{1}{x\ln|x|}}$.

Övning 3.12 (Sid. 63)

Lösning

Se nästföljande sida.

a) $D\,2^x = D(e^{\ln 2})^x = e^{x\ln 2} \cdot D(x\ln 2) = \underline{2^x \ln 2}$.

Allmänt gäller: $D a^x = a^x \ln a$.

b) $D\,10^x = 10^x \ln 10$.

c) $D(x^{\sqrt{2}} + x^{-\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1} - \sqrt{2}x^{-\sqrt{2}-1} = \underline{\dfrac{\sqrt{2}(x^{\sqrt{2}} - x^{-\sqrt{2}})}{x}}$.

d) $D x^x = D e^{x\ln x} = e^{x\ln x} D\, x\ln x = \underline{x^x(\ln x + 1)}$.

e) $D(\ln x)^{1-x} = D e^{(1-x)\ln(\ln x)} = e^{(1-x)\ln(\ln x)} D(1-x)\ln(\ln x) =$

$= \underline{(\ln x)^{1-x}\left(-\ln(\ln x) + \dfrac{1-x}{x\ln x}\right)}$.

Övning 3.13 (Sid. 63)

Lösning

$D|u| = \dfrac{|u|}{u} = \begin{cases} 1, & u > 0 \\ -1, & u < 0 \end{cases}$; bra att kunna.

$D\ln|f(x)| = \dfrac{1}{|f(x)|} D|f(x)| = \dfrac{1}{|f(x)|} \cdot \dfrac{|f(x)|}{f(x)} Df(x) = \underline{\dfrac{f'(x)}{f(x)}}$.

Generalisering

$f(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x) \Rightarrow |f(x)| = |f_1(x) \cdot f_2(x) \cdots f_n(x)| =$

$= |f_1(x)| \cdot |f_2(x)| \cdot \ldots \cdot |f_n(x)| \Rightarrow \ln|f(x)| = \sum_{k=1}^{n} \ln|f_k(x)| \Rightarrow$

$\Rightarrow D\ln|f(x)| = \sum_{k=1}^{n} D\ln|f_k(x)| \Leftrightarrow \dfrac{f'(x)}{f(x)} = \sum_{k=1}^{n} \dfrac{f_k'(x)}{f_k(x)}$.

Anm Beviset kompletteras med induktion; denna behandlas inte systematiskt i grundboken.

Övning 3.14 (Sid. 63)

Lösning

$f(x) = \dfrac{e^{x^2}(\arcsin x)^2 \cdot x \cdot \sqrt{\cos x}}{(\ln x)^6 \cdot \sin^2 x}$, $D_f =]0, 1[$.

$\ln f(x) = \ln(e^{x^2}(\arcsin x)^2 \cdot x \cdot \sqrt{\cos x}) - \ln(\ln^6 x \cdot \sin^2 x) =$

$= \ln e^{x^2} + \ln(\arcsin x)^2 + \ln x + \frac{1}{2}\ln(\cos x) -$

$- 6\ln(\ln x) - 2\ln\sin x =$

$= x^2 + 2\ln(\arcsin x) + \ln x + \frac{1}{2}\ln(\cos x) -$

$- \ln(\ln x) - 2\ln(\sin x)$;

$\dfrac{f'(x)}{f(x)} = 2x + \dfrac{2}{\arcsin x} \cdot \dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{x} - \dfrac{\tan x}{2} - \dfrac{6}{x \cdot \ln x} - 2\cot x \Leftrightarrow$

$\Leftrightarrow \underline{f'(x) = f(x) \cdot \left(2x + \dfrac{2}{\sqrt{1-x^2} \cdot \sin^{-1} x} + \dfrac{1}{x} - \dfrac{\tan x}{2} - \dfrac{6}{x\ln x} - 2\cot x\right)}$.

Övning 3.15 (Sid. 63)

Lösning

a) $\underline{f(x) = \ln\sqrt{x}} = \ln x^{1/2} = \frac{1}{2}\ln x \Rightarrow f'(x) = \frac{1}{2x} \Rightarrow f'(1) = \frac{1}{2}$;

$y = f(1) + f'(1)(x-1) = 0 + \frac{1}{2}(x-1) \Leftrightarrow \underline{t: y = \frac{1}{2}x - \frac{1}{2}}$.

b) $\underline{f(x) = 2^{-x}} = (2^{-1})^x \Rightarrow f'(x) = f(x) \cdot \ln 2^{-1} \Rightarrow f'(0) = -\ln 2$;

$y = f(0) + f'(0)(x-0) = 1 - (\ln 2)x \Leftrightarrow \underline{t: y = -(\ln 2)x + 1}$.

Övning 3.16 (Sid. 64)

Se nästa sida.

Lösning

$f(x) = \ln(x+\sqrt{x^2+1}) \Rightarrow f'(x) = \frac{1}{\sqrt{x^2+1}} \Rightarrow f'(0) = 1 = k_t = -\frac{1}{k_n}$.

Tangenten är $y=x$ och normalen $y=-x$.

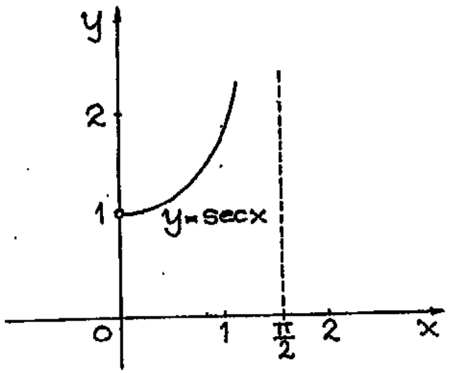
Övning 3.17 (Sid. 64)

Lösning

a) $f(x) = \sec x,\ 0<x<\pi/2$.

$f'(x) = \frac{\sin x}{\cos^2 x} > 0 \Rightarrow f$ strängt växande $\Rightarrow$ värdetabell...

x	0,5	0,8	1	1,2	1,5
y	1,14	1,44	1,85	2,76	14,1

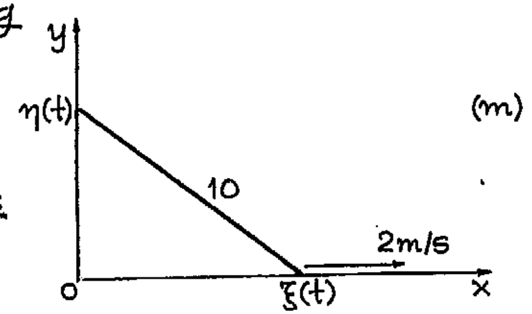


f är injektiv (den är ju växande), så den har en invers; dess definitionsmängd är $D_{f^{-1}}: y>1$ och dess värdemängd $V_{f^{-1}}: 0<x<\pi/2$.

b) $\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{\cos^2 x}{\sin x} = \frac{\cos^2 x}{\sqrt{1-\cos^2 x}} = \frac{y^{-2}}{\sqrt{1-y^{-2}}} = \frac{1}{y\sqrt{y^2-1}},\ y>1$.

Övning 3.18 (Sid. 64)

Lösning



När kontaktpunkten med x-axeln har koordinaten 6, har kontaktpunkten med y-axeln koordinaten 8 (triangeln är då egyptisk).

Pythagoras' sats ger vid en tidpunkt t:

$\xi(t)^2 + \eta(t)^2 = 10^2 \Rightarrow 2\xi(t)\xi'(t) + 2\eta(t)\eta'(t) = 0 \Leftrightarrow \eta'(t) = -\frac{\xi(t)}{\eta(t)}\xi'(t) \Rightarrow \frac{d\eta}{dt} = -\frac{6}{8}\cdot 2 = -1{,}5$.

Svar: Stegens överdel faller med 1,5 m/s.

Övning 3.19 (Sid. 64)

Lösning

$p\cdot V^{1,4} = k,\ p=p(t),\ V=V(t);\ p(t_0)=5,\ V(t_0)=56,\ V'(t_0)=4$.

$\ln pV^{1,4} = \ln k \Leftrightarrow \ln p + 1{,}4\ln V = \ln k \Rightarrow \frac{p'(t)}{p(t)} + 1{,}4\frac{V'(t)}{V(t)} = 0 \Rightarrow \frac{p'(t_0)}{p(t_0)} + 1{,}4\frac{V'(t_0)}{V(t_0)} = 0 \Rightarrow p(t_0) = -p'(t_0)\cdot\frac{V(t_0)}{1{,}4V'(t_0)} = -0{,}5$.

$\Rightarrow \forall n \geq 1: f^{(n)}(x) = (-3)^n f(x)$ (induktivt).

Bevis:

(i) $\begin{cases} VL_1 = f^{(1)}(x) = f'(x) \\ HL_1 = (-3)^1 f(x) = -3f(x) \end{cases} \Rightarrow VL_1 = HL_1$.

(ii) Antag att $VL_\nu = HL_\nu$, dvs. $f^{(\nu)}(x) = (-3)^\nu f(x)$.

(iii) $VL_{\nu+1} = f^{(\nu+1)}(x) = \frac{d}{dx} f^{(\nu)}(x) \overset{(ii)}{=} \frac{d}{dx}(-3)^\nu f(x) = (-3)^\nu f'(x) =$

$= (-3)^\nu \cdot (-3) f(x) = (-3)^{\nu+1} f(x) = HL_{\nu+1}$.

Induktionen är därmed genomförd.

b) Jag tillämpar Leibniz' formel och får

$f(x) = x^3 e^x \Rightarrow f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (D^k x^3) D^{n-k} e^x =$

$= \sum_{k=0}^{n} \binom{n}{k} (D^k x^3) \cdot e^x = e^x \sum_{k=0}^{n} \binom{n}{k} D^k x^3 = e^x \binom{n}{0} x^3 +$

$+ e^x \binom{n}{1} 3x^2 + e^x \binom{n}{2} \cdot 6x + e^x \binom{n}{3} \cdot 6 = e^x \cdot x^3 + e^x \cdot n \cdot 3x^2 +$

$+ e^x \cdot \frac{n(n-1)}{2} \cdot 6x + e^x \frac{n(n-1)(n-2)}{6} \cdot 6 = \underline{e^x(x^3 + 3nx^2 +}$

$\underline{+ 3n(n-1)x + n(n-1)(n-2))}$.

Övning 3.25 (Sid. 65)

Lösning

a) $De^{ix} = D(\cos x + i \sin x) = -\sin x + i\cos x = i(\cos x + i \sin x) = ie^{ix}$.

b) $De^{-ix} = D\overline{e^{ix}} = \overline{De^{ix}} = \overline{ie^{ix}} = \overline{(i)} \cdot \overline{e^{ix}} = -ie^{-ix}$.

c) $De^{(1+i)x} = De^x e^{ix} = e^x \cdot e^{ix} + e^x \cdot ie^{ix} = (1+i)e^x \cdot e^{ix} =$

$= (1+i)e^{(1+i)x}$.

Anm. Derivatoperatorn $D = \frac{d}{dx}$ är reell.

Övning 3.26 (Sid. 65)

Lösning

$f(x) = e^{2x} - 2x \Rightarrow f'(x) = 2e^{2x} - 2 \Rightarrow f'(h) = 2(e^{2h} - 1) = k_t$

$\Leftrightarrow k_n = -\frac{1}{f'(h)} = -\frac{1}{2(e^{2h}-1)} \Rightarrow y = f(h) - \frac{1}{f'(h)}(x-h) =$

$= e^{2h} - 2h - \frac{1}{2(e^{2h}-1)}(x-h)$ (normalens ekvation).

$x = 0 \Rightarrow y_p = f(h) = e^{2h} - 2h + \frac{h}{2(e^{2h}-1)} \xrightarrow[h \to 0]{!} 1 + \frac{1}{4} = \frac{5}{4}$.

Anm. I ! har jag utnyttjat resultatet i 2.5 b).

Svar: Gränspunkten är $(0, \frac{5}{4})$.

Övning 3.27 (Sid. 65)

Lösning

$f(x) = \arctan(e^x) + \arctan(e^{-x}) \Rightarrow f'(x) = \frac{1}{1+(e^x)^2} \cdot (e^x)' +$

$+ \frac{1}{1+(e^{-x})^2} \cdot (e^{-x})' = \frac{e^x}{1+e^{2x}} + \frac{-e^{-x}}{1+e^{-2x}} = \frac{e^x}{1+e^{2x}} - \frac{e^x}{e^{2x}+1} = 0$.

Övning 3.28 (Sid. 65)

Lösning

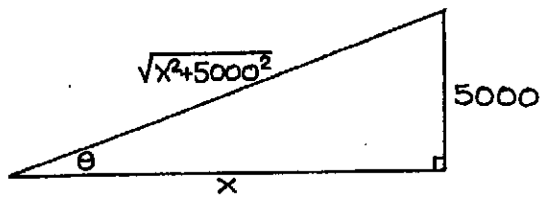
Se nästa sida.

Svar: Trycket sjunker med 0,5 atm $= 5{,}06625 \cdot 10^4$ Pa.

Övning 3.20 (Sid. 64)

Lösning

(m)



$$\tan\theta(t) = \frac{5000}{x(t)} \Rightarrow \frac{1}{\cos^2\theta(t)} \cdot \theta'(t) = -\frac{5000}{x(t)^2}\, x'(t) \Leftrightarrow$$

$$\Leftrightarrow \theta'(t) = -\cos^2\theta(t) \cdot \frac{5000}{x(t)^2}\, x'(t) = -\frac{x^2}{x^2+5000^2} \cdot \frac{5000}{x^2}\, x'(t) =$$

$$= -\frac{5000}{x(t)^2+5000^2}\, x'(t) \Rightarrow \theta'(t_0) \overset{!}{=} -\frac{5000}{x(t_0)^2+5000^2} \cdot \left(-\frac{500}{3}\right) =$$

$$= \frac{5 \cdot 10^3}{250 \cdot 10^6} \cdot \frac{500}{3} = \frac{1}{300}\,\frac{\text{rad}}{\text{sek}} = 3{,}3 \cdot 10^{-3}\,\frac{\text{rad}}{\text{sek}} = \underline{0{,}19°/\text{s}.}$$

Övning 3.21 (Sid. 65)

Lösning

$$\pi r^2(t) h(t) = k \Rightarrow \frac{d}{dt} r^2(t) h(t) = 0 \Rightarrow 2r(t) r'(t) h(t) +$$

$$+ r^2(t) h'(t) = 0\ ;\ \underline{h'(t) = 2a h(t)} \Rightarrow r'(t) h(t) + r(t) a h(t) =$$

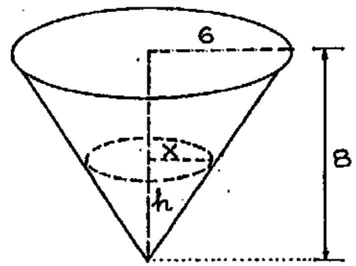
$$= 0 \Rightarrow \underline{r'(t) = -a r(t).}\ (*)$$

Anm. Jag har satt proportionalitetskonstanten i $h'(t) \propto h(t)$ lika med $2a$ för snyggt resultat (*).

Övning 3.22 (Sid. 65)

Lösning

(m)



$$\frac{x}{6} = \frac{h}{8} \Rightarrow x = \frac{3}{4}h \Rightarrow V = \frac{1}{3}\pi x^2 h = \frac{\pi}{3} \cdot \frac{9}{16} h^2 \cdot h = \frac{3\pi}{16} h^3.$$

$$V(t) = \frac{3\pi}{16} h^3(t) \Rightarrow V'(t) = \frac{3\pi}{16} \cdot 3h^2(t) h'(t) = \frac{9\pi}{16} h^2(t) h'(t)$$

$$\Leftrightarrow h'(t) = \frac{16}{9\pi h^2(t)} V'(t) \Rightarrow \frac{dh}{dt} = \frac{16}{9\pi \cdot 4^2} \cdot 0{,}1 = \frac{1}{90\pi}\ \text{m/min}.$$

Svar: Vattenytan stiger med 3,5 mm/min.

Övning 3.23 (Sid. 65)

Lösning

Ökningen tecknas $f'(t)$; den avtar, så $(f'(t))' =$

$= \underline{f''(t) \leq 0} \Leftrightarrow \text{sgn}(f''(t)) = -1 \vee \text{sgn}(f''(t)) = 0.$

Svar: $f''(t)$ är icke-positiv.

Övning 3.24 (Sid. 65)

Lösning

a) $f(x) = e^{-3x} \Rightarrow f'(x) = -3f(x) \Rightarrow f''(x) = -3f'(x) = (-3)^2 f(x) \Rightarrow \cdots$

Övning 3.29 (Sid. 65)

Lösning

a) $D\frac{x}{x+1} = \frac{1}{(x+1)^2}$ (Se 3.8 c)).

b) $De^{x^2/(1+x)} = e^{x^2/(1+x)} D\frac{x^2}{x+1} = e^{x^2/(1+x)} \cdot \frac{2(x+1)x - x^2}{(x+1)^2} =$

$= e^{x^2(x+1)^{-1}} \cdot \frac{x^2+2x}{(x+1)^2}.$

c) $f(x) = \frac{2x+3}{\sqrt{4x^2+12x+10}} \Rightarrow \left\{ \begin{array}{l} f(x) = \frac{u}{\sqrt{u^2+1}} \Rightarrow f'(x) = \frac{1}{(u^2+1)^{3/2}} \frac{du}{dx} \\ u = 2x+3 \Rightarrow \frac{du}{dx} = 2 \end{array} \right\} \Rightarrow$

$\Rightarrow f'(x) = \frac{2}{(4x^2+12x+10)^{3/2}}.$

d) $D(x^2+1)\sqrt{x^2+1} = D(x^2+1)^{3/2} = \frac{3}{2}(x^2+1)^{1/2} \cdot 2x = 3x\sqrt{x^2+1}.$

Övning 3.30 (Sid. 65)

Lösning

a) $D(A\cos(\omega x+\delta)) = A \cdot D\cos(\omega x+\delta) = A \cdot (-\sin(\omega x+\delta)) \cdot \omega =$

$= -A\omega\sin(\omega x+\delta).$

b) $De^{-x}\sin x = (-e^{-x})\sin x + e^{-x} \cdot \cos x = e^{-x}(\cos x - \sin x).$

c) $De^{\sin x} = e^{\sin x} D\sin x = e^{\sin x} \cos x.$

d) $D(-x+\tan x) = -1 + \frac{1}{\cos^2 x} = -1 + \tan^2 x - 1 = \tan^2 x.$

e) $D\cot\sqrt{x} = -\frac{1}{\sin^2\sqrt{x}} D\sqrt{x} = \frac{-1}{\sin^2\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = -\frac{1+\cot^2\sqrt{x}}{2\sqrt{x}}.$

f) $D\sin^5 3x = 5\sin^4 3x \cdot D\sin 3x = 5\sin^4 3x \cdot \cos 3x \cdot D(3x) =$

$= 5\sin^4 3x \cdot \cos 3x \cdot 3 = 15\sin^4 3x \cdot \cos 3x.$

g) $D\tan^3 x = 3\tan^2 x \cdot D\tan x = 3\tan^2 x \cdot \frac{1}{\cos^2 x} = \frac{3\sin^2 x}{\cos^4 x}.$

h) $D\sin(\cos 2x) = \cos(\cos 2x) D\cos 2x = \cos(\cos 2x)(-2\sin 2x).$

Övning 3.31 (Sid. 65)

Lösning

a) $y = \sinh x = \frac{e^x - e^{-x}}{2} \Leftrightarrow e^x - e^{-x} = 2y \Leftrightarrow e^x(e^x - e^{-x}) = 2ye^x$

$\Leftrightarrow (e^x)^2 - 1 = 2ye^x \Leftrightarrow (e^x)^2 - 2ye^x = 1 \Leftrightarrow e^x = y + \sqrt{y^2+1}$

$\Leftrightarrow x = \ln(y+\sqrt{y^2+1}) = f^{-1}(y) \Rightarrow f^{-1}(x) = \ln(x+\sqrt{x^2+1}).$

Anm. $f^{-1}(x) = \operatorname{arsinh} x$; areasinus hyperbolicus.

b) $f^{-1}(x) = \ln(x+\sqrt{x^2+1}) \Rightarrow Df^{-1}(x) = \frac{1}{\sqrt{x^2+1}} \Rightarrow Df^{-1}(a) = \frac{1}{\sqrt{a^2+1}}.$

c) $y = \sinh x \Leftrightarrow x = \ln(y+\sqrt{y^2+1})$

$1 = \frac{d}{dy}\sinh x = \cosh x \frac{dx}{dy} \overset{!}{=} \sqrt{\sinh^2 x + 1}\, \frac{dx}{dy} = \sqrt{y^2+1}\, \frac{dx}{dy}$

$\Leftrightarrow \frac{dx}{dy} = \frac{1}{\sqrt{y^2+1}} \Rightarrow \left.\frac{dx}{dy}\right|_{y=a} = \frac{1}{\sqrt{a^2+1}} \Leftrightarrow Df^{-1}(a) = \frac{1}{\sqrt{a^2+1}}.$

Anm. $\cosh^2 x - \sinh^2 x = 1$ (hyperboliska ettan).

Övning 3.32 (Sid. 66)

Lösning

Cos-satsen $\Rightarrow b^2 = x^2 + R^2 - 2xR\cos\omega t$, $x = x(t)$.

$0 = \frac{d}{dt}(x^2+R^2-2xR\cos\omega t) = 2x\frac{dx}{dt} - 2R\frac{dx}{dt}\cos\omega t +$

$+2xR\omega\sin\omega t \overset{\omega t=\frac{\pi}{2}}{\Rightarrow} 0=2x\left(\frac{\pi}{2\omega}\right)\frac{dx}{dt}+2\omega R x\left(\frac{\pi}{2\omega}\right) \Leftrightarrow \frac{dx}{dt}=-R\omega.$

Resultat: Kolvens fart är $R\omega$ (nedåtriktad).

Övning 3.33 (Sid. 66)

Lösning

Linans längd är $2h$; y är avståndet från golvet; x är som i figuren. Pythagoras' sats ger

$x^2+h^2=(2h-y)^2 \Rightarrow 2x\frac{dx}{dt}=2(2h-y)\left(-\frac{dy}{dt}\right) \Leftrightarrow x\frac{dx}{dt}=$

$=-(2h-y)\frac{dy}{dt} \Rightarrow xv_0=(y-2h)\frac{dy}{dt} \Leftrightarrow xv_0=\sqrt{x^2+h^2}\,\frac{dy}{dt} \Leftrightarrow$

$\Leftrightarrow \frac{dy}{dt}=\frac{x}{\sqrt{x^2+h^2}}v_0.$

Övning 3.34 (Sid. 66)

Lösning

a) $f(x)=x\cdot\arctan x-\ln\sqrt{1+x^2}=x\arctan x-\frac{1}{2}\ln(x^2+1) \Rightarrow$

$\Rightarrow f'(x)=\arctan x+x\cdot\frac{1}{x^2+1}-\frac{1}{2}\frac{1}{x^2+1}\cdot 2x=\arctan x.$

b) $y=\arcsin\frac{e^{2x}-1}{e^{2x}+1} \Leftrightarrow \sin y=\frac{e^{2x}-1}{e^{2x}+1}=1-\frac{2}{e^{2x}+1} \Rightarrow$

$\Rightarrow \cos y\cdot y'=\frac{4e^{2x}}{(e^{2x}+1)^2} \Leftrightarrow \sqrt{1-\sin^2 y}\,\frac{dy}{dx}=\frac{4e^{2x}}{(e^{2x}+1)^2}$; (*)

$1-\sin^2 y=1-\frac{e^{4x}-2e^{2x}+1}{e^{4x}+2e^{2x}+1}=\frac{4e^{2x}}{(e^{2x}+1)^2} \Rightarrow \sqrt{1-\sin^2 y}=\frac{2e^x}{e^{2x}+1}$;

(*) $\Rightarrow \frac{2e^x}{e^{2x}+1}g'(x)=\frac{4e^{2x}}{e^{2x}+1} \Leftrightarrow g'(x)=\frac{2e^x}{e^{2x}+1}=\frac{2}{e^x+e^{-x}}=\frac{1}{\cosh x}.$

Övning 3.35 (Sid. 66)

Lösning

$f(x)=x^2;\quad x_0=a.$

$f'(x)=2x \Rightarrow k_t=f'(a)=2a=-1/k_n \Leftrightarrow k_n=-\frac{1}{2a},\ a\neq 0$

Normalens ekvation blir $y=-\frac{1}{2a}x+a^2+\frac{1}{2}$.

Skärningspunkten kallas $P:(p,p^2)$.

$p^2=-\frac{1}{2a}p+a^2+\frac{1}{2} \Leftrightarrow p=-\frac{1}{4a}\pm\sqrt{\frac{1}{16a^2}+a^2+\frac{1}{2}}=\frac{-1\pm(4a^2+1)}{4a}$

$\Leftrightarrow p=a \vee p=-\frac{2a^2+1}{2a} \Rightarrow \underline{P_1:(a,a^2)} \vee \underline{P_2:\left(-\frac{2a+1}{2a},\ \frac{(2a+1)^2}{4a^2}\right)}.$

Övning 3.36 (Sid. 67)

Lösning

$V(t)=\frac{\pi}{3}(60h^2(t)-h^3(t)) \Rightarrow V'(t)=\pi(40h(t)\ \ h^2(t))h'(t)$

$\Rightarrow V'(t_0)=\pi(40h(t_0)-h^2(t_0))h'(t_0)=\pi(40\cdot 10-100)\cdot 0{,}03=$

$=\pi\,300\cdot 0{,}03=9\pi\approx \underline{28{,}3\,cm^3/s}.$

4. Användningar av derivator

Övning 4.1 (Sid. 74)

Lösning

a) $f(x)=\frac{1}{9}\frac{x^3}{x+2}$, $x\neq -2$.

$f'(x)=\frac{3x^2(x+2)-x^3}{9(x+2)^2}=\frac{3x^3+6x^2-x^3}{9(x+2)^2}=\frac{2x^3+6x^2}{9(x+2)^2}=\frac{2x^2(x+3)}{9(x+2)^2}$;

x		-3		-2		0	
$\operatorname{sgn}(f'(x))$	$-$	0	$+$		$+$	0	$+$
$f(x)$	↘	3	↗		↗	0	↗

$x=-3$ och $x=0$ är stationära (kritiska).

$x=-3$ är en lokal minimipunkt. (Minimipunkt = en punkt som ger lokalt minimum).

b) $f(x)=x^3-3x$

$f'(x)=3x^2-3=3(x^2-1)=3(x+1)(x-1)$

x		-1		1	
$\operatorname{sgn}(f'(x))$	$+$	0	$-$	0	$+$
$f(x)$	↗	2	↘	-2	↗

$x=-1$ och $x=1$ är stationära; $x=-1$ är en lokal maximipunkt och $x=1$ en lokal minimipunkt.

c) $f(x)=x^2e^x \Rightarrow f'(x)=2xe^x+x^2e^x=x(x+2)e^x$

Derivatans teckenschema är enligt följande:

x		-2		0	
$\operatorname{sgn}(f'(x))$	$+$	0	$-$	0	$+$
$f(x)$	↗	$\frac{4}{e^2}$	↘	0	↗

$x=-2$ och $x=0$ är stationära; $x=-2$ är en lokal maximipunkt och $x=0$ är en lokal minimipunkt.

d) $f(x)=(2+\sin x)^5 \Rightarrow f'(x)=5(2+\sin x)^4\cdot\cos x$;

f är periodisk med fundamentalperioden 2π, så vi studerar restriktionen $f_{2\pi}$.

$f'(x)=0 \Rightarrow \cos x=0 \Leftrightarrow x=\frac{\pi}{2} \vee x=\frac{3\pi}{2}$.

x		$\pi/2$		$3\pi/2$	
$\operatorname{sgn}(f'(x))$	$+$	0	$-$	0	$+$
$f(x)$	↗	243	↘	1	↗

Stationära är $\frac{\pi}{2}+n\pi$, $n\in\mathbb{Z}$; $\frac{\pi}{2}+k.2\pi$ är lokala maximipunkter; $\frac{3\pi}{2}+l.2\pi$ är lokala min/pkr.

e) $f(x)=\frac{x^3}{x^2-1}$, $x\neq\pm 1$.

$f'(x)=\frac{3x^2(x^2-1)-x^3\cdot 2x}{(x^2-1)^2}=\frac{x^4-3x^2}{(x^2-1)^2}=\frac{x^2(x+\sqrt{3})(x-\sqrt{3})}{(x^2-1)^2}$;

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$\operatorname{sgn}(f'(x))$	$+$	0	$-$		$-$	0	$-$		$-$	0	$+$
$f(x)$	↗	$-\frac{3\sqrt{3}}{2}$	↘		↘	0	↘		↘	$\frac{3\sqrt{3}}{2}$	↗

Kritiska (stationära) är $x=-\sqrt{3}, 0, \sqrt{3}$; $x=-\sqrt{3}$ är en lokal max/pkt; $x=\sqrt{3}$ är en lokal min/pkt.

Övning 4.2 (Sid. 74)

Lösning

a) $f(x) = \frac{1}{9}\frac{x^3}{x+2}$, $x \neq -2$.

$f(-3) = 3$ är ett lokalt maximivärde.

b) $f(x) = x^3 - 3x$

$f(-1) = 2$ är ett lokalt maximivärde.

$f(1) = -2$ är ett lokalt minimivärde.

c) $f(x) = x^2 e^x$

$f(-2) = 4e^{-2}$ är ett lokalt maximivärde.

$f(0) = 0$ är ett lokalt minimivärde.

d) $f(x) = (2+\sin x)^5$

$f(\frac{\pi}{2} + k2\pi) = 243$ är lokala maximivärden.

$f(\frac{3\pi}{2} + l2\pi) = 1$ är lokala minimivärden.

e) $f(x) = \frac{x^3}{x^2-1}$, $|x| \neq 1$.

$f(-\sqrt{3}) = -\frac{3\sqrt{3}}{2}$ är lokalt maximivärde.

$f(\sqrt{3}) = \frac{3\sqrt{3}}{2}$ är lokalt minimivärde.

Anm. Jag har utnyttjat teckentabellerna i den föregående övningen.

Övning 4.3 (Sid. 74)

Lösning

$f(x) = xe^{-1/x}$

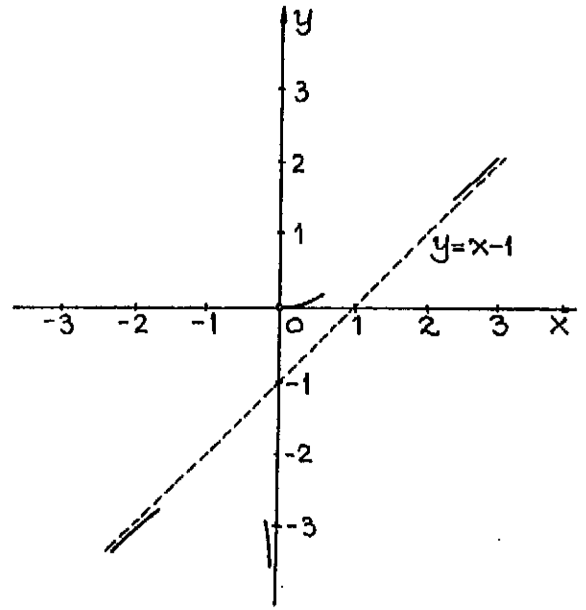
(i) f är definierad för $x \neq 0$.

$$\left.\begin{array}{l} \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} xe^{-1/x} = -\infty \\ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} xe^{-1/x}\ [u = \frac{1}{x}] = \lim_{u\to\infty} \frac{u}{e^u} = 0^+ \end{array}\right\} \Rightarrow \text{y-axeln}$$

är lodrät asymptot i $-\infty$.

(ii) $\lim_{|x|\to\infty} f(x)/x = \lim_{|x|\to\infty} e^{-1/x} = 1 \Rightarrow \lim_{x\to\pm\infty} (f(x)-x) = \lim_{x\to\pm\infty} x(e^{-1/x}-1) =$

$= [v = \frac{1}{x}] = \lim_{v\to 0^\pm} \frac{e^{-v}-1}{v} = -1^\pm \Rightarrow y = x-1$ asymptot i $\pm\infty$.



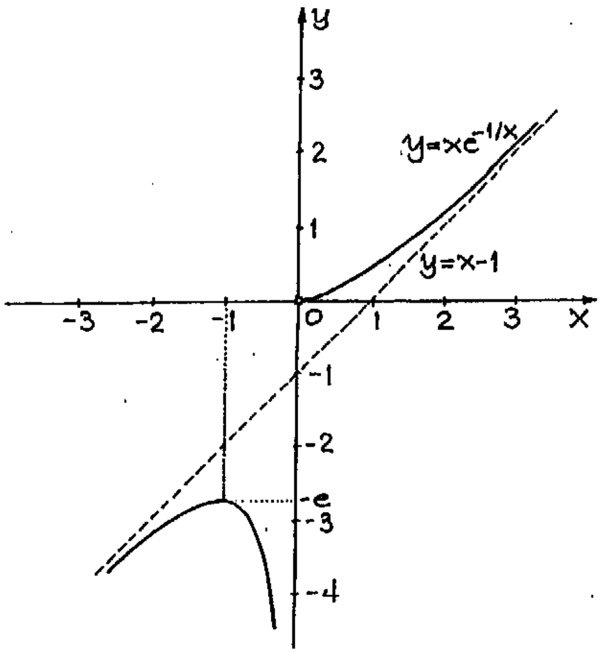
forts.

(iii) $f'(x) = e^{-1/x} + xe^{-1/x} \cdot \frac{1}{x^2} = e^{-1/x}\left(1+\frac{1}{x}\right) = \frac{x+1}{x}e^{-1/x}$;

x		-1		0	
$\operatorname{sgn}(f'(x))$	$+$	0	$-$		$+$
$f(x)$	$\nearrow$	$-e$	$\searrow$		$\nearrow$

$f(-1) = -e$ är ett lokalt maximivärde. Största och minsta värde saknas.

x	-2	$-1,5$	$-0,5$	$-0,2$	$0,2$	$0,5$	1
y	$-3,3$	$-2,9$	$-3,7$	-29	0^+	$0,1$	$0,4$



Övning 4.4 (Sid. 74)

Lösning

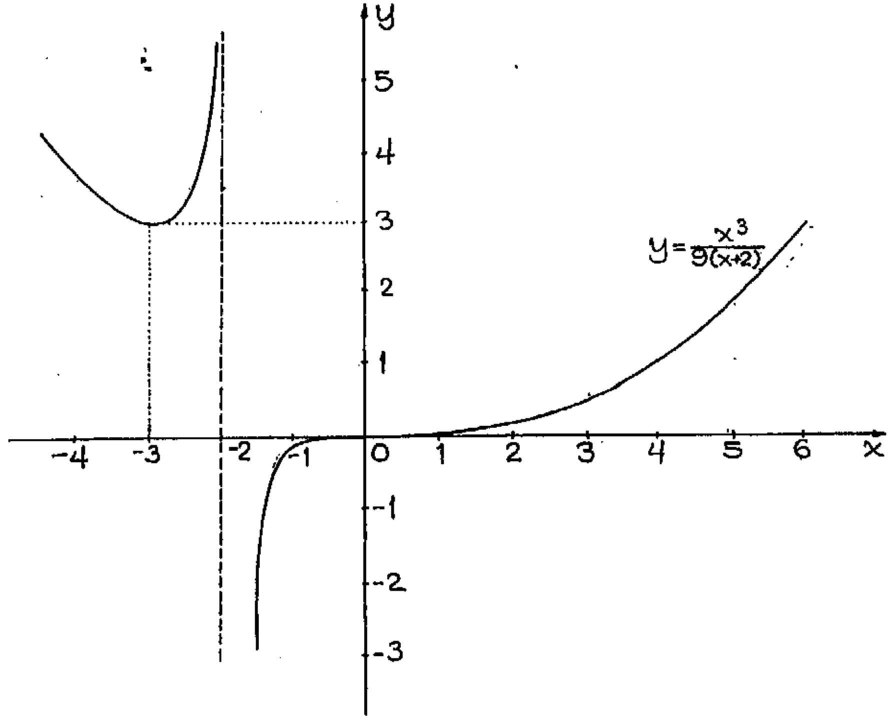
a) $f(x) = \frac{1}{9}\frac{x^3}{x+2}$:

(i) $\lim_{x\to -2^-} f(x) = \infty \wedge \lim_{x\to -2^+} f(x) = -\infty \Rightarrow x=-2$ asymptot i $\pm\infty$.

(ii) Studium av derivatan finns i Ö. 4.1 a).

(iii)

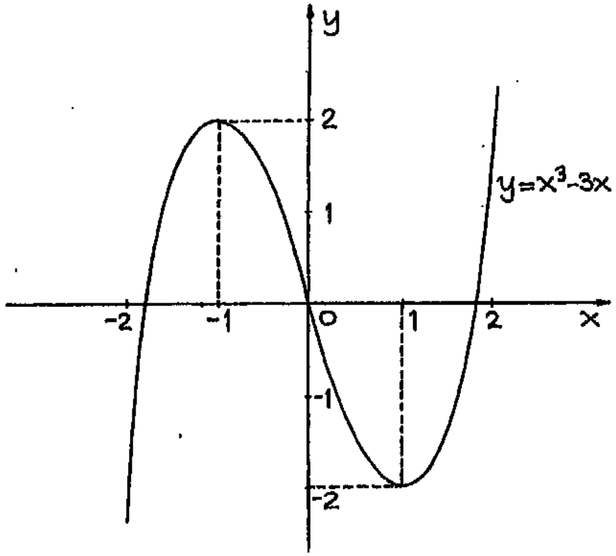
x	-4	$-3,5$	$-2,5$	$-1,5$	-1	$0,5$	1	3	5
y	$3,6$	$3,2$	$3,5$	$-1,9$	$-0,1$	0	$0,03$	$0,6$	$1,74$



b) $f(x) = x^3 - 3x$

f är en polynomfunktion så det finns inget mer än det som står i 4.1 b) att göra. En värdetabell...

x	-2	$-1,5$	$-0,5$	$0,5$	$1,5$	2	$2,5$
y	-2	$1,1$	$1,4$	$-1,4$	$-1,1$	2	$8,1$

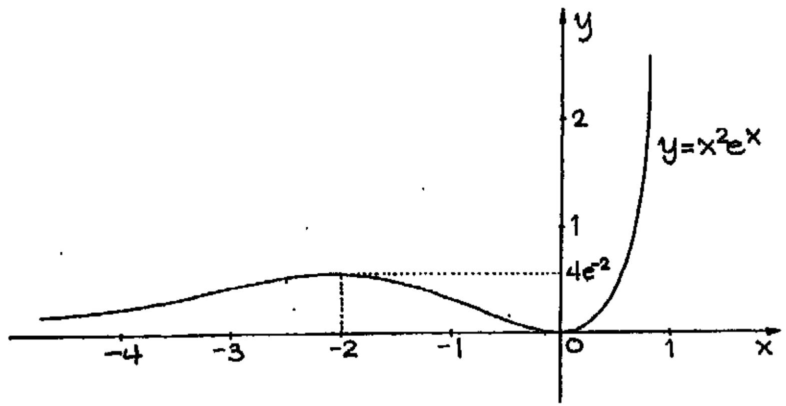


c) $\underline{f(x) = x^2 e^x}$

$\lim_{x \to -\infty} x^2 e^x = \lim_{u \to \infty} u^2 e^{-u} = 0^+ \Rightarrow$ x-axeln asymptot i $-\infty$.

En värdetabell är vad som behövs. (Se f.ö. 4.1c)).

x	-4	-3	-2,5	-1,5	-1	-0,5	0,5
y	0,29	0,49	0,51	0,50	0,37	0,15	0,41



d) $\underline{f(x) = x + \frac{1}{x}}$.

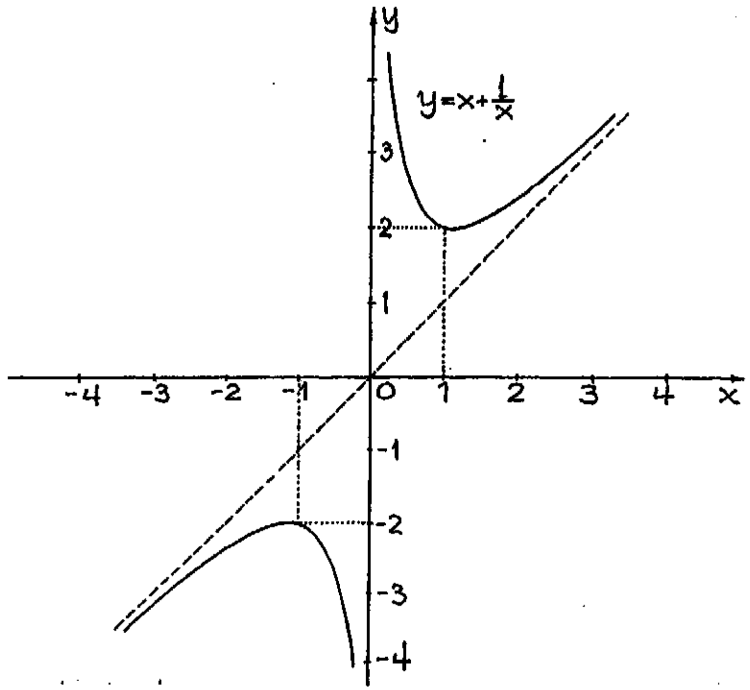
$\lim_{x \to 0^+} f(x) = \infty \wedge \lim_{x \to 0^-} f(x) = -\infty \Rightarrow$ y-axeln asymptot.

$$\left\{ \begin{array}{l} \lim_{x \to \infty} (f(x) - x) = \lim_{x \to \infty} \frac{1}{x} = 0^+ \\ \lim_{x \to -\infty} (f(x) - x) = \lim_{x \to -\infty} \frac{1}{x} = 0^- \end{array} \right. \Rightarrow y = x \text{ (sned) asymptot.}$$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x} = \frac{(x+1)(x-1)}{x}.$$

	-1		0		1	
sgn(f'(x))	+	0	−	−	0	+
f(x)	↗	-2	↘	↘	2	↗

x	±0,2	±0,5	±0,7	±1,2	±1,5	±2	±3	±4
y	±5,2	±2,5	±2,1	±2,0	±2,2	±2,5	±3,3	±4,3



e) $f(x) = e^x/x^2,\ x \neq 0.$

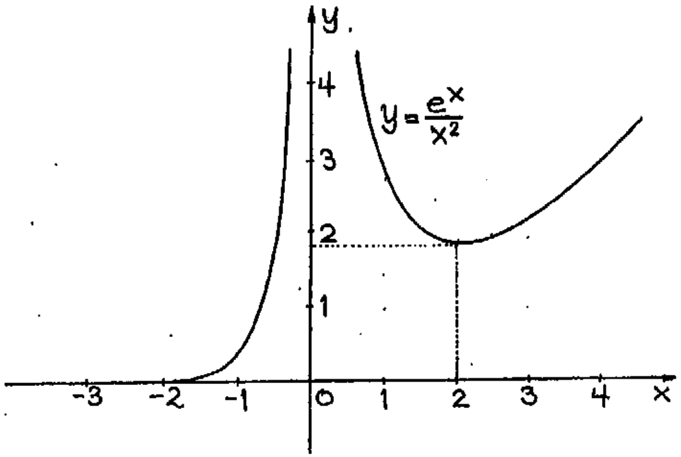
$\lim_{x\to\infty} f(x) = \infty \wedge \lim_{x\to-\infty} f(x) = 0^+ \Rightarrow$ x-axeln asymptot i $-\infty$.

$\lim_{x\to 0^+} f(x) = \infty = \lim_{x\to 0^-} f(x) \Rightarrow$ y-axeln asymptot i ∞.

$f'(x) = \frac{x-2}{x^3} e^x;$

x		0		2	
sgn($f'(x)$)	+	|	−	0	+
$f(x)$	↗	|	↘	$\frac{e^2}{4}$	↗

x	-2	-1	-0,5	0,5	1	1,5	2,5
y	0,03	0,37	2,43	6,59	2,72	1,99	1,94



Övning 4.5 (Sid. 74)

Lösning

a) $f(x) = \frac{x^2}{(x+1)^2} = 1 - \frac{2x+1}{(x+1)^2}$

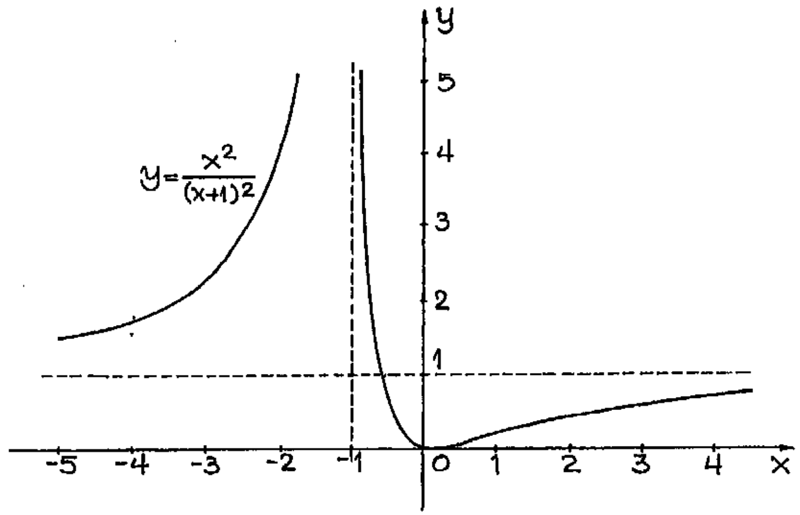
$\lim_{x\to\infty} f(x) = 1^+ \wedge \lim_{x\to-\infty} f(x) = 1^- \Rightarrow y = 1$ är asymptot i $\pm\infty$.

$\lim_{x\to-1^+} f(x) = \infty = \lim_{x\to-1^-} f(x) \Rightarrow x = -1$ asymptot i ∞.

$f'(x) = 2\frac{x}{x+1}\cdot\left(\frac{x}{x+1}\right)' = 2\cdot\frac{x}{x+1}\cdot\frac{1}{(x+1)^2} = \frac{2x}{(x+1)^3};$

x		-1		0	
sgn($f'(x)$)	+	|	−	0	+
$f(x)$	↗	|	↘	0	↗

x	-5	-4	-3	-2	-1,5	-0,5	0,5	1	3
y	1,56	1,78	2,25	4	9	1	0,11	0,25	0,56



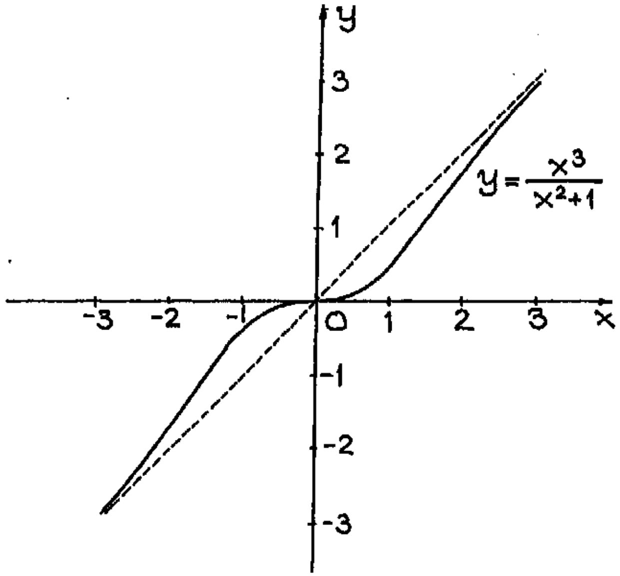
b) $f(x) = \frac{x^3}{x^2+1}$

$\lim_{|x|\to\infty} \frac{f(x)}{x} = \lim_{|x|\to\infty} \frac{x^2}{x^2+1} = 1 \Rightarrow \begin{cases} \lim_{x\to\infty}(f(x)-x) = 0^- \\ \lim_{x\to-\infty}(f(x)-x) = 0^- \end{cases} \Rightarrow$

$\Rightarrow y = x$ är (sned) asymptot i $\pm\infty$.

$f'(x) = \frac{3x^2(x^2+1) - 2x\cdot x^3}{(x^2+1)^2} = \frac{2x^4+3x^2}{(x^2+1)^2} \geq 0 \Rightarrow f$ växande.

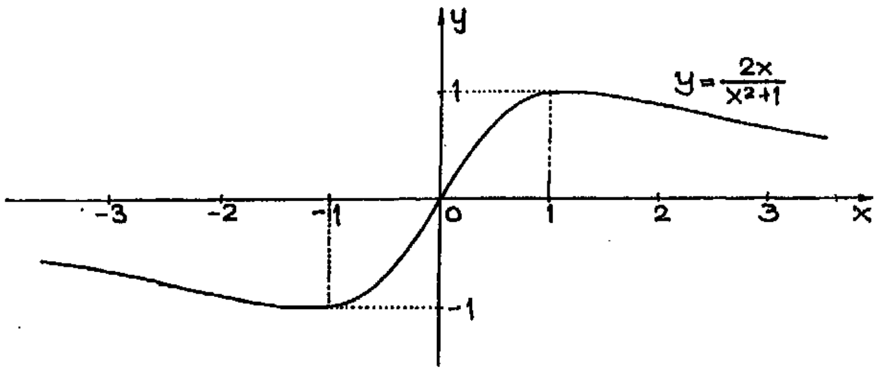
x	-4	-3	-2	-1	1	2	3	4
y	-3,8	-2,7	-1,6	-0,5	0,5	1,6	2,7	3,8



c) $f(x)=\frac{2x}{x^2+1} \Rightarrow f'(x)=\frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}=\frac{2(1+x)(1-x)}{(x^2+1)^2}$;

		-1		1		x
sgn $f'(x)$	$-$	0	$+$	0	$-$	
$f(x)$	↘	-1	↗	1	↘	

$\lim\limits_{x\to\infty} f(x)=0^+ \wedge \lim\limits_{x\to-\infty} f(x)=0^- \Rightarrow$ x-axeln asymptot.



d) $f(x)=\frac{2x}{x^2-1}=\frac{2x}{(x+1)(x-1)}$

		-1		0		1		x
sgn $f(x)$	$-$	|	$+$	0	$-$	|	$+$	

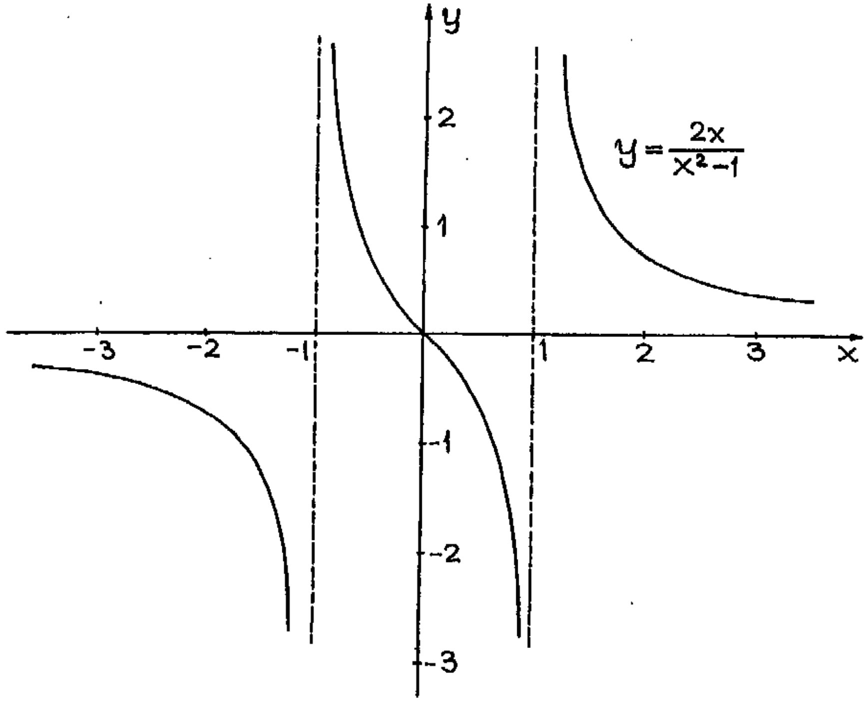
$\lim\limits_{x\to-1^-} f(x)=-\infty \wedge \lim\limits_{x\to-1^+} f(x)=\infty \Rightarrow x=-1$ asymptot i $\pm\infty$.

$\lim\limits_{x\to1^-} f(x)=-\infty \wedge \lim\limits_{x\to1^+} f(x)=\infty \Rightarrow x=1$ asymptot i $\pm\infty$.

$\lim\limits_{x\to\infty} f(x)=0^+ \wedge \lim\limits_{x\to-\infty} f(x)=0^- \Rightarrow$ x-axeln asymptot.

$f'(x)=2\frac{x^2-1-x\cdot 2x}{(x^2-1)^2}=2\frac{-1-x^2}{(x^2-1)^2}=-2\frac{1+x^2}{(1-x^2)^2}<0 \Rightarrow f$ avtagande.

x	-3	-2	$-1,5$	$-0,5$	0	$0,5$	$1,5$	2	3
y	$-0,37$	$-0,67$	$-1,2$	$0,67$	0	$-0,67$	$1,2$	$0,67$	$0,37$



Övning 4.6 (Sid. 74)

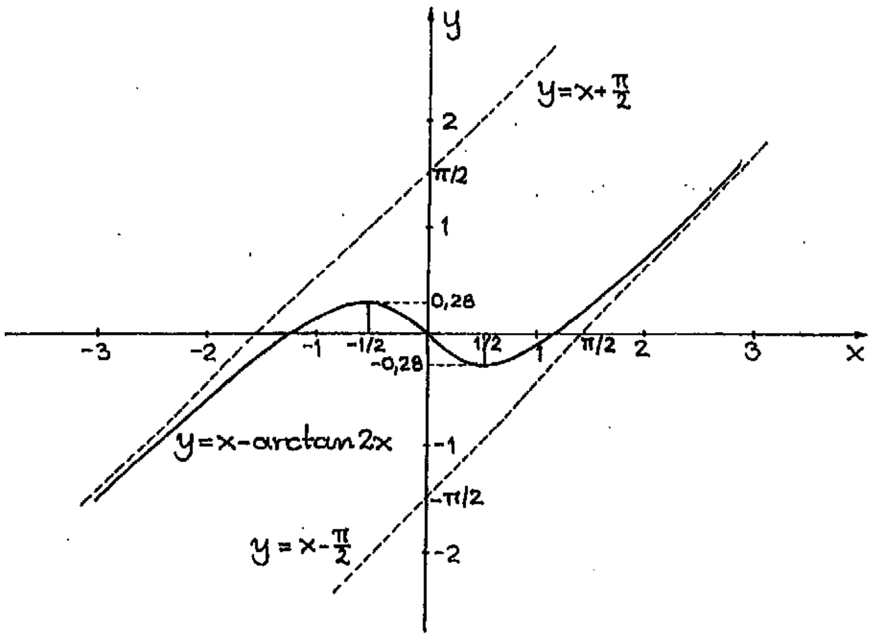
Lösning

a) $\begin{cases} \lim\limits_{x\to\infty}(f(x)-x) = -\frac{\pi}{2} \Rightarrow y = x - \frac{\pi}{2} \text{ asymptot (uppifrån).} \\ \lim\limits_{x\to\infty}(f(x)-x) = +\frac{\pi}{2}^{-} \Rightarrow y = x + \frac{\pi}{2} \text{ asymptot (underifrån).} \end{cases}$

$f'(x) = 1 - \frac{2}{1+4x^2} = \frac{4x^2-1}{4x^2+1} = 4\frac{(x-1/2)(x+1/2)}{4x^2+1}$;

x		$-1/2$		$1/2$	
$\text{sgn}(f'(x))$	+	0	−	0	+
$f(x)$	↗	$\frac{\pi}{4}-\frac{1}{2}$	↘	$-\frac{\pi}{4}+\frac{1}{2}$	↗

x	±0,2	±0,7	±1	±1,2	±1,5	±2	±3
y	∓0,2	∓0,25	∓0,11	±0,02	±0,25	±0,67	±1,59



b) $f(x) = \frac{x}{\ln x}$, $x \neq 1$, $x > 0$.

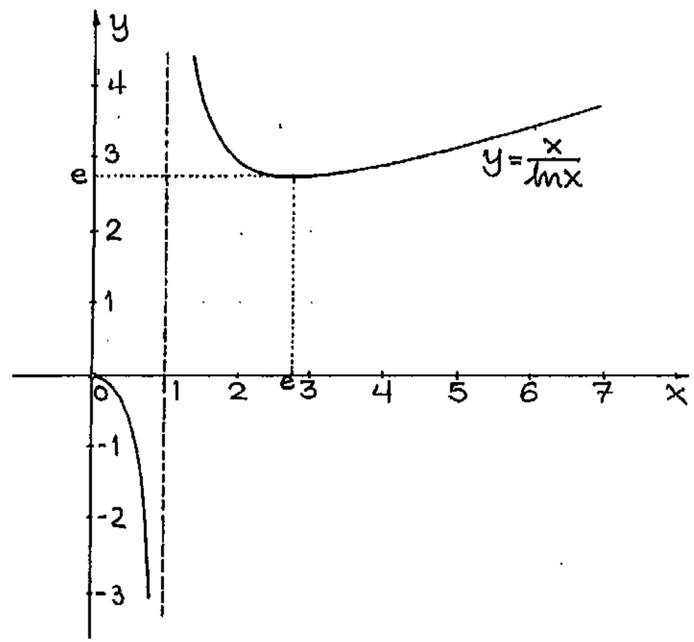
$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\frac{x}{\ln x} = \frac{0}{-\infty} = 0^-$.

$\lim\limits_{x\to 1^-} f(x) = -\infty \wedge \lim\limits_{x\to 1^+} f(x) = \infty \Rightarrow x = 1$ asymptot i $\pm\infty$.

$f'(x) = \frac{\ln x - 1}{\ln^2 x}$;

x	0		1		e		∞
$\text{sgn} f'(x)$		−		−	0	+	
$f(x)$	0^-	↘ $-\infty$	∞	↘	e	↗	∞

x	0,1	0,2	0,5	0,7	1,1	1,5	2	3	6
y	0^-	-0,12	-0,72	-1,96	11	3,70	2,89	2,73	3,35



c) $f(x) = \frac{2x}{\sqrt{x^2+1}} - \arctan x$

$f(-x) = -f(x)$, dvs f är en udda funktion; geometriskt innebär detta att grafen är symmetrisk m.a.p. origo.

(ii) $\lim_{x\to+\infty} f(x) = 2-\pi/2 \Rightarrow y = 2-\frac{\pi}{2} \approx 0{,}429^{+}$ asymptot i ∞.

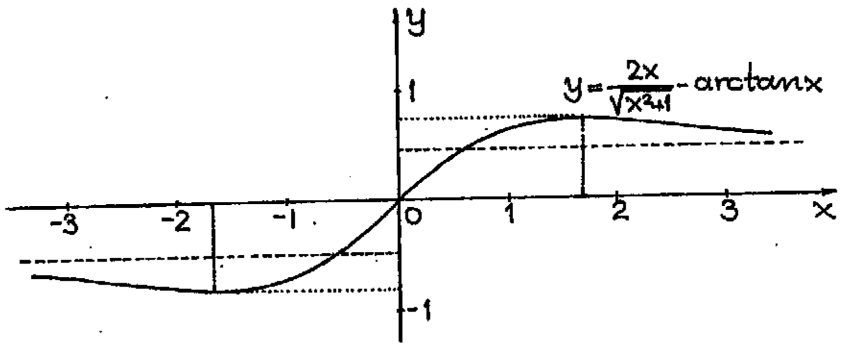
$\lim_{x\to-\infty} f(x) = -2+\pi/2 \Rightarrow y = -2+\frac{\pi}{2} \approx -0{,}429^{-}$ asymptot i $-\infty$.

(iii) $f'(x) = \frac{2}{\sqrt{x^2+1}} - \frac{2x^2}{(x^2+1)^{3/2}} - \frac{1}{x^2+1} = \frac{2(x^2+1)-2x^2-\sqrt{x^2+1}}{(x^2+1)^{3/2}} =$

$= \frac{2-\sqrt{x^2+1}}{(x^2+1)^{3/2}} = 0 \Leftrightarrow \sqrt{x^2+1} = 2 \Leftrightarrow x^2 = 3 \Leftrightarrow \underline{x=-\sqrt{3}} \vee \underline{x=\sqrt{3}}.$

x		$-\sqrt{3}$		$\sqrt{3}$	
$\mathrm{sgn}(f'(x))$	$-$	0	$+$	0	$-$
$f(x)$	↘	$-0{,}7$	↗	$0{,}7$	↘

x	$\pm0{,}2$	$\pm0{,}4$	$\pm0{,}8$	±1	$\pm1{,}5$	±2	$\pm2{,}5$	±3	$\pm3{,}5$
y	$\pm0{,}2$	$\pm0{,}4$	$\pm0{,}6$	$\pm0{,}6$	$\pm0{,}7$	$\pm0{,}7$	$\pm0{,}6$	$\pm0{,}6$	$\pm0{,}5$



d) $f(x) = \frac{\sin 3x}{1+\frac{1}{2}\cos 3x}$

$f(x+P) = f(x) \Rightarrow 3P = 2\pi \Leftrightarrow P = \frac{2\pi}{3}$ fundamentalperiod.

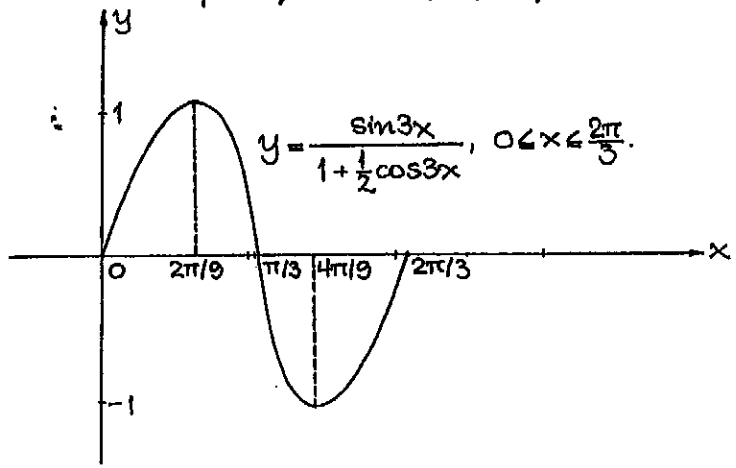
Låt oss alltså bestämma restriktionen $f_{[0,2\pi/3]}$.

$f'(x) = \frac{3\cos 3x(1+0{,}5\cos 3x) - 1{,}5\sin^2 3x}{(1+0{,}5\cos 3x)^2}$;

$f'(x) = 0 \Rightarrow 3\cos 3x + 1{,}5\cos^2 3x - 1{,}5\sin^2 3x = 0 \Rightarrow \cos 3x = -\frac{1}{2}$

$\Leftrightarrow 3x = \frac{2\pi}{3} \vee 3x = \frac{4\pi}{3} \Leftrightarrow x = \frac{2\pi}{9} \vee x = \frac{4\pi}{9}$ $(0 \leq x \leq 2\pi/3)$

x	0		$2\pi/9$		$4\pi/9$		$2\pi/3$
$\mathrm{sgn}(f'(x))$	2	$+$	0	$-$	0	$+$	2
	0	↗	$2/\sqrt{3}$	↘	$-2/\sqrt{3}$	↗	0



Den totala grafen erhålles genom utvidgning av kurvbågen ovan.

Övning 4.7 (Sid. 74)

Lösning

a) $\underline{f(x) = x^2+2x,\ -2 \leq x \leq 1}$

$f'(x) = 2x+2 = 0 \Rightarrow x = -1$; $f(-2) = 0$, $f(-1) = -1$, $f(1) = 3$.

$f_{min} = f(-1) = -1$, $f_{max} = f(1) = 3$.

b) $\underline{f(x) = 6x-x^3,\ 0 \leq x \leq 2.}$ (*)

$f'(x) = 6-3x^2 = -3(x^2-2) = 0 \Leftrightarrow x^2 = 2 \overset{(*)}{\Leftrightarrow} x = \sqrt{2}.$

$f(0)=0,\ f(\sqrt{2})=4\sqrt{2},\ f(2)=4.$

$f_{max}=f(\sqrt{2})=4\sqrt{2},\ \ f_{min}=f(0)=0.$

c) $\underline{f(x)=xe^{-x},\ 0\le x\le 2.}$

$f'(x)=(1-x)e^{-x}=0 \Leftrightarrow 1-x=0 \Leftrightarrow x=1.$

$f(0)=0,\ f(1)=e^{-1},\ f(2)=2e^{-2}.$

$f_{max}=f(1)=e^{-1},\ f_{min}=f(0)=0.$

Övning 4.8 (Sid. 75)

Lösning

$\underline{f(x)=x^3e^{-x}} \Rightarrow f'(x)=3x^2e^{-x}-x^3e^{-x}=x^2(3-x)e^{-x};$

x	$-\infty$		0		3		∞
$sgn(f'(x))$		$+$	0	$+$	0	$-$	
$f(x)$	$-\infty$	$\nearrow$	0	$\nearrow$	$\frac{27}{e^3}$	$\searrow$	0^+

$f_{max}=f(0)=27e^{-3}\approx 1{,}344,\ \ f_{min}$ saknas.

Övning 4.9 (Sid. 75)

Lösning

a) $\underline{f(x)=x^4e^{-x}} \Rightarrow f'(x)=4x^3e^{-x}-x^4e^{-x}=x^3(4-x)e^{-x}.$

x	$-\infty$		0		4		∞
$sgn(f'(x))$		$-$	0	$+$	0	$-$	
$f(x)$	∞	$\searrow$	0	$\nearrow$	$\frac{256}{e^4}$	$\searrow$	0^+

$f_{min}=f(0)=0,\ f_{max}$ saknas; $f(4)=256e^{-4}\approx 4{,}688$ är ett lokalt maximum.

b) $\underline{f(x)=\frac{x}{\ln x},\ x>0,\ x\neq 1.}$ Se Ö. 4.6 b).

f_{max} och f_{min} saknas; $f(e)=e$ är ett lokalt minimum.

c) $\underline{f(x)=x^2e^{x^2/2-3x+1}}=x^2\exp\{\frac{x^2}{2}-3x+1\},\ x\in\mathbb{R}.$

$f'(x)=(2x+x^2(x-3))\exp\{\frac{1}{2}x^2-3x+1\}=x(x-1)(x-2)e^{\frac{x^2}{2}-3x+1};$

x	$-\infty$		0		1		2		∞
$sgn(f'(x))$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$	∞	$\searrow$	0	$\nearrow$	$\frac{1}{3\sqrt{3}}$	$\searrow$	$\frac{4}{e^3}$	$\nearrow$	∞

$f_{min}=f(0)=0,\ f_{max}$ saknas; $f(1)=\frac{1}{3\sqrt{3}}$ är ett lokalt maximum och $f(2)=4e^{-3}$ ett lokalt minimum.

d) $\underline{f(x)=\frac{x+2}{x^2+1}+2\arctan x,\ x\in\mathbb{R}.}$

$f'(x)=\frac{x^2+1-2x(x+2)}{(x^2+1)^2}+\frac{2}{x^2+1}=\frac{x^2+1-2x^2-4x+2x^2+2}{(x^2+1)^2}=$

$=\frac{x^2-4x+3}{(x^2+1)^2}=\frac{(x-1)(x-3)}{(x^2+1)^2},$

x	$-\infty$		1		3		∞
$sgn(f'(x))$		$+$	0	$-$	0	$+$	
$f(x)$	$-\pi^+$	$\nearrow$	$3{,}07$	$\searrow$	$3{,}0$	$\nearrow$	π^-

f_{max} och f_{min} saknas; $f(1)=\frac{3}{2}+\frac{\pi}{2}\approx 3{,}07$ är ett lokalt maximum och $f(3)=\frac{1}{2}+2\arctan 3\approx 2{,}998$ är ett lokalt minimum.

e) $\underline{f(x)=x\ln x+(x\ln x)^2,\ 0<x\leq 1/2.}$

$f'(x)=\ln x+1+2(x\ln x)\cdot(\ln x+1)=(1+2x\ln x)(\ln x+1)$;

$0<x<\frac{1}{2} \Rightarrow x\ln x>-\frac{1}{2} \Rightarrow 2x\ln x+1>0$;

x	0		$1/e$		$1/2$
$\mathrm{sgn}(f'(x))$		$-$	0	$+$	
$f(x)$	∞	$\searrow$	$\frac{1-e}{e^2}$	$\nearrow$	$\frac{\ln^2 2-\ln 4}{4}$

f_{max} saknas; $f(\frac{1}{2})=\frac{\ln^2 2-\ln 4}{4}\approx -0{,}226$ är ett lokalt maximum; $f_{min}=f(e^{-1})=\frac{1-e}{e^2}\approx -0{,}233$.

Övning 4.10 (Sid. 75)

Lösning

a) $\underline{f(x)=x^2+2x,\ D_f: -2\leq x\leq 1.}$

$f_{min}=-1 \wedge f_{max}=3 \Rightarrow \underline{V_f: -1\leq y\leq 3.}$

b) $\underline{f(x)=6x-x^3,\ D_f: 0\leq x\leq 2.}$

$f_{min}=0 \wedge f_{max}=4\sqrt{2} \Rightarrow \underline{V_f: 0\leq y\leq 4\sqrt{2}.}$

c) $\underline{f(x)=xe^{-x},\ D_f: 0\leq x\leq 2.}$

$f_{min}=0 \wedge f_{max}=e^{-1} \Rightarrow \underline{V_f: 0\leq y\leq e^{-1}.}$

Anm. En kontinuerlig funktion avbildar ett slutet intervall på ett slutet intervall.

Övning 4.11 (Sid. 75)

Lösning

$\underline{f(x)=xe^{-x},\ D_f=\mathbb{R}.}$

$\lim_{x\to-\infty} f(x)=-\infty \wedge f_{max}=\frac{27}{e^3} \Rightarrow \underline{V_f: y\leq 27e^{-3}.}$

Övning 4.12 (Sid. 75)

Lösning

a) $\underline{f(x)=\ln x,\ g(x)=x-1,\ x>0.}$

Jag ska visa att $\mathcal{F}(x)=f(x)-g(x)\leq 0$ för $x>0$.

$\mathcal{F}(x)=\ln x-x+1 \Rightarrow \mathcal{F}'(x)=\frac{1}{x}-1=\frac{1-x}{x}$;

$$\left\{\begin{array}{l} 0<x<1 \Rightarrow \mathcal{F}'(x)>0 \Rightarrow \mathcal{F} \text{ växande} \\ x>1 \Rightarrow \mathcal{F}'(x)<0 \Rightarrow \mathcal{F} \text{ avtagande} \end{array}\right. \Rightarrow \mathcal{F}(x)\leq\mathcal{F}(1)=0.$$

b) $\underline{f(x)=e^x,\ g(x)=1+x,\ x\neq 0.}$ $\quad \mathcal{F}(x)=f(x)-g(x)$.

$\mathcal{F}(x)=e^x-x-1 \Rightarrow \mathcal{F}'(x)=e^x-1$;

$$\left\{\begin{array}{l} x<0 \Rightarrow \mathcal{F}'(x)<0 \Rightarrow \mathcal{F} \text{ avtagande} \\ x>0 \Rightarrow \mathcal{F}'(x)>0 \Rightarrow \mathcal{F} \text{ växande} \end{array}\right. \Rightarrow \mathcal{F}(x)\geq\mathcal{F}(0)=0;$$

$x\neq 0 \Rightarrow \mathcal{F}(x)>0 \Leftrightarrow f(x)>g(x) \Leftrightarrow e^x>1+x.$

c) $\underline{f(x)=\ln(1+4x),\ g(x)=\arctan 3x,\ x>0;}$ $\mathcal{F}(x)=f(x)-g(x)$.

$\mathcal{F}'(x)=\frac{4}{1+4x}-\frac{3}{1+9x^2}=\frac{36x^2-12x+1}{(1+4x)(1+9x^2)}=\frac{(6x-1)^2}{(1+4x)(1+9x^2)}\geq 0$ (för $x>0$)

$\Rightarrow \mathcal{F}$ strängt växande $\Rightarrow \mathcal{F}(x) > \mathcal{F}(0) = 0 \Leftrightarrow f(x) > g(x)$.

d) $\underline{f(x) = \ln(1+x),\ g(x) = x - \frac{1}{2}x^2,\ x > 0}$; $\mathcal{F}(x) = f(x) - g(x)$.

$\mathcal{F}(x) = \ln(1+x) - x + \frac{1}{2}x^2 \Rightarrow \mathcal{F}'(x) = \frac{1}{1+x} - 1 + x = \frac{1-(1+x)(1-x)}{1+x} =$

$= \frac{1-(1-x^2)}{1+x} = \frac{x^2}{1+x} > 0 \Rightarrow \mathcal{F}$ strängt växande $\Rightarrow \mathcal{F}(x) > \mathcal{F}(0) = 0$

$\Leftrightarrow f(x) > g(x)$ (för $x > 0$).

e) $\underline{f(x) = \ln x,\ g(x) = \sqrt{x} - \frac{1}{\sqrt{x}},\ x \geq 1}$; $\mathcal{F}(x) = f(x) - g(x)$.

$\mathcal{F}(x) = \ln x - x^{1/2} + x^{-1/2} \Rightarrow \mathcal{F}'(x) = \frac{1}{x} - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{x} - \frac{1}{2\sqrt{x}} -$

$- \frac{1}{2x\sqrt{x}} = -\frac{x - 2\sqrt{x} + 1}{2x\sqrt{x}} = -\frac{(\sqrt{x}-1)^2}{2x\sqrt{x}} \leq 0 \Rightarrow \mathcal{F}$ avtagande $\Rightarrow$

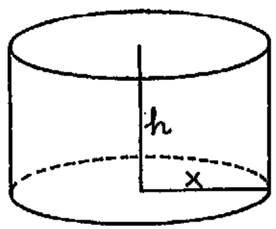
$\Rightarrow \mathcal{F}(x) \leq \mathcal{F}(1) = 0 \Leftrightarrow f(x) \leq g(x)$ för $x \geq 1$.

Anm. $\mathcal{F}$ växande: $x \geq a \Rightarrow \mathcal{F}(x) \geq \mathcal{F}(a)$.

$\mathcal{F}$ avtagande: $x \geq a \Rightarrow \mathcal{F}(x) \leq \mathcal{F}(a)$.

Övning 4.13 (Sid. 75)

Lösning



Volymen ges av $\pi x^2 h = V$ (givet) $\Leftrightarrow h = V/\pi x^2$.

Arean: $S = \pi x^2 + 2\pi x h = \pi x^2 + \frac{2V}{x} \Rightarrow S' = 2\pi x - \frac{2V}{x^2}$;

$S' = 0 \Rightarrow \pi x = \frac{2V}{x^2} \Leftrightarrow x^3 = \frac{2V}{\pi} \Leftrightarrow \underline{x = \left(\frac{2V}{\pi}\right)^{1/3} = h}$.

Övning 4.14 (Sid. 75)

Lösning

Körtiden är $\frac{300}{x}$ timmar.

Chaufförens lön blir $\frac{300}{x} \cdot 86 = \frac{25800}{x}$ kronor.

Olja och drivmedel kostar $\frac{300}{x}\left(2 + \frac{x^2}{300}\right) \cdot 6 = 6x + \frac{3600}{x}$.

De totala kostnaderna blir $\frac{29400}{x} + 6x$ kronor.

Vi studerar funktionen

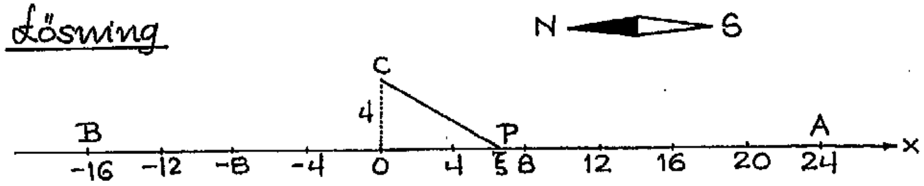
$$f(x) = \frac{29400}{x} + 6x, \quad 30 \leq x \leq 90.$$

$f'(x) = 6 - \frac{29400}{x^2} = 0 \Leftrightarrow x^2 = 4900 = 70^2 \Leftrightarrow x = 70$.

Resultat: Den mest ekonomiska körningen sker vid 70 km/h. Den totala kostnaden blir då 840 kronor.

Övning 4.15 (Sid. 75)

Lösning



Bensinförbrukningen är proportionell mot två färder till A och en färd till B. Det leder till

$$f(\xi) = 2(\underbrace{\sqrt{\xi^2+16}}_{CP} + \underbrace{24-\xi}_{PA}) + \underbrace{\sqrt{\xi^2+16}}_{CP} + \underbrace{\xi+16}_{PB}, \quad -16 \le \xi \le 24.$$

$$f'(\xi) = 2\left(\frac{\xi}{\sqrt{\xi^2+16}} - 1\right) + \frac{\xi}{\sqrt{\xi^2+16}} + 1 = \frac{3\xi}{\sqrt{\xi^2+16}} - 1 = \frac{3\xi - \sqrt{\xi^2+16}}{\sqrt{\xi^2+16}} =$$

$$= \frac{(3\xi-\sqrt{\xi^2+16})(3\xi+\sqrt{\xi^2+16})}{3\xi+\sqrt{\xi^2+16}} = \frac{(3\xi)^2-(\sqrt{\xi^2+16})^2}{3\xi+\sqrt{\xi^2+16}} =$$

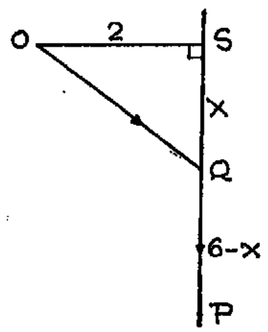
$$= \frac{9\xi^2-\xi^2-16}{3\xi+\sqrt{\xi^2+16}} = \frac{8\xi^2-16}{3\xi+\sqrt{\xi^2+16}} = \frac{8(\xi+\sqrt{2})(\xi-\sqrt{2})}{3\xi+\sqrt{\xi^2+16}};$$

x		$-\sqrt{2}$		$\sqrt{2}$	
$\operatorname{sgn}(f'(\xi))$	+	0	−	0	+
$f(\xi)$	↗	78,1	↘	75,3	↗

Svar: Rakt mot en punkt belägen 22,6 km (eller $(24-\sqrt{2})$ km) norr om A

Övning 4.16 (Sid. 76)

Lösning



$t_{O\to Q} = \frac{\sqrt{x^2+4}}{6}, \quad t_{Q\to P} = \frac{6-x}{10};$

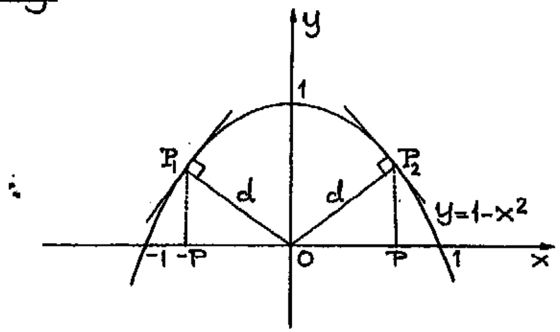
$f(x) = \frac{1}{6}\sqrt{x^2+16} + \frac{1}{10}(6-x), \quad 0<x<6.$

$f'(x) = \frac{1}{6}\frac{x}{\sqrt{x^2+16}} - \frac{1}{10} = 0 \Leftrightarrow \dots \Leftrightarrow x = 1,5$

Svar: Han kan sätta kursen rakt mot en punkt belägen 1,5 km från S.

Övning 4.17 (Sid. 76)

Lösning



Det finns tydligen 2 olika punkter med samma egenskap: $d^2 = p^2 + (1-p^2)^2 = p^4 - p^2 - 1$;

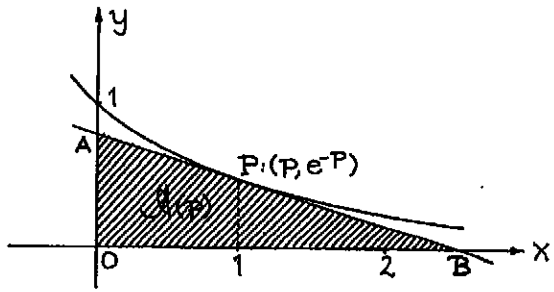
$$f(p) = p^4 - p^2 + 1 \Rightarrow f'(p) = 4p^3 - 2p = 4p(p^2 - \tfrac{1}{2}) = 0 \Leftrightarrow \begin{cases} p_1 = -1/\sqrt{2} \\ p_2 = 0 \\ p_3 = 1/\sqrt{2} \end{cases},$$

p		$-1/\sqrt{2}$		0		$1/\sqrt{2}$	
$\operatorname{sgn}(f'(p))$	−	0	+	0	−	0	+
$f(p)$	↘	1/2	↗	1	↘	1/2	↗

Svar: Punkterna $P_1: (-\frac{1}{\sqrt{2}}, \frac{1}{2})$ och $P_2: (\frac{1}{\sqrt{2}}, \frac{1}{2})$.

Övning 4.18 (Sid. 76)

Lösning



Tangenten är $y = e^{-p} - e^{-p}(x-p)$.

A:s koordinat fås för $x=0$; den är $x_A = (p+1)e^{-p}$.

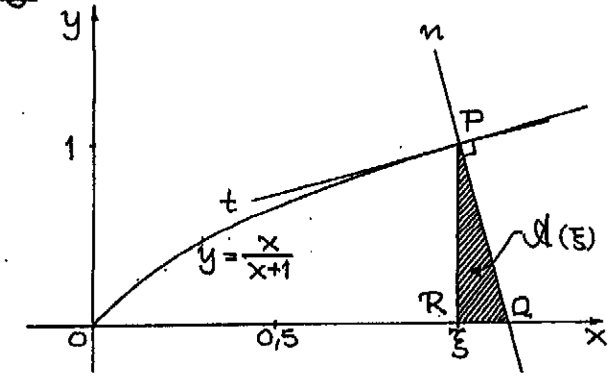
B:s koordinat fås för $y=0$; den är $x_B = p+1$.

$\mathcal{A}(p) = \frac{1}{2}(p+1)^2 e^{-p} \Rightarrow \mathcal{A}'(p) = (1+p)(1-p)e^{-p}$; Obs! $p>0$.

Svar: Den största arean en sådan triangel kan ha är $\mathcal{A}(1) = \frac{4}{e} \approx 1{,}47$ ae. $(P:(1,\frac{1}{e}))$.

Övning 4.19 (Sid. 76)

Lösning



Koordinaterna för P är (ξ,η), $\eta = \frac{\xi}{\xi+1}$.

$f(x) = \frac{x}{x+1} \Rightarrow f'(x) = \frac{1}{(x+1)^2} \Rightarrow k_t = \frac{1}{(\xi+1)^2} \Rightarrow k_n = -(\xi+1)^2$.

Normalens ekvation blir $y = \frac{\xi}{\xi+1} - (\xi+1)^2(x-\xi)$.

Q:s koordinat fås för $y=0$; $x_Q = \xi + \frac{\xi}{(\xi+1)^2}$.

Den skuggade triangeln har arean $\mathcal{A}(\xi)$;

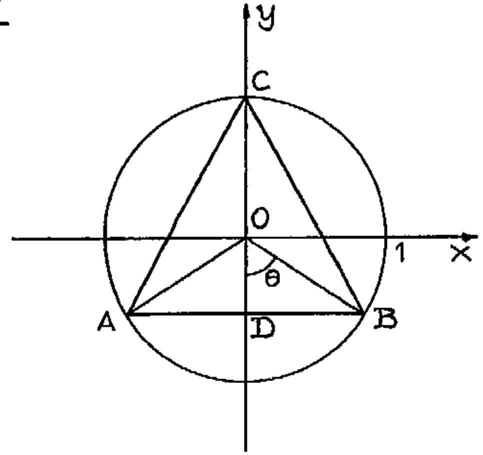
$\mathcal{A}(\xi) = \frac{1}{2}\frac{\xi^2}{(\xi+1)^4} \Rightarrow \mathcal{A}'(\xi) = \frac{\xi}{(\xi+1)^4} - \frac{2\xi^2}{(\xi+1)^5} = \frac{\xi^2+\xi-2\xi^2}{(\xi+1)^5} = \frac{\xi(1-\xi)}{(\xi+1)^5}$.

$$\left.\begin{array}{l} 0<\xi<1 \Rightarrow \mathcal{A}'(\xi)>0 \Rightarrow \mathcal{A} \text{ växande} \\ \xi>1 \Rightarrow \mathcal{A}'(\xi)<0 \Rightarrow \mathcal{A} \text{ avtagande} \end{array}\right\} \Rightarrow \mathcal{A}_{max} = \mathcal{A}(1) = \frac{1}{32}.$$

Resultat: $P:(1,\frac{1}{2})$ ger störst area.

Övning 4.20 (Sid. 76)

Lösning



$\mathcal{A}(\theta) = \frac{1}{2}\cdot AB\cdot CD = \frac{1}{2}\cdot 2\sin\theta(1+\cos\theta) = \sin\theta + \frac{1}{2}\sin 2\theta$;

$\mathcal{A}'(\theta) = \cos\theta + \cos 2\theta = \cos\theta + 2\cos^2\theta - 1 = 0 \Leftrightarrow \cos^2\theta + \frac{1}{2}\cos\theta =$

$= \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow \mathcal{A}_{max} = \mathcal{A}(\frac{\pi}{3}) = \frac{3\sqrt{3}}{4}$ ae.

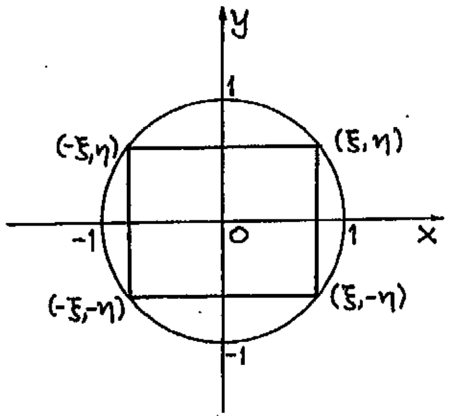
Resultat: Den största arean, $\frac{3\sqrt{3}}{4}$ ae, har den liksidiga triangeln. (Sidan av triangeln är $\sqrt{3}$ le).

Anm. $\mathcal{A}(\xi) = \xi\sqrt{1-\xi^2}$, $0<\xi<1$, är en annan väg...

Övning 4.21 (Sid. 76)

Lösning

Jag ritar rektangeln med sidorna axelparallella.



$x^2+y^2=1 \Leftrightarrow y^2=1-x^2 \Leftrightarrow y=\pm\sqrt{1-x^2} \Rightarrow \eta=\sqrt{1-\xi^2}, \quad 0\le\xi\le 1$

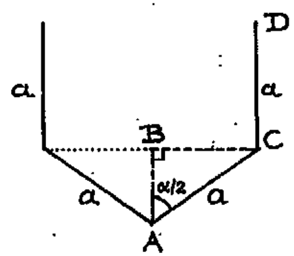
$\mathcal{A}(\xi)=\frac{1}{2}\cdot 2\xi\cdot 2\eta=2\xi\sqrt{1-\xi^2} \Rightarrow \mathcal{A}'(\xi)=2\sqrt{1-\xi^2}-2\frac{\xi}{\sqrt{1-\xi^2}}=\frac{4(1-2\xi^2)}{\sqrt{1-\xi^2}}.$

$\mathcal{A}'(\xi)=0 \Rightarrow \xi^2=\frac{1}{2} \Leftrightarrow \xi=\pm 1/\sqrt{2} \Rightarrow \mathcal{A}_{max}=\mathcal{A}(\frac{1}{\sqrt{2}})=2.$

Svar: Den största arean (2 ae) antar kvadraten.

Övning 4.22 (Sid. 76)

Lösning



$\sin\frac{\alpha}{2}=\frac{BC}{AC}=\frac{BC}{a} \Leftrightarrow BC=a\cdot\sin\frac{\alpha}{2}$
$\cos\frac{\alpha}{2}=\frac{AB}{AC}=\frac{AB}{a} \Leftrightarrow AB=a\cos\frac{\alpha}{2}$
$\Big\} \Rightarrow \mathcal{A}=2\cdot\frac{AB\cdot BC}{2}+2BC\cdot CD=$

$=a\cdot\sin\frac{\alpha}{2}\cdot a\cos\frac{\alpha}{2}+2a\sin\frac{\alpha}{2}\cdot a=a^2\cdot(2a\sin\frac{\alpha}{2}+\frac{1}{2}\sin\alpha)$

Låt oss studera funktionen

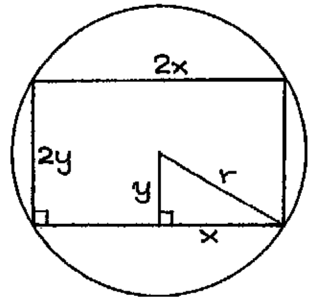
$$f(\alpha)=\frac{1}{2}\sin\alpha+2\sin\frac{\alpha}{2}, \quad 0<\alpha<\pi.$$

$f'(\alpha)=\frac{1}{2}\cos\alpha+\cos\frac{\alpha}{2}=\frac{1}{2}(2\cos^2\frac{\alpha}{2}-1)+\cos\frac{\alpha}{2}=\cos^2\frac{\alpha}{2}+\cos\frac{\alpha}{2}-\frac{1}{2}=0$

$\Leftrightarrow \cos\frac{\alpha}{2}=\frac{\sqrt{3}-1}{2}=0{,}336 \Leftrightarrow \underline{\alpha=2{,}392\,\text{rad}=137^\circ}.$

Övning 4.23 (Sid. 77)

Lösning



$x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2 \Rightarrow W(x)=\frac{1}{6}xy^2=\frac{1}{6}(r^2x-x^3);$

$W'(x)=\frac{1}{6}(r^2-3x^2)=0 \Rightarrow x=r/\sqrt{3} \Rightarrow y=\sqrt{2r/3}$

Svar: Bredd $\frac{2\sqrt{3}}{3}r$, höjd $\frac{2\sqrt{6}}{3}r$.

Anm. Koordinatsystemet är överflödigt här.

Övning 4.24 (Sid. 77)

Lösning

$\frac{x^2}{9}+\frac{y^2}{4}=1 \Leftrightarrow y^2=4-\frac{4}{9}x^2 \Rightarrow \underline{r^2=f(x)=(x-1)^2-\frac{4}{9}x^2+4}$;

$f'(x)=2(x-1)-\frac{8}{9}x=\frac{10}{9}x-2=0 \Leftrightarrow x=\frac{9}{5} \Rightarrow y=\frac{8}{5}$

$r^2=f(\frac{9}{5})=\frac{16}{25}+\frac{64}{25}=\frac{80}{25}=5\cdot\frac{16}{25} \Rightarrow \underline{r=\frac{4\sqrt{5}}{5}\approx 1{,}789}$.

Övning 4.25 (Sid. 77)

Lösning

$r^2=(x-2)^2-\frac{4}{9}x^2+4=g(x) \Rightarrow g'(x)=2(x-2)-\frac{8}{9}x=\frac{10}{9}x-4$;

$g'(x)=0 \Rightarrow x=\frac{18}{5} \Rightarrow r^2=g(\frac{18}{5})=\frac{64}{25}-\frac{144}{25}+4=\frac{20}{25} \Rightarrow \underline{r=\frac{2\sqrt{5}}{5}}$.

Övning 4.26 (Sid. 77)

Lösning

$r^2=(x-\frac{5}{3})^2-\frac{4}{9}x^2+4=h(x) \Rightarrow h'(x)=2(x-\frac{5}{3})-\frac{8}{9}x=\frac{10}{9}x-\frac{10}{3}$;

$h'(x)=0 \Rightarrow x=3 \Rightarrow r^2=\frac{16}{9}-\cancel{4}+\cancel{4} \Rightarrow \underline{r=\frac{4}{3}}$.

Övning 4.27 (Sid. 77)

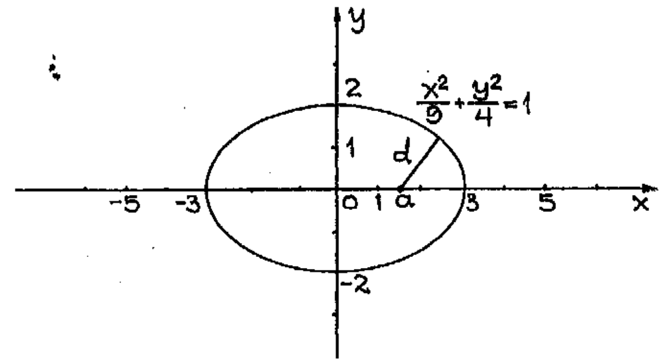
Lösning

$d^2=F(x)=(x-a)^2-\frac{4}{9}x^2+4 \Rightarrow F'(x)=2(x-a)-\frac{8}{9}x=\frac{10}{9}x-2a$

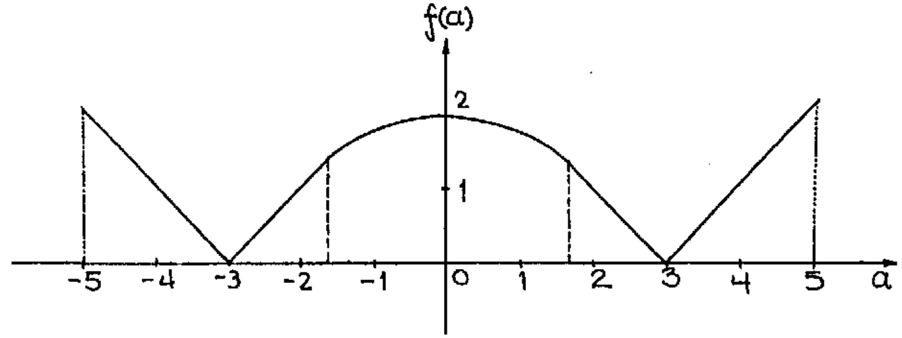
$F'(x)=0 \Rightarrow x=\frac{9a}{5} \Rightarrow F(\frac{9a}{5})=4-\frac{4a^2}{9} \Rightarrow f(a)=2\sqrt{1-\frac{a^2}{9}}$,

Detta gäller så länge $-\frac{5}{3}\le a\le\frac{5}{3}$; jfr Ö. 4.25.

När $\frac{5}{3}<a\le 5$, så är $r=|a-3|$ och när $-5\le a<\frac{5}{3}$ så är $r=|a+3|$ (Se fig.)



$$d=f(a)=\begin{cases} |a-3|, & 5/3<a\le 5 \\ 2\sqrt{1-a^2/9}, & -5/3\le a\le 5/3. \\ |a+3|, & -5\le a<-5/3 \end{cases}$$



f(a) är kontinuerlig för $-5\le a\le 5$; den är deriverbar för $|a|\neq\frac{5}{3}$. Läs även författarnas kommentar.

Anm. Övningarna 4.24-27 utgör en "enhet".

Övning 4.28 (Sid. 77)

Lösning

$\mathcal{F}(z) = \dfrac{\tan 37^\circ}{\tan 22^\circ} \cdot \dfrac{11z+10{,}5}{21z-4{,}5} = k \cdot \dfrac{11z+10{,}5}{21z-4{,}5}$, $k = \dfrac{\tan 37^\circ}{\tan 22^\circ} > 0$.

$\mathcal{F}'(z) = \dfrac{11 \cdot (21z-4{,}5) - 21(11z+10{,}5)}{(21z-4{,}5)^2} \cdot k = \dfrac{-270k}{(21z-4{,}5)^2} < 0 \Rightarrow$

$\Rightarrow \mathcal{F}$ avtagande $\Rightarrow \mathcal{F}(3) < \mathcal{F}(z) < \mathcal{F}(1{,}5)$.

Säkerheten är som minst för $z = 3$.

Övning 4.29 (Sid. 77)

Lösning

$f(x) = \dfrac{e^x}{x^2-3}$

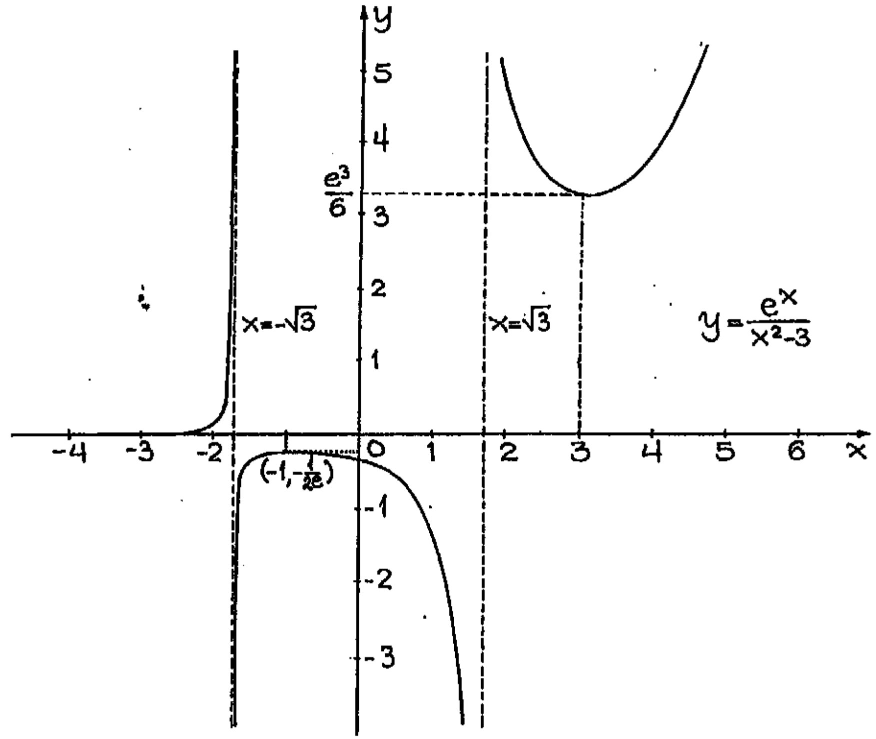
a) f är definierad för alla x utom $x = \pm\sqrt{3}$.

$f'(x) = \dfrac{e^x}{x^2-3} - \dfrac{2xe^x}{(x^2-3)^2} = \dfrac{x^2-2x-3}{(x^2-3)^2}e^x = \dfrac{(x+1)(x-3)}{(x^2-3)^2}e^x$;

x	$-\infty$		$-\sqrt{3}$		-1		$\sqrt{3}$		3		∞
sgn $f'(x)$		+		+	0	−		−	0	+	
$f(x)$	0^+	↗	$\infty \vert -\infty$	↗	$-\frac{1}{2e}$	↘	$-\infty \vert \infty$	↘	$\frac{e^3}{6}$	↗	∞

f har lokalt maximum för $x = -1$ och lokalt minimum för $x = 3$; $x = \pm\sqrt{3}$ är asymptoter i $\pm\infty$ medan x-axeln är asymptot i $-\infty$.

x	-2	-1,5	-0,5	0,5	1	1,2	1,5	2	2,5	4	...
y	0,14	-0,30	-0,22	-0,6	-1,34	-2,13	-6,0	7,39	3,75	4,19	



Övning 4.30 (Sid. 78)

Lösning

$f(x) = \dfrac{x^3}{(x-1)^2}$, $x \neq 1$.

$\lim\limits_{x\to1^-} f(x) = \infty = \lim\limits_{x\to1^+} f(x) \Rightarrow x = 1$ asymptot i ∞.

$\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{x^2}{(x-1)^2} = 1$; (*)

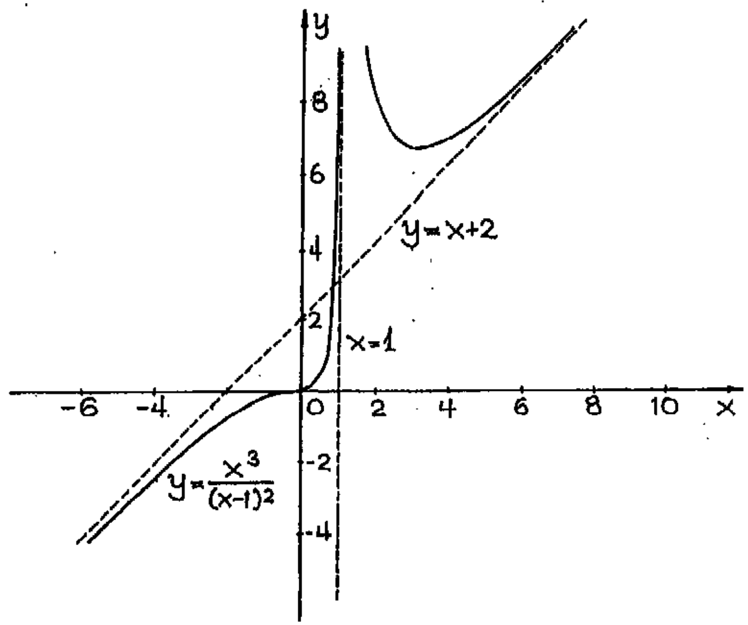
$\lim\limits_{x\to\infty} (f(x)-x) = \lim\limits_{x\to\infty} \left(2 + \dfrac{3x-2}{(x-1)^2}\right) = 2^+ \overset{(*)}{\Rightarrow} y \doteq x+2$ i $+\infty$.

$\lim\limits_{x\to-\infty} (f(x)-x) = \lim\limits_{x\to-\infty} \left(2 + \dfrac{3x-2}{(x-1)^2}\right) = 2^- \overset{(*)}{\Rightarrow} y \doteq x+2$ i $-\infty$.

$$f'(x)=\frac{3x^2(x-1)^2-2(x-1)x^3}{(x-1)^4}=\frac{x^3-3x^2}{(x-1)^3}=\frac{x^2(x-3)}{(x-1)^3};$$

x		0		1		3		∞
$\operatorname{sgn}(f'(x))$	$+$	0	$+$		$-$	0	$+$	
$f(x)$	$-\infty \nearrow$	0	$\nearrow \infty$		$-\infty \searrow$	$\frac{27}{4}$	$\nearrow$	∞

x	-4	-3	-2	$-1{,}5$	$-0{,}5$	$0{,}5$	$1{,}5$	2	3	4	5	6
y	$-2{,}6$	$-1{,}7$	$-0{,}9$	$-0{,}5$	$-0{,}1$	$0{,}25$	$13{,}25$	8	$6{,}75$	$7{,}1$	$7{,}8$	$8{,}64$



Övning 4.31 (Sid. 78)

Lösning

$f(x)=-x\ln x-(1-x)\ln(1-x),\ 0<x<1.$

$$f'(x)=-\ln x-1+\ln(1-x)+1=\ln(1-x)-\ln x=\ln\frac{1-x}{x}=0\Leftrightarrow$$

$$\Leftrightarrow \frac{1-x}{x}=1\Leftrightarrow 1-x=x\Leftrightarrow 2x=1\Leftrightarrow x=\frac{1}{2};$$

$$\lim_{x\to 0^+}f(x)=0^-=\lim_{x\to 1^-}f(x);\quad f(\tfrac{1}{2})=\ln 2;\qquad V_f:\ 0<y<\ln 2.$$

Övning 4.32 (Sid. 78)

Lösning

$f(x)=\ln(\sin x),\ 0<x<\pi.$

$$f'(x)=\underbrace{\cot x=1}_{k_t=1}\Leftrightarrow x=\frac{\pi}{4}\Rightarrow f(\tfrac{\pi}{4})=\ln\frac{1}{\sqrt{2}}=-\frac{1}{2}\ln 2.$$

$$y=f(\tfrac{\pi}{4})+f'(\tfrac{\pi}{4})(x-\tfrac{\pi}{4})\Rightarrow y=x-\frac{\ln 2}{2}-\frac{\pi}{4}.$$

Övning 4.33 (Sid. 78)

Lösning

$f(x)=\arctan\frac{1}{x}+\arctan x,\ x\neq 0.$

$$f'(x)=\frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right)+\frac{1}{1+x^2}=-\frac{1}{x^2+1}+\frac{1}{x^2+1}=0\Leftrightarrow f(x)=C;$$

$$\begin{cases}\lim\limits_{x\to\infty}f(x)=0+\frac{\pi}{2}=\frac{\pi}{2}\\ \lim\limits_{x\to-\infty}f(x)=0-\frac{\pi}{2}=-\frac{\pi}{2}\end{cases}\Rightarrow f(x)=\begin{cases}\frac{\pi}{2},\ x>0\\ -\frac{\pi}{2},\ x<0\end{cases}=\frac{\pi}{2}\cdot\operatorname{sgn}(x).$$

Övning 4.34 (Sid. 78)

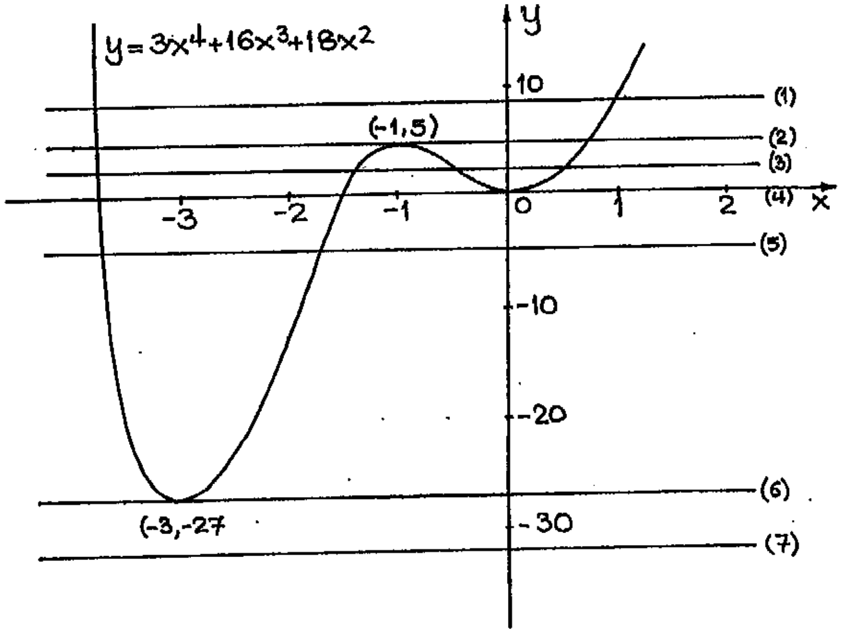
Lösning

Se nästföljande sida.

$\mathcal{N}(a)$ = antalet skärningar mellan grafen till $y=p(x)$ och linjen $y=a$, för olika a (se fig.)

$p'(x) = 12x^3+48x^2+36x = 12x(x^2+4x+3) = 12x(x+1)(x+3)$.

x		-3		-1		0	
$\mathrm{sgn}(p'(x))$	$-$	0	$+$	0	$-$	0	$+$
$p(x)$	$\searrow$	-27	$\nearrow$	5	$\searrow$	0	$\nearrow$



Resultat: $\mathcal{N}(a) = \begin{cases} 4, & 0<a<5 & (3) \\ 3, & a=0,\ a=3 & (2),(4) \\ 2, & -27<a<0,\ a>5 & (1),(5) \\ 1, & a=-27 & (6) \\ 0, & a<-27 & (7) \end{cases}$

Antalet nollställen till $p(x)$ är enligt ovan.

Anm. 0-ställe = nollställe osv. $\mathcal{N}(0)=3$.

Övning 4.35 (Sid. 78)

Lösning

a) $e^{x+y}=e^x+e^y \Leftrightarrow e^x\cdot e^y - e^y = e^x \Leftrightarrow e^y(e^x-1)=e^x \Leftrightarrow$

$\Leftrightarrow e^y = \frac{e^x}{e^x-1} = (1-e^{-x})^{-1} \Leftrightarrow y = \ln(1-e^{-x})^{-1} = -\ln(1-e^{-x})$;

$D_{\ln} = \mathbb{R}_+ \Rightarrow 1-e^{-x}>0 \Leftrightarrow e^{-x}<1 \Leftrightarrow x>0 \Rightarrow \underline{D_f=\mathbb{R}_+}$.

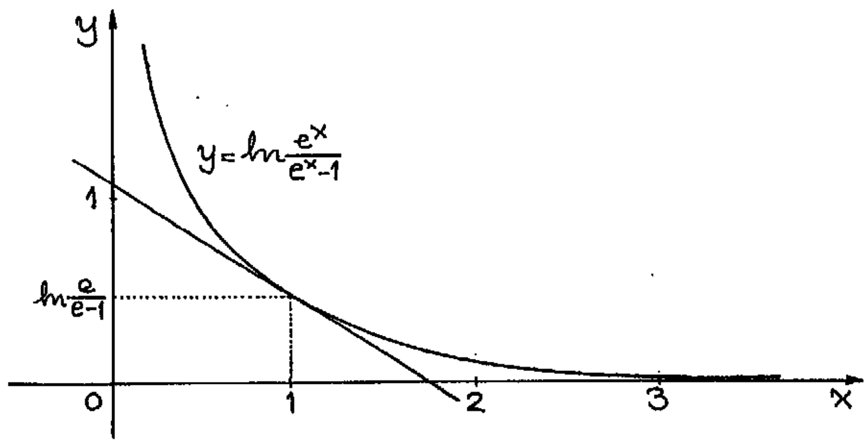
b) $f(x) = \ln\frac{e^x}{e^x-1} = \ln e^x - \ln(e^x-1) = x-\ln(e^x-1)$, $x>0$;

$f'(x) = 1-\frac{e^x}{e^x-1} = -\frac{1}{e^x-1} < 0 \Rightarrow f$ avtagande.

$\lim_{x\to 0^+} f(x) = \infty \Rightarrow y$-axeln asymptot i ∞.

$\lim_{x\to\infty} f(x) = 0^+ \Rightarrow x$-axeln asymptot i ∞.

x	0,2	0,4	0,6	0,8	1	2	3	4
y	1,71	1,11	0,80	0,60	0,46	0,15	0,05	0^+



c) $y = f(1)+f'(1)(x-1) = 1-\ln(e-1)-\frac{1}{e-1}(x-1) = 1+\frac{1}{e-1}-\ln(e-1)-\frac{x}{e-1} \Leftrightarrow y = -\frac{1}{e-1}x+1+(e-1)^{-1}-\ln(e-1)$.

Övning 4.36 (Sid. 78)

Lösning

$f(x) = \dfrac{x}{x-1}e^{1/x}, \quad x \neq 0, 1.$

(i) $\left.\begin{array}{l} \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{x}{x-1}e^{1/x}\ [u = \tfrac{1}{x}] = \lim\limits_{u\to -\infty} \dfrac{1}{1-u}e^{u} = 0^+ \\ \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{x}{x-1}e^{1/x}\ [u = \tfrac{1}{x}] = \lim\limits_{u\to \infty} \dfrac{1}{1-u}e^{u} = -\infty \end{array}\right\} \Rightarrow$

$\Rightarrow$ y-axeln asymptot i $-\infty$.

(ii) $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \dfrac{x}{x-1}e^{1/x} = -\infty \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \dfrac{x}{x-1}e^{1/x} = \infty \end{array}\right\} \Rightarrow x = 1$ asymptot i $\pm\infty$.

(iii) $\left.\begin{array}{l} \lim\limits_{x\to \infty} f(x) = \lim\limits_{x\to \infty} \dfrac{1}{1+1/x}e^{1/x}\ [v = \tfrac{1}{x}] = \lim\limits_{v\to 0^+} \dfrac{1}{1-v}e^{v} = 0^+ \\ \lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} \dfrac{1}{1-1/x}e^{1/x}\ [v = \tfrac{1}{x}] = \lim\limits_{v\to 0^-} \dfrac{1}{1-v}e^{v} = 0^- \end{array}\right\} \Rightarrow$

$\Rightarrow y = 1$ asymptot i $\pm\infty$

Några andra asymptoter finns inte.

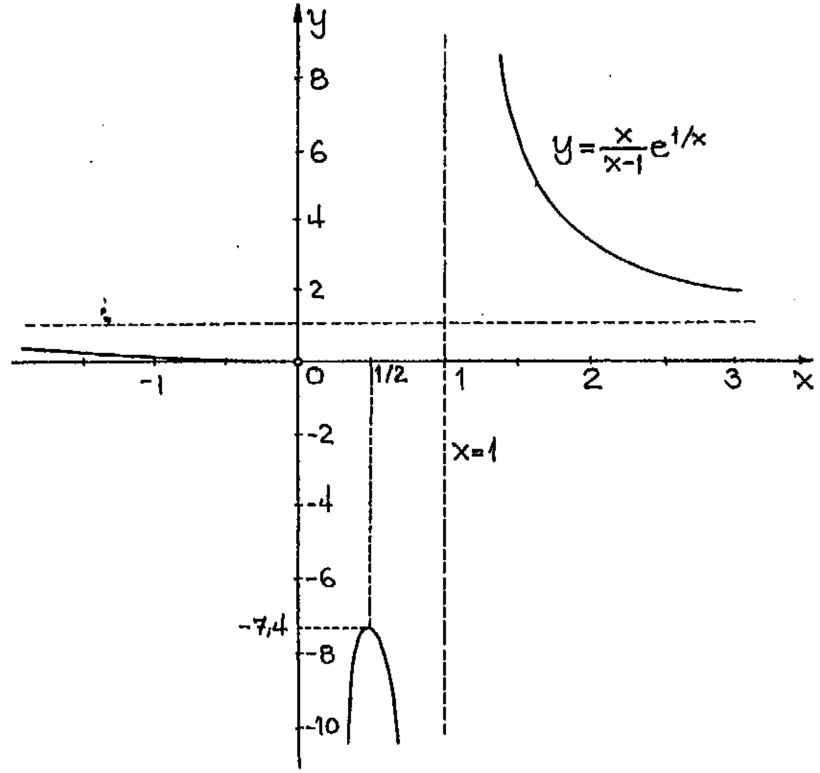
(iv) $f'(x) = -\dfrac{1}{(x-1)^2}e^{1/x} - \dfrac{1}{x(x-1)}e^{1/x} = -2\,\dfrac{x-1/2}{x(x-1)}e^{1/x};$

x		0		1/2		1	
sgn($f'(x)$)	−		+	0	−		−
$f(x)$	1^- ↘	0^+ | $-\infty$	↗	$-7{,}4$	↘ $-\infty$		∞ ↘ 1^+

f uppvisar ett lokalt maximum för $x = 1/2$.

x	-3	-2	-1	-0,5	-0,2	0⁺	0,2	0,3	0,5	0,6
y	0,53	0,4	0,18	0,05	0	−∞	-24	-12	-7,4	-7,9

…	0,8	0,9	1,2	1,4	1,6	2	3	4	5	6
	-14	-27	13,8	7,1	5	3,3	2,1	1,7	1,5	1,4



Övning 4.37 (Sid. 79)

Lösning

Fall I: Radien som parameter (ober. variabel).

$2\pi r = 10 \Leftrightarrow r = \frac{5}{\pi} \Rightarrow h = 10(1 - \frac{1}{\pi}) \Rightarrow V = \pi r^2 h = \frac{250(\pi - 1)}{\pi^2};$

Fall II: Höjden som parameter.

$2r = 10 - h \Leftrightarrow r = 5 - \frac{h}{2} \Rightarrow V = \frac{\pi}{4}(10-h)^2 h, \quad 0 < h < 10.$

$V' = \frac{\pi}{4}(3h^2 - 40h + 100) = 0 \Leftrightarrow h = \frac{10}{3} \Rightarrow V''(\frac{10}{3}) < 0 \Rightarrow \max.$

I detta fall blir $r=\frac{10}{3}$. För att tillverka botten och lock krävs det $4r=\frac{40}{3}>10$, vilket är orimligt.

Resultat: Radien ska vara $\frac{5}{\pi}\approx 1{,}60$ cm och höjden $10-\frac{10}{\pi}\approx 6{,}8$ cm.

Övning 4.38 (Sid. 79)

Lösning

$V(t)=\pi r^2(t)h(t) \Rightarrow \frac{dV}{dt}=\pi\cdot 2r(t)r'(t)h(t)+\pi r^2(t)h'(t)$; (*)

$V'(t_0)=7\,m^3/min$, $r(t_0)=100\,m$, $h(t_0)=0{,}005$ och $r'(t_0)=2\,m/min$.

(*) $\Rightarrow \pi r(t_0)^2 h'(t_0)=V'(t_0)-2\pi r(t_0)r'(t_0)h(t_0) \Rightarrow$

$\Rightarrow \pi\cdot 100^2 h'(t_0)=7-2\pi\cdot 100\cdot 0{,}005\cdot 2=7-\pi>0 \Rightarrow h'(t_0)=$

$=\frac{7-\pi}{\pi}\cdot 10^{-4}\,m/min=1{,}23\cdot 10^{-4}\,m/min.$

Resultat: Tjockleken ökar 0,12 mm i minuten.

Övning 4.39 (Sid. 79)

Lösning

$x=\sin\frac{y}{2}+\cos\frac{y}{2}=\sqrt{2}\sin\left(\frac{y}{2}+\frac{\pi}{4}\right) \Leftrightarrow \sin\left(\frac{y}{2}+\frac{\pi}{4}\right)=\frac{x}{\sqrt{2}}$;

$|x|\le\sqrt{2} \Rightarrow \frac{y}{2}+\frac{\pi}{4}=\arcsin\frac{x}{\sqrt{2}} \Leftrightarrow y=2\arcsin\frac{x}{\sqrt{2}}-\frac{\pi}{2} \Rightarrow$

$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{2}}{\sqrt{1-x^2/2}}=\frac{2}{\sqrt{2-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{4}{2-x^2},\ |x|<\sqrt{2}.$

Övning 4.40 (Sid. 79)

Lösning

$f(x)=\arcsin\frac{x^2-1}{x^2+1}-2\arctan x,\ x\in\mathbb{R}.$

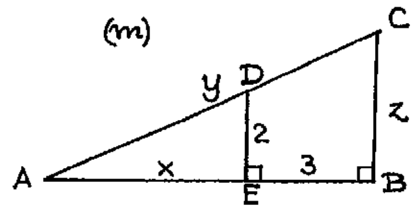
$f'(x)=\left(1-\left(\frac{x^2-1}{x^2+1}\right)^2\right)^{-1/2}\cdot\frac{4x}{(x^2+1)^2}-\frac{2}{x^2+1}=\frac{1}{\sqrt{4x^2}}\cdot\frac{4x}{x^2+1}-\frac{2}{x^2+1}=$

$=\frac{4x}{2|x|}\cdot\frac{1}{x^2+1}-\frac{2}{x^2+1}=2\left(\frac{|x|}{x}-1\right)\frac{1}{x^2+1}\le 0 \Rightarrow f$ avtagande $\Rightarrow$

$\Rightarrow f(\infty)<f(x)<f(-\infty) \Rightarrow -\frac{\pi}{2}<f(x)<\frac{3\pi}{2} \Rightarrow V_f=\left]-\frac{\pi}{2},\frac{3\pi}{2}\right[.$

Övning 4.41 (Sid. 70)

Lösning



$AC=y=$ stegens längd.

$DE=2=$ plankets höjd.

$\Delta AED\sim\Delta ABC \Rightarrow \frac{2}{x}=\frac{z}{x+3} \Leftrightarrow z=\frac{2(x+3)}{x} \Rightarrow y^2=(x+3)^2+z^2=$

$=\frac{4(x+3)^2}{x^2}+(x+3)^2=f(x) \Rightarrow f'(x)=\frac{8x(x+3)-8(x+3)^2}{x^3}+2(x+3)=$

$=(x+3)\left(\frac{8x-8(x+3)}{x^3}+2\right)=(x+3)\left(2-\frac{24}{x^3}\right);\ f'(x)=0 \Rightarrow x=\sqrt[3]{12} \Rightarrow$

$\Rightarrow f(\sqrt[3]{12})=(3+\sqrt[3]{12})^2\cdot\left(1+\frac{4}{\sqrt[3]{12^2}}\right)=(2^{2/3}+3^{2/3})^{3/2}\approx 7{,}02\,m.$

Resultat: Den kortaste längden på stegen är $(2^{2/3}+3^{2/3})^{3/2}\approx 7{,}02$ meter.

5. Primitiva funktioner

Övning 5.1 (Sid. 93)

Lösning

a) $\int x^4 dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$.

b) $\int \frac{1}{x} dx = \ln|x| + C$.

c) $\int e^x dx = e^x + C$.

d) $\int \cos x dx = \sin x + C$.

e) $\int \sin x dx = -\cos x + C$.

f) $\int \frac{1}{\cos^2 x} dx = \tan x + C$.

g) $\int \frac{1}{\sin^2 x} dx = -\cot x + C$.

h) $\int \frac{1}{1+x^2} dx = \arctan x + C$.

i) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C = -\arccos x + C'$.

j) $\int \frac{1}{\sqrt{x^2+\alpha}} dx = \ln|x+\sqrt{x^2+\alpha}| + C$.

Övning 5.2 (Sid. 93)

Lösning

a) $f(x) = x \Rightarrow F(x) = \frac{1}{2}x^2$; $\quad (F(x) = \int f(x) dx = D^{-1} f(x))$.

b) $f(x) = x^2 \Rightarrow F(x) = \frac{1}{3}x^3 + 2$. $\quad$ ($C = 2$, specialfall).

c) $f(x) = x^3 \Rightarrow F(x) = \frac{1}{4}x^4 + 5$. $\quad (C = 5)$.

d) $f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow F(x) = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x}$. $\quad (C = 0)$.

e) $f(x) = \frac{1}{x^3} = x^{-3} \Rightarrow F(x) = \frac{x^{-3+1}}{-3+1} + 9 = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} + 9$. $\quad (C = 9)$.

f) $f(x) = \frac{1}{x} \Rightarrow F(x) = \ln|x| + 3$. $\quad (C = 3)$.

g) $f(x) = \sqrt{x} = x^{1/2} \Rightarrow F(x) = \frac{x^{1/2+1}}{1/2+1} + 4 = \frac{x^{3/2}}{3/2} + 4 = \frac{2}{3}x\sqrt{x} + 4$.

h) $f(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2} \Rightarrow F(x) = \frac{x^{3/2+1}}{3/2+1} = \frac{x^{5/2}}{5/2} = \frac{2}{5}x^2\sqrt{x}$.

i) $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow F(x) = \frac{x^{-1/2+1}}{-1/2+1} = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$.

j) $f(x) = \frac{1}{x\sqrt{x}} = x^{-3/2} \Rightarrow F(x) = \frac{x^{-3/2+1}}{-3/2+1} + 5 = \frac{x^{-1/2}}{-1/2} + 5 = -\frac{2}{\sqrt{x}} + 5$.

Övning 5.3 (Sid. 93)

Lösning

$\int f(x) dx = F(x) \Rightarrow \int f(x+\alpha) dx = F(x+\alpha)$ (att memorera).

a) $\int \frac{1}{x} dx = \ln|x|$. (Jfr 5.2 f)).

b) $\int \frac{1}{x-2} dx = \ln|x-2| + 3$. $\quad (\alpha = -2 ;\ C = 3)$.

c) $\int \frac{1}{1-x} dx = \int \frac{1}{-(x-1)} dx = -\int \frac{1}{x-1} dx = -\ln|x-1| - 8$.

d) $\int \frac{2}{x+1} dx = 2\int \frac{1}{x+1} dx = 2\ln|x+1|$.

e) $\int \frac{1}{2x-1} dx = \int \frac{1}{2(x-1/2)} dx = \int \frac{1}{2} \frac{1}{x-1/2} dx = \frac{1}{2}\ln|x-\frac{1}{2}|$.

f) $\int \frac{2}{1-3x} dx = \int \frac{2}{-3(x-1/3)} dx = \int (-\frac{2}{3} \frac{1}{x-1/3}) dx = -\frac{2}{3}\ln|x-\frac{1}{3}|$.

g) $\int \frac{1}{x^2} dx = -\frac{1}{x}$. (Jfr 5.2 d)).

h) $\int \frac{1}{(x-2)^2}dx = -\frac{1}{x-2}$. (C=0)

i) $(1-x)^2 = 1-2x+x^2 = x^2-2x+1 = (x-1)^2 \Rightarrow \int \frac{1}{(1-x)^2}dx = \frac{1}{1-x}$.

j) $\int \frac{2}{(x+1)^2}dx = 2\int \frac{1}{(x+1)^2}dx = -\frac{2}{x+1}$. (C=0).

k) $(2x+1)^2 = (2(x+\frac{1}{2}))^2 = 4(x+\frac{1}{2})^2 \Rightarrow \int \frac{1}{(2x+1)^2}dx = -\frac{1}{4}\frac{1}{x+1/2} =$

$= -\frac{1}{2}\frac{1}{2x+1}$. (C=0).

l) $\int \frac{2}{(1-3x)^2}dx = 2\int \frac{1}{(3x-1)^2}dx = 2\int \frac{1}{9(x-1/3)^2}dx = -\frac{2}{9}\frac{1}{x-1/3} =$

$= \frac{2}{3}\cdot\frac{1}{-3(x-1/3)} = \frac{2}{3}\frac{1}{1-3x}$.

<u>Att memorera</u>: $\int f(ax+b)dx = \frac{1}{a}F(ax+b)$.

<u>Övning 5.4 (Sid. 93)</u>

<u>Lösning</u>

a) $\int (x^2+x-2)dx = \frac{x^3}{3}+\frac{(x-2)^2}{2}+C$. (Derivera!...)

b) $\int (\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3})dx = \int \frac{1}{x}dx - \int \frac{1}{x^2}dx + \int \frac{1}{x^3}dx =$ (Se 5.2) $=$

$= \ln|x| + \frac{1}{x} - \frac{1}{2x^2}+C$.

c) $\int (\sqrt{x}+\frac{1}{\sqrt{x}})dx = \frac{2}{3}x\sqrt{x}+2\sqrt{x}+C = \frac{2}{3}(x+3)\sqrt{x}+C$. (Se 5.2).

d) $\frac{3+5x^{2/3}}{x^3} = \frac{3}{x^3}+\frac{5x^{2/3}}{x^3} = 3x^{-3}+5x^{2/3-3} = 3x^{-3}+5x^{-7/3} \Rightarrow$

$\Rightarrow \int \frac{3+5x^{2/3}}{x^3}dx = 3\int x^{-3}dx + 5\int x^{-7/3}dx = 3\cdot\frac{x^{-2}}{-2}+5\frac{x^{-4/3}}{-4/3}+$

$+C = -\frac{3}{2x^2}-\frac{15}{4x^{4/3}}+C$.

e) $\int \frac{1}{(1+x)^2}dx = -\frac{1}{1+x}+C$. (Jfr 5.3 j)).

forts.

k) $\int \frac{1}{(2x+1)^2}dx = \frac{1}{4}\int \frac{1}{(x+1/2)^2}dx = -\frac{1}{4}\frac{1}{x+1/2}+C = -\frac{1}{2}\frac{1}{2x+1}+C$.

l) $\int (-\frac{1}{1+x^2})dx = -\arctan x + C = \operatorname{arccot} x + C'$.

<u>Övning 5.5 (Sid. 93)</u>

<u>Lösning</u>

a) $\int \sin x\,dx = -\cos x - 3$. (C=−3). (Se Anm. i 5.3).

b) $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x$. (C=0).

c) $\int \sin\frac{x}{3}dx = -3\cos\frac{x}{3}+1$ (C=1).

d) $\int \sin(2x+\frac{\pi}{3})dx = -\frac{1}{2}\cos(2x+\frac{\pi}{3})$.

e) $\int \cos x\,dx = \sin x + 10$. (C=10).

f) $\int \cos(1-x)dx = \int \cos(x-1)dx = \sin(x-1)$. (C=0).

g) $\int \cos(\frac{2}{3}x)dx = \frac{3}{2}\sin\frac{3x}{2}$.

h) $\int e^x dx = e^x - 1$.

i) $\int 2e^{3x}dx = 2\cdot\frac{1}{3}e^{3x} = \frac{2}{3}e^{3x}$.

j) $\int e^{-x}dx = \frac{e^{-x}}{-1} = -e^{-x}$.

k) $\int e^{2x+1}dx = \frac{1}{2}e^{2x+1}$.

l) $\int e^{x/3}dx = 3e^{3x}-3$.

<u>Att memorera</u>: $f(x) = g(ax+b) \Rightarrow F(x) = \frac{1}{a}G(ax+b)$.

$\int f(x)dx$ = alla primitiver till $f(x)$ (tyvärr)!

Övning 5.6 (Sid. 96)

Lösning

a) $f(x)=e^{x^2}\cdot 2x=e^{x^2}\cdot(x^2)'=(e^{x^2})' \Rightarrow F(x)=e^{x^2}$.

b) $f(x)=2xe^{x^2}=e^{x^2}2x \Rightarrow F(x)=e^{x^2}$ (enl. a) ovan).

c) $f(x)=e^{x^2}\cdot x=\frac{1}{2}e^{x^2}\cdot 2x=\frac{1}{2}(e^{x^2})' \Rightarrow F(x)=\frac{1}{2}e^{x^2}$.

d) $f(x)=xe^{x^2}=e^{x^2}\cdot x \Rightarrow F(x)=\frac{1}{2}e^{x^2}$.

e) $f(x)=\cos x^2\cdot 2x=\cos x^2\cdot(x^2)'=(\sin x^2)' \Leftrightarrow F(x)=\sin x^2$.

f) $f(x)=x\cdot\cos x^2=\frac{1}{2}\cdot\cos x^2\cdot 2x \Rightarrow F(x)=\frac{1}{2}\sin x^2$.

g) $f(x)=x\sin x^2=\frac{1}{2}\sin x^2\cdot(x^2)'=\frac{1}{2}(-\cos x^2)' \Rightarrow F(x)=-\frac{1}{2}\cos x^2$.

h) $f(x)=2x(x^2+5)^8=(x^2+5)^8(x^2+5)'=\left(\frac{(x^2+5)^9}{9}\right)' \Rightarrow F(x)=\frac{(x^2+5)^9}{9}$.

Övning 5.7 (Sid. 96)

Lösning

a) $f(x)=x^2\sin x^3=\frac{1}{3}\sin x^3\cdot 3x^2=\frac{1}{3}\sin x^3\cdot(x^3)'=\left(-\frac{\cos x^3}{3}\right)'$

$\Rightarrow F(x)=-\frac{1}{3}\cos x^3$.

b) $f(x)=\frac{1}{x^2}\cos\frac{1}{x}=-\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)=-\cos\left(\frac{1}{x}\right)\cdot\left(\frac{1}{x}\right)'=\left(-\cos\left(\frac{1}{x}\right)\right)'$

$\Rightarrow F(x)=-\cos\frac{1}{x}$.

c) $f(x)=\sin^2 x\cos x=(\sin x)^2\cdot(\sin x)'=\left(\frac{1}{3}\sin^3 x\right)' \Leftrightarrow$

$\Leftrightarrow F(x)=\frac{1}{3}\sin^3 x$.

d) $f(x)=\cos x\cdot\sin^3 x=(\sin x)^3(\sin x)'=\left(\frac{\sin^4 x}{4}\right) \Rightarrow F(x)=$

$=\frac{1}{4}\sin^4 x$.

e) $f(x)=\cos x\cdot\frac{1}{\sin x}=\frac{(\sin x)'}{\sin x}=(\ln|\sin x|)' \Rightarrow F(x)=\ln|\sin x|$.

f) $f(x)=\cos x\cdot\frac{1}{\sin^2 x}=\left(\frac{1}{\sin x}\right)^2\cdot(\sin x)'=\left(-\frac{1}{\sin x}\right)' \Rightarrow F(x)=-\frac{1}{\sin x}$.

g) $f(x)=e^x\cdot\sin(e^x)=\sin e^x\cdot(e^x)'=(-\cos e^x)' \Rightarrow F(x)=-\cos e^x$.

h) $f(x)=e^x(e^x+5)^8=(e^x+5)^8(e^x+5)'=\left(\frac{(e^x+5)^9}{9}\right)' \Rightarrow F(x)=\frac{(e^x+5)^9}{9}$.

Övning 5.8 (Sid. 94)

Lösning

a) $f(x)=2x\cdot\frac{1}{x^2+1}=\frac{(1+x^2)'}{1+x^2}=(\ln(1+x^2))' \Rightarrow F(x)=\ln(1+x^2)$.

b) $f(x)=\frac{2x}{x^2+1} \Rightarrow F(x)=\ln(x^2+1)$, enl. a) ovan.

c) $f(x)=\frac{x}{x^2+1}=\frac{1}{2}\frac{2x}{x^2+1} \Rightarrow F(x)=\frac{1}{2}\ln(x^2+1)$.

d) $f(x)=\frac{x^2}{x^3+1}=\frac{1}{3}\cdot\frac{3x^2}{x^3+1}=\frac{1}{3}\cdot\frac{(x^3+1)'}{x^3+1} \Rightarrow F(x)=\frac{1}{3}\ln(x^3+1)$.

e) $f(x)=\frac{\cos x}{\sin x}=\frac{(\sin x)'}{\sin x} \Rightarrow F(x)=\ln|\sin x|$. (Jfr. 5.7 e)).

f) $f(x)=\cot x=\frac{\cos x}{\sin x} \Rightarrow F(x)=\ln|\sin x|$.

g) $f(x)=\tan x=\frac{\sin x}{\cos x}=-\frac{-\sin x}{\cos x}=-\frac{(\cos x)'}{\cos x} \Rightarrow F(x)=-\ln|\cos x|$.

h) $f(x)=\frac{e^x}{e^x+1}=\frac{(e^x+1)'}{e^x+1} \Rightarrow F(x)=\ln(e^x+1)$.

Anm. Jag har i hela denna övning utnyttjat

$$\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|.$$

Övning 5.9 (Sid. 94)

Lösning

a) $f(x) = \cot x \Leftrightarrow \mathcal{F}(x) = \ln|\sin x| + C$. (Se 5.8 e-f)).

b) $f(x) = \dfrac{3x^2}{x^3+1} = \dfrac{(x^3+1)'}{x^3+1} \Leftrightarrow \mathcal{F}(x) = \ln|x^3+1| + C$.

c) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{(e^x+e^{-x})'}{e^x+e^{-x}} \Leftrightarrow \mathcal{F}(x) = \ln(e^x+e^{-x}) + C$.

d) $f(x) = \dfrac{1}{\sqrt{1-x^2}\cdot \arcsin x} = \dfrac{(\arcsin x)'}{\arcsin x} \Leftrightarrow \mathcal{F}(x) = \ln|\arcsin x| + C$.

e) $f(x) = \dfrac{1}{(1+x^2)\arctan x} = \dfrac{(\arctan x)'}{\arctan x} \Leftrightarrow \mathcal{F}(x) = \ln|\arctan x| + C$.

f) $f(x) = \dfrac{1-\tan x}{1+\tan x} = \dfrac{\cos x - \sin x}{\sin x + \cos x} = \dfrac{(\sin x + \cos x)'}{\sin x + \cos x} \Leftrightarrow$

$\Leftrightarrow \mathcal{F}(x) = \ln|\sin x + \cos x| + C$.

Övning 5.10 (Sid. 94)

Lösning

a) $f(x) = \frac{1}{x}(\ln x)^2 = (\ln x)^2 \cdot (\ln x)' = \left(\frac{\ln^3 x}{3}\right)' \Rightarrow \mathcal{F}(x) = \frac{\ln^3 x}{3}$.

b) $f(x) = \frac{1}{x}(\ln x) = (\ln x)\cdot(\ln x)' = \left(\frac{\ln^2 x}{2}\right)' \Rightarrow \mathcal{F}(x) = \frac{\ln^2 x}{2}$.

c) $f(x) = \frac{(\ln x)^2}{x} = (\ln x)^2 \cdot (\ln x)' = \left(\frac{\ln^3 x}{3}\right)' \Rightarrow \mathcal{F}(x) = \frac{1}{3}\ln^3 x$.

d) $f(x) = \frac{\ln x}{x} = \ln x \cdot (\ln x)' = \left(\frac{\ln^2 x}{2}\right)' \Rightarrow \mathcal{F}(x) = \frac{1}{2}\ln^2 x$.

e) $f(x) = \frac{1}{x}\cdot \sin(\ln x) = \sin(\ln x)\cdot(\ln x)' = (-\cos \ln x)' \Rightarrow$

$\Rightarrow \mathcal{F}(x) = -\cos(\ln x)$.

f) $f(x) = \frac{1}{x}\frac{1}{\ln x} = \frac{(\ln x)'}{\ln x} \Rightarrow \mathcal{F}(x) = \ln|\ln x|$. Obs! $\begin{cases} f \Leftrightarrow g \\ e \Leftrightarrow h \end{cases}$

Övning 5.11 (Sid. 94)

Lösning

a) $f(x) = (\sin x)^5 \cos x = (\sin x)^5 \cdot (\sin x)' = \left(\frac{\sin^5 x}{5}\right)' \Rightarrow$

$\Rightarrow \mathcal{F}(x) = \frac{1}{5}\sin^5 x$.

b) $f(x) = \sin^5 x = \sin^4 x \cdot \sin x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x =$

$= (1 - 2\cos^2 x + \cos^4 x)\sin x = \sin x - 2\cos^2 x \sin x + \cos^4 x \sin x =$

$= (-\cos x)' + 2\left(\frac{\cos^3 x}{3}\right)' - \left(\frac{\cos^5 x}{5}\right)' = \left(-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right)'$

$\Leftrightarrow \mathcal{F}(x) = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$.

c) $f(x) = e^{x^2}$ saknar primitiv i elementära funktioner.

d) $f(x) = e^{x^2}\cdot 2x \Rightarrow \mathcal{F}(x) = e^{x^2}$.

e) $f(x) = \frac{1}{x}(\ln x)^3 \Rightarrow \mathcal{F}(x) = \frac{1}{4}\ln^4 x$.

f) $f(x) = (\ln x)^3$; är nog svårt att bestämma $\mathcal{F}(x)$.

g) $f(x) = \dfrac{4x^3}{x^4+2} = \dfrac{(x^4+2)'}{x^4+2} \Rightarrow \mathcal{F}(x) = \ln(x^4+2)$.

h) $f(x) = \dfrac{1}{x^4+2}$; svårt att finna/bestämma $\mathcal{F}(x)$.

Övning 5.12 (Sid. 94)

Lösning

a) $\int x(1+x^2)^5 dx = \int (1+x^2)^5 x\,dx = \frac{1}{2}\int (1+x^2)^5 (x^2+1)' dx =$

$= \frac{1}{2}\int (x^2+1)^5 d(x^2+1)\ [u = x^2+1] = \left\{\frac{1}{2}\int u^5 du\right\}_{u=x^2+1} =$

$= \left\{ \frac{1}{2} \cdot \frac{u^6}{6} + C \right\}_{u=x^2+1} = \frac{1}{12}(x^2+1)^6 + C.$

b) $\int \frac{1}{x \ln x} dx \begin{bmatrix} u = \ln x \\ du = dx/x \end{bmatrix} = \left\{ \int \frac{du}{u} \right\}_{u=\ln x} = \ln|\ln x| + C.$

c) $\int \sin x \cdot \cos^{4/3} x \, dx = -\int \cos^{-4/3} x \cdot (\cos x)' dx \begin{bmatrix} u = \cos x \\ du = -\sin x dx \end{bmatrix} =$

$= \left\{ -\int u^{-4/3} du \right\}_{u=\cos x} = \left\{ -\frac{u^{-1/3}}{-1/3} + C \right\}_{u=\cos x} = \frac{3}{(\cos x)^{1/3}} + C.$

d) $\int \frac{1}{\sqrt{x}} \sin\sqrt{x} \, dx \left[u = \sqrt{x} \Rightarrow du = \frac{dx}{2\sqrt{x}} \right] = \left\{ 2 \int \sin u \, du \right\}_{u=\sqrt{x}} =$

$= \left\{ -2\cos u + C \right\}_{u=\sqrt{x}} = -2\cos\sqrt{x} + C.$

e) $\int x \sqrt{7x^2+5} \, dx \left[u = 7x^2+5 \Rightarrow du = 14x dx \right] = \left\{ \frac{1}{14} \int \sqrt{u} \, du \right\}_{u=7x^2+5}$

$= \left\{ \frac{1}{14} \cdot \frac{2}{3} u^{3/2} + C \right\}_{u=7x^2+5} = \frac{1}{21}(7x^2+5)^{3/2} + C.$

f) $\int \frac{x}{\sqrt{x^2+5}} dx \left[u = x^2+5 \Rightarrow du = 2x dx \right] = \left\{ \int \frac{1}{2} \frac{1}{\sqrt{u}} du \right\}_{u=x^2+5} =$

$= \left\{ \sqrt{u} + C \right\}_{u=x^2+5} = \sqrt{x^2+5} + C.$

Övning 5.13 (Sid. 95)

Lösning

a) $\int \frac{1}{e^x + e^{-x}} dx = \int \frac{e^x}{e^{2x}+1} dx \begin{bmatrix} u = e^x \\ du = e^x dx \end{bmatrix} = \left\{ \int \frac{1}{u^2+1} \right\}_{u=e^x} =$

$= \left\{ \arctan u + C \right\}_{u=e^x} = \arctan(e^x) + C.$

b) $\int x\sqrt{x+1} \, dx \begin{bmatrix} x+1 = u^2 \\ dx = 2u du \end{bmatrix} = \left\{ \int (u^2-1) u \cdot 2u du \right\}_{u=\sqrt{x+1}} =$

$= \left\{ 2 \int (u^4 - u^2) du \right\}_{u=\sqrt{x+1}} = \left\{ 2\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C \right\}_{u=\sqrt{x+1}} =$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.$

c) $\int \frac{x}{\sqrt{2x+5}} dx \begin{bmatrix} 2x+5 = u^2 \\ x = (u^2-5)/2 \\ dx = u du \end{bmatrix} = \left\{ \int \frac{\frac{u^2-5}{2}}{u} u du \right\}_{u=\sqrt{2x+5}} =$

$= \left\{ \frac{1}{2} \int (u^2-5) du \right\}_{u=\sqrt{2x+5}} = \left\{ \frac{1}{2}\left(\frac{u^3}{3} - 5u \right) + C \right\}_{u=\sqrt{2x+5}} =$

$= \left\{ \frac{1}{6}(u^2-15)u + C \right\}_{u=\sqrt{2x+5}} = \frac{1}{6}(2x-10)\sqrt{2x+5} + C =$

$= \frac{1}{3}(x-5)\sqrt{2x+5} + C$

d) $\int \frac{1}{x + x^{1/3}} dx \begin{bmatrix} x = u^3 \\ dx = 3u^2 du \end{bmatrix} = \left\{ 3 \int \frac{u^2}{u^3+u} du \right\}_{u=x^{1/3}} = \left\{ \frac{3}{2} \int \frac{2u du}{u^2+1} \right\}_{x=u^3} =$

$= \left\{ \frac{3}{2} \ln(u^2+1) + C \right\}_{u=x^{1/3}} = \frac{3}{2} \ln(x^{2/3}+1) + C.$

Övning 5.14 (Sid. 95)

Lösning

a) $\int x^2 \ln x \, dx = \int \left(\frac{x^3}{3}\right)' \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^3 \cdot (\ln x)' dx =$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C = \frac{x^3}{9}(3\ln x - 1) + C.$

b) $\int x e^{-x} dx = \int (-e^{-x})' x \, dx = -xe^{-x} + \int e^{-x} dx = -(x+1)e^{-x} + C$

Anm. $\int f(x) g(x) dx = F(x) g(x) - \int F(x) g'(x) dx$ osv.

c) $f(x) = \sqrt{x} \Rightarrow F(x) = \frac{2}{3} x\sqrt{x} \Rightarrow \int \sqrt{x} \ln x \, dx = \frac{2}{3} x\sqrt{x} \ln x -$

$- \frac{2}{3} \int \sqrt{x} \, dx = \frac{2}{3} x\sqrt{x} \ln x - \frac{4}{9} x\sqrt{x} + C = \frac{2}{3} x\sqrt{x} \cdot \left(\ln x - \frac{2}{3}\right) + C.$

d) $\int \arctan x \, dx = \int 1 \cdot \arctan x \, dx = x \arctan x - \int \frac{x}{x^2+1} dx =$

$= x \arctan x - \frac{1}{2}\ln(x^2+1) + C$ (Se 5.8 c)).

e) $\int x \arctan x \, dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1} dx = \frac{x^2}{2} \tan^{-1} x -$

$-\frac{1}{2}\int(1-\frac{1}{x^2+1})dx = \frac{1}{2}x^2\arctan x - \frac{1}{2}(x-\arctan x)+C =$

$= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C.$

f) $\int \ln(x+1)dx = (x+1)\ln(x+1) - \int 1\cdot dx = (x+1)\ln(x+1)-(x+1)+C$

$= (x+1)(\ln(x+1)-1)+C.$

g) $\int(\ln^2 x)dx = \int 1\cdot(\ln x)^2 dx = x\cdot\ln^2 x - \int x\cdot 2\ln x\cdot\frac{1}{x}dx =$

$= x\ln^2 x - 2\int \ln x\, dx = x\ln^2 x - 2(\int 1\cdot\ln x\, dx) = x\cdot\ln^2 x -$

$-2(x\ln x - \int 1\cdot dx) = x\ln^2 x - 2(x\ln x - x)+C =$

$= x\cdot\ln^2 x - 2x\ln x + 2x + C.$

h) $\int x^2\sin x\, dx = \int(-\cos x)'x^2 dx = -x^2\cos x + \int 2x\cdot\cos x\, dx =$

$= -x^2\cos x + 2(\int(\sin x)'x\, dx) = -x^2\cos x + 2(x\sin x -$

$-\int\sin x\, dx) = -x^2\cos x + 2(x\sin x + \cos x)+C = 2x\sin x +$

$+(2-x^2)\cos x + C.$

i) $\int x(x+1)^9 dx = \int(\frac{(x+1)^{10}}{10})'x\, dx = \frac{1}{10}x(x+1)^{10} - \frac{1}{10}\int(x+1)^{10}dx =$

$= \frac{1}{10}x(x+1)^{10} - \frac{1}{10}\cdot\frac{1}{11}(x+1)^{11}+C = \frac{x(x+1)^{10}}{10} - \frac{(x+1)^{11}}{110}+C.$

j) $\int\frac{x}{\cos^2 x}dx = \int(\tan x)'x\, dx = x\tan x - \int\tan x\, dx = x\tan x -$

$+\ln|\cos x| + C$ (Se 5.8 g)).

Övning 5.15 (Sid. 95)

Lösning

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)dx.$$

a) $I = \int e^x\sin x\, dx = e^x\sin x - \int e^x\cos x\, dx = e^x\sin x -$

$-(e^x\cos x + \int e^x\sin x\, dx) = e^x\sin x - e^x\cos x - \int e^x\sin x\, dx =$

$= e^x(\sin x - \cos x) - I \Leftrightarrow 2I = e^x(\sin x - \cos x) + 2C \Leftrightarrow$

$\Leftrightarrow I = \int e^x\sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$

b) $J = \int e^{2x}\sin 3x\, dx = \frac{1}{2}e^{2x}\sin 3x - \frac{1}{2}\int e^{2x}(3\cos 3x)dx =$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\, dx = \frac{1}{2}e^{2x}\sin 3x -$

$-\frac{3}{2}(\frac{1}{2}e^{2x}\cos 3x - \frac{1}{2}\int e^{2x}\cdot(-3\sin 3x)dx) = \frac{1}{2}e^{2x}\sin 3x -$

$-\frac{3}{2}\cdot(\frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\, dx) = \frac{1}{2}e^{2x}\sin 3x -$

$-\frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}J \Leftrightarrow J + \frac{9}{4}J = e^{2x}\cdot\frac{e^{2x}(2\sin 3x - 3\cos 3x)}{4}$

$\Leftrightarrow \frac{13}{4}J = \frac{e^{2x}(2\sin 3x - 3\cos 3x)}{4} + \frac{13}{4}C \Leftrightarrow J = \int e^{2x}\sin 3x\, dx =$

$= \frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$

Övning 5.16 (Sid. 95)

Lösning

a) $\int e^{\sqrt{x}}dx\,[x=u^2 \Rightarrow dx = 2u\,du] = \{\int e^u\cdot 2u\,du\}_{x=u^2} =$

$= \{2\int e^u\cdot u\,du\}_{u^2=x} = \{2e^u u - 2\int e^u du\}_{x=u^2} =$

$= \{2ue^u - 2e^u + C\}_{u=\sqrt{x}} = 2(\sqrt{x}-1)e^{\sqrt{x}} + C.$

Att memorera: $\int xe^x dx = (x-1)e^x + C.$

b) $\int x^3 \sin x^2 dx \; [u = x^2 \Rightarrow du = 2x dx] = \{\frac{1}{2}\int u \sin u \, du\} =$

$= \{\frac{1}{2}\int u(-\cos u)' du\}_{u=x^2} = \{-\frac{1}{2}u\cos u + \frac{1}{2}\int \cos u \, du\}_{u=x^2} =$

$= \{-\frac{1}{2}u\cos u + \frac{1}{2}\sin u + C\}_{x^2=u} = \frac{1}{2}(\sin x^2 - x^2\cos x^2) + C.$

c) $\int e^x \ln(1+e^x) dx \; [u = e^x \Rightarrow du = e^x dx] = \{\int \ln(1+u) du\}_{u=e^x}$

$= \{\int 1 \cdot \ln(1+u) du\}_{u=e^x} \overset{!}{=} \{(1+u)\ln(1+u) - \int du\}_{u=e^x} =$

$= \{(1+u)\ln(1+u) - (1+u) + C\}_{u=e^x} = (1+e^x)(\ln(1+e^x) - 1) + C.$

Anm. Jag använder primitiven $F(x) = x+1$ till $f(x) = 1$ för att förenkla integrationen i $\overset{!}{=}$.

Övning 5.17 (Sid. 95)

Lösning

a) $u = x-1 \Leftrightarrow x = u+1 \Rightarrow x^2+4 = (u+1)^2+4 = u^2+2u+5 \Rightarrow$

$\Rightarrow \frac{x^2+4}{x-1} = \frac{u^2+2u+5}{u} = u+2+\frac{5}{u} = x+1+\frac{5}{x-1} \Rightarrow \int \frac{x^2+4}{x-1}dx =$

$= \int (x+1+\frac{5}{x-1})dx = \frac{1}{2}(x+1)^2 + 5\ln|x-1| + C.$

b)

$$
\begin{array}{rl}
 & x^2+2x-4 = (x+1)^2-5 \\
\hline
x^3+5x^2+2x-1 & \boxed{x+3} \\
(-)\; \underline{x^3+3x^2} & \\
2x^2+2x-1 & \\
(-)\; \underline{2x^2+6x} & \\
-4x-1 & \\
(-)\; \underline{-4x-12} & \\
11 &
\end{array}
$$

$\int \frac{x^3+5x^2+2x-1}{x+3}dx = \int((x+1)^2 - 5 + \frac{11}{x+3})dx = \frac{(x+1)^3}{3} - 5x + 11\ln|x+3| + C.$

Övning 5.18 (Sid. 95)

Lösning

Bra att memorera: $\alpha \neq \beta \Rightarrow \int \frac{dx}{(x-\alpha)(x-\beta)} = \frac{1}{\alpha-\beta}(\frac{1}{x-\alpha} - \frac{1}{x-\beta})$.

a) $\int \frac{1}{x^2-4}dx = \int \frac{1}{(x-2)(x+2)}dx = \frac{1}{4}\int(\frac{1}{x-2} - \frac{1}{x+2})dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C.$

b) $\int \frac{1}{x^2-4x-5}dx = \int \frac{1}{(x+1)(x-5)}dx = \frac{1}{6}\int(\frac{1}{x+1} - \frac{1}{x-5})dx = \frac{1}{6}\ln\left|\frac{x+1}{x-5}\right| + C.$

c) $f(x) = x^3 - 6x^2 + 11x - 6 \Rightarrow f(1) = 0 \Leftrightarrow (x-1)$ faktor i $f(x)$.

$$
\begin{array}{rl}
 & x^2-5x+6 = (x-2)(x-3) \\
\hline
x^3-6x^2+11x-6 & \boxed{x-1} \\
(-)\; \underline{x^3-x^2} & \\
-5x^2+11x-6 & \\
(-)\; \underline{-5x^2+5x} & \\
6x-6 & \\
(-)\; \underline{6x-6} & \\
0 &
\end{array}
$$

$\frac{5x^2-7x+13}{x^3-6x^2+11x-6} = \frac{5x^2-7x+13}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = \frac{A(x-2)(x-3)}{f(x)} +$

$+ \frac{B(x-1)(x-3)}{f(x)} + \frac{C(x-1)(x-2)}{f(x)} =$

$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{x^3-6x^2+11x-6}, \quad x \neq 1, 2, 3$

$\Leftrightarrow A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = 5x^2 - 7x + 13;$

$\left.\begin{cases} x = 1 \Rightarrow 2A = 11 \\ x = 2 \Rightarrow -B = 19 \\ x = 3 \Rightarrow 2C = 37 \end{cases}\right\} \Rightarrow \frac{5x^2-7x+13}{x^3-6x^2+11x-6} = \frac{11}{2}\frac{1}{x-1} - 19\frac{1}{x-2} + \frac{37}{2}\frac{1}{x-3} \Rightarrow$

$$\Rightarrow \int \frac{5x^2-7x+13}{x^3-6x^2+11x-6}dx = \frac{11}{2}\ln|x-1| - 19\ln|x-2| + \frac{37}{2}\ln|x-3| + C.$$

Övning 5.19 (Sid. 95)

Lösning

a) $\frac{1}{x(x-3)^2} = \frac{1}{(x-3)\cdot x}\cdot\frac{1}{x-3} = \frac{1}{3}\left(\frac{1}{x-3}-\frac{1}{x}\right)\frac{1}{x-3} = \frac{1}{3}\frac{1}{(x-3)^2} - \frac{1}{3}\frac{1}{(x-3)\cdot x} =$

$= \frac{1}{3}\frac{1}{(x-3)^2} - \frac{1}{9}\left(\frac{1}{x-3}-\frac{1}{x}\right) \Rightarrow \int\frac{1}{x(x-3)^2}dx = \frac{1}{3}\int\frac{1}{(x-3)^2}dx -$

$-\frac{1}{9}\int\left(\frac{1}{x-3}-\frac{1}{x}\right)dx = -\frac{1}{3}\frac{1}{x-3} - \frac{1}{9}\ln\left|\frac{x-3}{x}\right| + C.$

b) $u = x+2 \Leftrightarrow x = u-2 \Rightarrow \frac{3x+11}{(x+2)^2} = \frac{3u+5}{u^2} = \frac{3}{u} + \frac{5}{u^2} = \frac{3}{x+2} + \frac{5}{(x+2)^2}$

$\Rightarrow \int\frac{3x+11}{(x+2)^2}dx = 3\int\frac{1}{x+2}dx + 5\int\frac{1}{(x+2)^2}dx = 3\ln|x+2| - 5\frac{1}{x+2} + C.$

c) $\frac{1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} =$

$= \frac{Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2}{x^2(x-1)^3} \Leftrightarrow$

$\Leftrightarrow Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2 = 1$; (*)

(i) $x = 0 \Rightarrow \underline{B = -1} \Rightarrow Ax(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2 =$

$= (x-1)^3 + 1 = x^3 - 3x^2 + 3x \Leftrightarrow \underline{A(x-1)^3 + Cx(x-1)^2 + Dx(x-1) +}$

$\underline{+ Ex = x^2 - 3x + 3}$;

(ii) $x = 0 \Rightarrow -A = 3 \Leftrightarrow \underline{A = -3} \Rightarrow Cx(x-1)^2 + Dx(x-1) + Ex =$

$= x^2 - 3x + 3 + 3(x-1)^3 = 3x^3 - 8x^2 + 6x \Leftrightarrow \underline{C(x-1)^2 + D(x-1) + E =}$

$\underline{= 3x^2 - 8x + 6}$;

(iii) $x = 1 \Rightarrow \underline{E = 1} \Rightarrow C(x-1)^2 + D(x-1) = 3x^2 - 8x + 5 = (x-1)(3x-5)$

$\Leftrightarrow \underline{C(x-1) + D = 3x-5}$;

(iv) $x = 1 \Rightarrow \underline{D = -2} \Rightarrow C(x-1) = 3x - 3 = 3(x-1) \Leftrightarrow \underline{C = 3.}$

$\int\frac{dx}{x^2(x-1)^3} = -3\int\frac{dx}{x} + 3\int\frac{dx}{x^2} - \int\frac{dx}{x-1} - 2\int\frac{dx}{(x-1)^2} + \int\frac{dx}{(x-1)^3} = -3\ln|x| -$

$-\frac{3}{x} + \ln|x-1| + \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C.$

d) $x^3 + 2x^2 + x = x(x^2 + 2x + 1) = (x+1)^2 x$;

$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x+1)}\cdot\frac{1}{x+1} = \left(\frac{1}{x} - \frac{1}{x+1}\right)\frac{1}{x+1} = \frac{1}{x(x+1)} - \frac{1}{(x+1)^2} =$

$= \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \Rightarrow \int\frac{1}{x^3+2x^2+x}dx = \int\left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx =$

$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C = \ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + C.$

Övning 5.20 (Sid. 95)

Lösning

a) $\int\frac{1}{x^2+1}dx = \arctan x + C.$

b) $\int\frac{1}{(2x)^2+1}dx \; [u = 2x \Rightarrow du = 2dx] = \left\{\int\frac{2^{-1}}{u^2+1}du\right\}_{u=2x} =$

$= \left\{\frac{1}{2}\arctan u + C\right\}_{u=2x} = \frac{1}{2}\arctan(2x) + C.$

c) $\int\frac{1}{(x/3)^2+1}dx \; [u = \frac{x}{3} \Rightarrow dx = 3du] = \left\{3\int\frac{1}{u^2+1}du\right\}_{u=x/3} =$

$= \left\{3\arctan u + C\right\}_{u=x/3} = 3\arctan\frac{x}{3} + C.$

Övning 5.21 (Sid. 95)

Se nästa sida.

Lösning

a) $\int \frac{dx}{4x^2+1} = \int \frac{dx}{(2x)^2+1} = \frac{1}{2}\arctan(2x)+C$ (Se Ö. 5.20 b)).

b) $\int \frac{dx}{x^2/9+1} = \int \frac{dx}{(x/3)^2+1} = 3\arctan\frac{x}{3}+C$ (Se Ö. 5.2 c)).

c) $\int \frac{dx}{x^2+1/4} = \int \frac{4}{4x^2+1}dx = 4\cdot\frac{1}{2}\arctan(2x)+C = 2\arctan(2x)+C.$

d) $\int \frac{1}{x^2+9}dx = \frac{1}{9}\int\frac{dx}{x^2/9+1} = \frac{1}{9}\cdot 3\arctan\frac{x}{3}+C = \frac{1}{3}\arctan\frac{x}{3}+C.$

Övning 5.22 (Sid. 95)

Lösning

Att memorera: $\int \frac{1}{(x-\alpha)^2+\beta^2}dx = \frac{1}{\beta}\arctan\frac{x+\alpha}{\beta}.$

a) $\int \frac{1}{x^2-2x+2}dx = \int\frac{1}{(x-1)^2+1}dx = \arctan(x-1)+C.$

b) $\int \frac{1}{x^2+4x+5}dx = \int\frac{1}{(x+2)^2+1}dx = \arctan(x+2)+C.$

c) $\int \frac{1}{x^2+2x+5}dx = \int\frac{1}{(x+1)^2+2^2}dx = \frac{1}{2}\arctan\frac{x+1}{2}+C.$

d) $\int \frac{1}{x^2+4x+6}dx = \int\frac{1}{(x+2)^2+(\sqrt{2})^2}dx = \frac{1}{\sqrt{2}}\arctan\frac{x+2}{\sqrt{2}}+C.$

Övning 5.23 (Sid. 96)

Lösning

Att memorera: $\int\frac{f'(x)}{f(x)}dx = \ln|f(x)|.$

a) $\frac{x+1}{x^2-2x+2} = \frac{x-1+2}{(x-1)^2+1} = \frac{x-1}{(x-1)^2+1}+\frac{2}{(x-1)^2+1} \Rightarrow \int\frac{x+1}{x^2-2x+2}dx =$

$= \int\frac{x-1}{(x-1)^2+1}dx + 2\int\frac{1}{(x-1)^2+1}dx = \frac{1}{2}\ln(x^2-2x+2)+2\arctan(x-1).$

b) $\frac{x+1}{x^2+4x+5} = \frac{x+2-1}{(x+2)^2+1} = \frac{x+2}{(x+2)^2+1} - \frac{1}{(x+2)^2+1} \Rightarrow \int\frac{x+1}{x^2+4x+5}dx =$

$= \int\frac{x+2}{(x+2)^2+1}dx - \int\frac{1}{(x+2)^2+1}dx = \frac{1}{2}\ln(x^2+4x+5) - \arctan(x+2)+C.$

c) $\frac{x+1}{x^2-2x+5} = \frac{x+1}{(x-1)^2+4} = \frac{x-1}{(x-1)^2+2^2}+\frac{2}{(x-1)^2+2^2} \Rightarrow \int\frac{x+1}{x^2-2x+5}dx =$

$= \int\frac{x+1}{(x-1)^2+2^2}dx + 2\int\frac{1}{(x-1)^2+2^2}dx = \frac{1}{2}\ln(x^2-2x+5)+\arctan\frac{x-1}{2}+C.$

d) $\frac{x+1}{x^2+4x+6} = \frac{x+2-1}{(x+2)^2+2} = \frac{x+2}{x^2+4x+6} - \frac{1}{(x+2)^2+2} \Rightarrow \int\frac{x+1}{x^2+4x+6}dx =$

$= \int\frac{x+2}{x^2+4x+6}dx - \int\frac{1}{(x+2)^2+(\sqrt{2})^2}dx = \frac{1}{2}\ln(x^2+4x+6) - \arctan\frac{x+2}{\sqrt{2}}$

Övning 5.24 (Sid. 96)

Lösning

a) $\frac{x^5-x^4-x^2+2x+1}{x^4-x^3-x+1} = x - \frac{x+1}{x^4-x^3-x+1} = x + \frac{x+1}{(x-1)^2(x^2+x+1)};$

$\frac{x+1}{(x-1)^2(x^2+x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+x+1} = \frac{A(x-1)+B}{(x-1)^2}+\frac{Cx+D}{x^2+x+1}$

$\Leftrightarrow (A(x-1)+B)(x^2+x+1)+(x-1)^2(Cx+D) = x+1;$

(i) $x=1 \Rightarrow 3B = 2 \Leftrightarrow \underline{B=\frac{2}{3}} \Rightarrow A(x-1)(x^2+x+1)+(x-1)^2(Cx+D) =$

$= x+1-\frac{2}{3}(x^2+x+1) = -\frac{2}{3}(-\frac{3}{2}x-\frac{3}{2}+x^2+x+1) = -\frac{2}{3}(x^2-\frac{1}{2}x-\frac{1}{2}) =$

$= -\frac{2}{3}(x-1)(x+\frac{1}{2}) \Leftrightarrow \underline{A(x^2+x+1)+(x-1)(Cx+D) = -\frac{2}{3}(x+\frac{1}{2})};$

(ii) $x=1 \Rightarrow 3A = -1 \Leftrightarrow \underline{A=-\frac{1}{3}} \Rightarrow (x-1)(Cx+D) = \frac{1}{3}(x^2+x+1)-$

$-\frac{2}{3}(x+\frac{1}{2}) = \frac{1}{3}(x^2+x+1-2x-1) = \frac{1}{3}(x^2-x) \Leftrightarrow \underline{Cx+D = +\frac{1}{3}x};$

$\frac{x^5-x^4-x^2+2x+1}{x^4-x^3-x+1} = x - \frac{1}{3}\frac{1}{x-1}+\frac{2}{3}\frac{1}{(x-1)^2}+\frac{1}{6}\frac{2x+1-1}{x^2+x+1} = x-$

$-\frac{1}{3}\frac{1}{x-1}+\frac{2}{3}\frac{1}{(x-1)^2}+\frac{1}{6}\frac{2x+1}{x^2+x+1}-\frac{1}{6}\cdot\frac{1}{(x+1/2)^2+(\sqrt{3}/2)^2} \Rightarrow$

$$\Rightarrow \int \frac{x^5-x^4-x^2+2x+1}{x^4-x^3-x+1}dx = \int\left(x-\frac{1/3}{x-1}+\frac{2/3}{(x-1)^2}+\frac{1}{6}\frac{(x^2+x+1)'}{x^2+x+1}-\right.$$

$$\left.-\frac{1}{6}\frac{1}{(x+1/2)^2+(\sqrt{3}/2)^2}\right)dx = \frac{1}{2}x^2-\frac{1}{3}\ln|x-1|-\frac{2}{3}\frac{1}{x-1}+$$

$$+\frac{1}{6}\ln(x^2+x+1)-\frac{1}{6}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C = \frac{1}{2}x^2-\frac{2}{3}\frac{1}{x-1}+$$

$$+\frac{1}{3}\ln\frac{\sqrt{x^2+x+1}}{|x-1|}-\frac{1}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.$$

Övning 5.25 (Sid. 96)

Lösning

a) $f(x)=\frac{2}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}=\frac{(Ax+B)(x-1)+C(x^2+1)}{(x-1)(x^2+1)} \Leftrightarrow$

$$\Leftrightarrow (Ax+B)(x-1)+C(x^2+1)=2\ ;$$

$x=1 \Rightarrow 2C=2 \Leftrightarrow C=1 \Rightarrow (Ax+B)(x-1)=2-x^2-1=-(x-1)(x+1)$

$\Leftrightarrow Ax+B=-x-1 \quad \Rightarrow f(x)=\frac{-x}{x^2+1}+\frac{1}{x-1}-\frac{1}{x^2+1}$;

$$\int\frac{2}{(x^2+1)(x-1)}dx=\int\left(\frac{1}{x-1}-\frac{x}{x^2+1}-\frac{1}{x^2+1}\right)dx=\ln|x-1|-$$

$$-\frac{1}{2}\ln(x^2+1)-\arctan x+C=\ln\frac{|x-1|}{\sqrt{x^2+1}}-\arctan x+C.$$

b) $\frac{1}{x^4-1}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{2}\left(\frac{1}{x^2-1}-\frac{1}{x^2+1}\right)=\frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)-$

$$-\frac{1}{2}\frac{1}{x^2+1} \Rightarrow \int\frac{1}{x^4-1}dx=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+C.$$

<u>Att memorera</u>: $\alpha\neq\beta \Rightarrow \frac{1}{(x-\alpha)(x-\beta)}=\frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right)$

och allmänt $\frac{1}{(f(x)-\alpha)(f(x)-\beta)}=\frac{1}{\alpha-\beta}\left(\frac{1}{f(x)-\alpha}-\frac{1}{f(x)-\beta}\right)$.

c) $\frac{5x^3+12x^2+12x+10}{(x^2+4)(x^2+2x+1)}=\frac{Ax+B}{x^2+4}+\frac{C}{x+1}+\frac{D}{(x+1)^2}=\frac{Ax+B}{x^2+4}+\frac{C(x+1)+D}{(x+1)^2}$

$\Leftrightarrow \forall x\neq 1: (Ax+B)(x+1)^2+(C(x+1)+D)(x^2+4)=5x^3+12x^2+12x+10$;

(i) $x=-1 \Rightarrow 5D=5 \Rightarrow \underline{D=1} \Rightarrow (Ax+B)(x+1)^2+C(x+1)(x^2+4)=$

$=5x^3+12x^2+12x+10-x^2-4=5x^3+11x^2+12x+6=$

$=(x+1)(5x^2+6x+6) \Leftrightarrow \underline{(Ax+B)(x+1)+C(x^2+4)=5x^2+6x+6.}$

(ii) $x=-1 \Rightarrow 5C=5 \Leftrightarrow \underline{C=1} \Rightarrow (Ax+B)(x+1)=5x^2+6x+6-x^2-4=$

$=4x^2+6x-2=(x+1)(4x-2) \Leftrightarrow \underline{Ax+B=4x-2.}$

$$\int\frac{5x^3+12x^2+12x+10}{(x^2+4)(x+1)^2}dx=2\int\frac{2x}{x^2+4}dx-2\int\frac{1}{x^2+4}dx+\int\frac{1}{x+1}dx+$$

$$+\int\frac{1}{(x+1)^2}dx=2\ln(x^2+4)-\arctan\frac{x}{2}+\ln|x+1|-\frac{1}{x+1}+C.$$

Övning 5.26 (Sid. 96)

Lösning

a) $\frac{x^5+1}{x^4+x^3+x^2}=\frac{x^5+1}{x^2(x^2+x+1)}=x-1+\frac{x^2+1}{x^2(x^2+x+1)}$;

$$\frac{x^2+1}{x^2(x^2+x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+x+1}=\frac{Ax(x^2+x+1)+B(x^2+x+1)+(Cx+D)x^2}{x^2(x^2+x+1)}$$

$$\Leftrightarrow Ax(x^2+x+1)+B(x^2+x+1)+x^2(Cx+D)=x^2+1\ ;$$

(i) $x=0 \Rightarrow \underline{B=1} \Rightarrow Ax(x^2+x+1)+x^2(Cx+D)=-x \Leftrightarrow \underline{A(x^2+x+1)+}$

$\underline{+x(Cx+D)=-1.}$

(ii) $x=0 \Rightarrow \underline{A=-1} \Rightarrow x(Cx+D)=x^2+x \Leftrightarrow \underline{Cx+D=x+1.}$

$$\frac{x^5+1}{x^4+x^3+x^2}=-\frac{1}{x}+\frac{1}{x^2}+\frac{x+1/2}{x^2+x+1}+\frac{1/2}{(x+1/2)^2+(\sqrt{3}/2)^2};$$

$$\int\frac{1}{x^4+x^3+x^2}dx=-\int\frac{1}{x}dx+\int\frac{1}{x^2}dx+\frac{1}{2}\int\frac{(x^2+x+1)'}{x^2+x+1}dx+$$

$$+\frac{1}{2}\int\frac{dx}{(x+1/2)^2+(\sqrt{3}/2)^2}=\ln\frac{\sqrt{x^2+x+1}}{|x|}-\frac{1}{x}+\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.$$

b) $\frac{x^2+8x+4}{x^2+4x+8}=1+\frac{4x-4}{x^2+4x+8}=1+2\frac{2(x+2)}{(x+2)^2+2^2}-\frac{12}{(x+2)^2+2^2}$;

$\int\frac{x^2+8x+4}{x^2+4x+8}dx=\int(1+2\frac{(x^2+4x+8)'}{x^2+4x+8}-\frac{12}{(x+2)^2+2^2})dx=x+2\ln(x^2+4x+8)-$

$-6\arctan\frac{x+2}{2}+C.$

c) $f(x)=\frac{x^3+4}{x^2+x}=x-1+\frac{x+4}{x(x+1)}=x-1+\frac{x+1+3}{x(x+1)}=(x-1)+\frac{1}{x}+\frac{3}{x(x+1)}=$

$=(x-1)+\frac{1}{x}-\frac{3}{x+1}+\frac{3}{x}=(x-1)+\frac{4}{x}-\frac{3}{x+1}\Rightarrow\int\frac{x^3+4}{x^2+x}dx=\frac{(x-1)^2}{2}+$

$+4\ln|x|-3\ln|x+1|+C.$

Övning 5.27 (Sid. 96)

Lösning

a) $x>-1\Rightarrow\int\frac{1+\sqrt{x+1}}{1-\sqrt{x+1}}du\,[u^2=x+1\Rightarrow dx=2u\,du]=\{\int\frac{1+u}{1-u}2u\,du\}$

$=\{\int(-2(u+2)+\frac{4}{1-u})du\}_{u=\sqrt{x+1}}=\{-(u+2)^2-4\ln|u-1|+C\}=$

$=-(\sqrt{x+1}+2)^2-4\ln(\sqrt{x+1}-1)+C.$

b) $x>2\Rightarrow\int\frac{\sqrt{x-2}}{x-1}dx\,[x=u^2+2\Rightarrow dx=2u\,du]=\{\int\frac{u}{u^2+2}2u\,du\}=$

$=\{\int2(1-\frac{1}{u^2+1})du\}_{u=\sqrt{x-2}}=2\sqrt{x-2}-2\arctan\sqrt{x-2}+C.$

c) $u=\sqrt{\frac{x-1}{x+1}}\Rightarrow\frac{x-1}{x+1}=1-\frac{2}{x+1}=u^2\Leftrightarrow x=\frac{1+u^2}{1-u^2}\Rightarrow dx=\frac{4u}{(1-u^2)^2}du$;

$x+1=\frac{2}{1-u^2}\Rightarrow(x+1)^2=\frac{4}{(1-u^2)^2}\Leftrightarrow\frac{1}{(x+1)^2}=\frac{(1-u^2)^2}{4}$;

$\int\frac{3}{(x+1)^2}\sqrt{\frac{x-1}{x+1}}dx\,[u=\sqrt{\frac{x-1}{x+1}}]=\{\int\frac{3}{4}(1-u^2)^2\frac{4u^2}{(1-u^2)^2}du\}_{u=\sqrt{\frac{x-1}{x+1}}}=$

$=\{\int3u^2du\}_{u=\sqrt{\frac{x-1}{x+1}}}=(\frac{x-1}{x+1})^{3/2}+C.$

d) Samma substitution som i c) gäller.

d) $\int(1+\sqrt{\frac{x-1}{x+1}})^2dx=\{\int(1+u)^2\cdot\frac{4u}{(1-u^2)^2}du\}=\{4\int\frac{u}{(1-u)^2}du\}=$

$=\{4\int(\frac{1}{(1-u)^2}-\frac{1}{1-u})du\}_{u=\sqrt{\frac{x-1}{x+1}}}=\{4\ln(1-u)+\frac{4}{1-u}+C\}_{u=\sqrt{\frac{x-1}{x+1}}}$

$=4\ln(1-\sqrt{\frac{x-1}{x+1}})+\frac{4\sqrt{x+1}}{\sqrt{x+1}-\sqrt{x-1}}+C.$

Övning 5.28 (Sid. 96)

Lösning

a) $\int\frac{1}{\sqrt{x^2+1}}dx\,[x=\sinh u\Rightarrow dx=\cosh u\,du]=\{\int du\}_{x=\sinh u}$

$=\{u+C\}_{x=\sinh u}=\sinh^{-1}x=\ln(x+\sqrt{x^2+1})+C.$

Anm. $x=\sinh u\Rightarrow x^2+1=\cosh^2u$; $\frac{d}{du}\sinh u=\cosh u$.

Se Ö. 1.84 c)!

b) $\int\frac{1}{\sqrt{x^2+4x+5}}dx=\int\frac{1}{\sqrt{(x+2)^2+1}}dx=\ln(x+2+\sqrt{x^2+4x+5})+C.$

c) $\int\frac{x}{\sqrt{x^2+1}}dx\,[u=\sqrt{x^2+1}\Rightarrow du=\frac{x}{\sqrt{x^2+1}}dx]=\{\int du\}_{u=\sqrt{x^2+1}}=$

$=\sqrt{x^2+1}+C$

d) $\int\frac{x+1}{\sqrt{x^2+4x+5}}dx=\int(\frac{x+2}{\sqrt{x^2+4x+5}}-\frac{1}{\sqrt{x^2+4x+5}})dx\,[u=x+2]=$

$=\{\int\frac{u}{\sqrt{u^2+1}}-\frac{1}{\sqrt{u^2+1}})du\}_{u=x+2}=\{\sqrt{u^2+1}-\ln(u+\sqrt{u^2+1})+C\}_{u=x+2}$

$=\sqrt{x^2+4x+5}-\ln(x+2+\sqrt{x^2+4x+5})+C$

Övning 5.29 (Sid. 96)

Lösning

a) $\int\sqrt{x^2+1}\,dx=\int1\cdot\sqrt{x^2+1}\,dx=x\sqrt{x^2+1}-\int x\cdot\frac{x}{\sqrt{x^2+1}}dx=x\sqrt{x^2+1}-$

$$-\int \frac{x^2+1-1}{\sqrt{x^2+1}}dx = x\sqrt{x^2+1} - \int\sqrt{x^2+1}\,dx + \int\frac{dx}{\sqrt{x^2+1}} \Leftrightarrow 2\int\sqrt{x^2+1}\,dx =$$
$$= x\sqrt{x^2+1} + \ln(x+\sqrt{x^2+1}) \Leftrightarrow \int\sqrt{x^2+1}\,dx = \frac{x\sqrt{x^2+1}+\ln(x+\sqrt{x^2+1})}{2}+C$$

b) $\int\sqrt{1-x^2}\,dx = \int 1\cdot\sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int x\frac{-x}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} -$
$$-\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} - \int\sqrt{1-x^2}\,dx + \int\frac{1}{\sqrt{1-x^2}}dx \Leftrightarrow 2\int\sqrt{1-x^2}\,dx =$$
$$= x\sqrt{1-x^2} + \arcsin x + 2C \Leftrightarrow \int\sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}+\arcsin x}{2}+C.$$

c) $\int\sqrt{x^2+2x+3}\,dx = \int\sqrt{(x+1)^2+2}\,dx\ [x+1=\sqrt{2}u \Rightarrow dx=\sqrt{2}du] =$
$$=\left\{2\int\sqrt{u^2+1}\,du\right\}_{x+1=\sqrt{2}u} = \left\{u\sqrt{u^2+1}+\ln(u+\sqrt{u^2+1})+C\right\}_{x+1=\sqrt{2}u}$$
$$=\frac{x+1}{\sqrt{2}}\cdot\frac{\sqrt{x^2+2x+3}}{\sqrt{2}} + \ln\left(\frac{x+1+\sqrt{x^2+2x+3}}{\sqrt{2}}\right)+C = \frac{1}{2}(x+1)\sqrt{x^2+2x+3} +$$
$$+\ln(x+1+\sqrt{x^2+2x+3})+C'$$

Övning 5.31 (Sid. 96)

Lösning

$u=\tan\frac{x}{2}$; $\sin x = \frac{2u}{u^2+1}$, $\cos u = \frac{1-u^2}{1+u^2}$, $dx = \frac{2}{u^2+1}du$.

a) $4+5\sin x = 4+\frac{10u}{u^2+1} = \frac{2(2u^2+5u+1)}{u^2+1} = \frac{2(2u+1)(u+2)}{u^2+1} \Leftrightarrow$
$$\Leftrightarrow \frac{3}{4+5\sin x} = \frac{3(u^2+1)}{4(u+1/2)(u+2)} \Rightarrow \frac{3\,dx}{4+5\sin x} = \frac{3}{2}\frac{du}{(u+1/2)(u+2)};$$
$$\int\frac{3}{4+5\sin x}dx\ [u=\tan\tfrac{x}{2}] = \left\{\int\left(\frac{1}{u+1/2}-\frac{1}{u+2}\right)du\right\}_{u=\tan\frac{x}{2}} =$$
$$=\left\{\ln\left|\frac{u+1/2}{u+2}\right|+C\right\}_{u=\tan\frac{x}{2}} = \ln\left|\frac{\tan(x/2)+1/2}{\tan(x/2)+2}\right|+C.$$

b) $u=\tan\frac{x}{2} \Rightarrow 2+\sin x = 2+\frac{2u}{u^2+1} = \frac{2(u^2+u+1)}{u^2+1} \Rightarrow \frac{dx}{2+\sin x} =$
$$=\frac{u^2+1}{2(u^2+u+1)}\cdot\frac{2}{u^2+1}du = \frac{du}{u^2+u+1} = \frac{du}{(u+1/2)^2+(\sqrt{3}/2)^2};$$

$$\int\frac{dx}{2+\sin x}[u=\tan\tfrac{x}{2}] = \left\{\int\frac{du}{u^2+u+1}\right\} = \left\{\int\frac{du}{(u+1/2)^2+(\sqrt{3}/2)^2}\right\} =$$
$$=\left\{\frac{2}{\sqrt{3}}\arctan\frac{u-1/2}{\sqrt{3}/2}+C\right\}_{u=\tan\frac{x}{2}} = \frac{2}{\sqrt{3}}\arctan\frac{2\tan(x/2)+1}{\sqrt{3}}+C.$$

c) $\int\frac{dx}{\sin x} = \left\{\int\frac{u^2+1}{2u}\frac{2}{u^2+1}du\right\}_{u=\tan\frac{x}{2}} = \left\{\int\frac{du}{u}\right\}_{u=\tan\frac{x}{2}} =$
$$=\ln|\tan\tfrac{x}{2}|+C.$$

d) $\int\frac{dx}{\sin^3 x} = \int\frac{\cos^2 x+\sin^2 x}{\sin^3 x}dx = \int\frac{1}{\sin x}dx + \int\frac{\cos^2 x}{\sin^3 x}dx =$
$$=\ln|\tan\tfrac{x}{2}| + \int\left(-\frac{1}{2\sin^2 x}\right)'\cos x\,dx = \ln|\tan\tfrac{x}{2}| -$$
$$-\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\int\frac{(\cos x)'}{\sin^2 x}dx = \ln|\tan\tfrac{x}{2}| - \frac{\cos x}{2\sin^2 x} - \frac{1}{2}\int\frac{1}{\sin x}dx =$$
$$=\ln|\tan\tfrac{x}{2}| - \frac{1}{2}\frac{\cos x}{\sin^2 x} - \frac{1}{2}\ln|\tan\tfrac{x}{2}|+C = \frac{1}{2}\Big(\ln|\tan\tfrac{x}{2}| -$$
$$-\frac{\cos x}{\sin^2 x}\Big)+C.$$

e) $u=\tan\frac{x}{2} \Rightarrow \frac{3+4\cos x}{(1+\cos x)^2} = \frac{(7-u^2)/(u^2+1)}{4/(u^2+1)^2} = \frac{1}{4}(7-u^2)(u^2+1) \Rightarrow$
$$\Rightarrow \int\frac{3+4\cos x}{(1+\cos x)^2}dx = \left\{\int\frac{1}{2}(7-u^2)du\right\} = \left\{\frac{7}{2}u-\frac{1}{6}u^3+C\right\}_{u=\tan\frac{x}{2}} =$$
$$=\frac{7}{2}\tan\frac{x}{2} - \frac{1}{6}\tan^3\frac{x}{2}+C$$

Övning 5.32 (Sid. 96)

Lösning

a) $\int\frac{\sin x}{\sqrt{\cos^2 x+2\cos x+3}}dx = \int\frac{\sin x}{\sqrt{(\cos x+1)^2+2}}dx\ [u=\cos x+1] =$
$$=\left\{\int\frac{-1}{\sqrt{u^2+2}}du\right\}_{u=\cos x+1} = \left\{-\ln(u+\sqrt{u^2+2})+C\right\}_{u=\cos x+1} =$$
$$=-\ln(\cos x+1+\sqrt{\cos^2 x+2\cos x+3})+C.$$

b) $\int \frac{\sin 2x}{\cos^3 x}dx = 2\int \frac{\sin x}{\cos^2 x}dx = 2\int \left(\frac{1}{\cos x}\right)' dx = \frac{2}{\cos x} + C.$

c) $\int \frac{\cos x}{\sin x + \sin^2 x}dx\,[u = \sin x] = \left\{ \int \frac{1}{u+u^2}du \right\} = \left\{ \int \left(\frac{1}{u} - \frac{1}{u+1}\right)du \right\} =$

$= \left\{ \ln\left|\frac{u}{u+1}\right| + C \right\} = \ln\left|\frac{\sin x}{1+\sin x}\right| + C = \ln\frac{|\sin x|}{1+\sin x} + C.$

d) $\int \sin^9 x \cos x\,dx\,[u = \sin x \Rightarrow du = \cos x\,dx] = \left\{ \int u^9 du \right\}_{u=\sin x} =$

$= \frac{1}{10}\sin^{10} x + C.$

e) $\tan^3 x + \tan x = \tan x(\tan^2 x + 1) = \frac{\tan x}{\cos^2 x};$

$\int \frac{\tan^3 x + \tan x}{\tan^3 x + 3\tan^2 x + 2\tan x + 6}dx\left[u = \tan x \Rightarrow du = \frac{dx}{\cos^2 x}\right] =$

$= \left\{ \int \frac{u}{u^3+3u^2+2u+6}du \right\}_{u=\tan x};$

$f(u) = u^3+3u^2+2u+6 = u^2(u+3)+2(u+3) = (u+3)(u^2+2);$

$\frac{u}{u^3+3u^2+2u+6} = \frac{A}{u+3} + \frac{Bu+C}{u^2+2} = \frac{A(u^2+2)+(u+3)(Bu+C)}{(u+3)(u^2+2)} \Leftrightarrow$

$\Leftrightarrow A(u^2+2)+(u+3)(Bu+C) = u;$

$u = -3 \Rightarrow 11a = -3 \Leftrightarrow \underline{A = -3/11} \Rightarrow (u+3)(Bu+C) = u + \frac{3}{11}(u^2+2) =$

$= (u+3)\cdot\frac{3}{11}\left(u+\frac{2}{3}\right) \Leftrightarrow \underline{Bu+C = \frac{1}{11}(3u+2)};$

$\int \frac{u}{u^3+3u^2+2u+6}du = \frac{1}{11}\int\left(\frac{3u}{u^2+2} + \frac{2}{u^2+2} - \frac{3}{u+3}\right)du =$

$= \frac{1}{11}\left(3\ln\sqrt{u^2+2} + \sqrt{2}\arctan\frac{u}{\sqrt{2}} - 3\ln|u+3| + C = \frac{1}{11}\ln\frac{\sqrt{u^2+2}}{|u+3|} + \right.$

$+ \sqrt{2}\arctan\frac{u}{\sqrt{2}} + C = \{u = \tan x\} = \frac{1}{11}\ln\frac{\sqrt{\tan^2 x+2}}{|\tan x + 3|} +$

$+ \sqrt{2}\arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C.$

f) $\int \frac{1}{\sin^2 x + 2\cos^2 x}dx = \int \frac{1}{\tan^2 x + 2}\cdot\frac{dx}{\cos^2 x}\,[u = \tan x] =$

$= \left\{ \int \frac{du}{u^2+2} \right\}_{u=\tan x} = \left\{ \frac{1}{\sqrt{2}}\tan^{-1} u + C \right\} = \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C.$

Övning 5.33 (Sid. 96)

Lösning

a) $\int \sin^5 x\,dx = \int \sin^4 x\cdot\sin x\,dx = \int (\sin^2 x)^2\cdot\sin x\,dx =$

$= \int (1-\cos^2 x)^2 \sin x\,dx \begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix} = \left\{ -\int (1-u^2)^2 du \right\} =$

$= \left\{ \int (2u^2 - 1 - u^4)du \right\}_{u=\cos x} = \left\{ \frac{2}{3}u^3 - u - \frac{1}{5}u^5 + C \right\}_{u=\cos x} =$

$= \frac{2}{3}\cos^3 x - \cos x - \frac{1}{5}\cos^5 x + C.$

b) $\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) =$

$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \Rightarrow$

$\Rightarrow \int \sin^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

c) $\left. \begin{cases} \sin(5x+x) = \sin 6x = \sin 5x\cos x + \cos 5x \sin x \\ \sin(5x-x) = \sin 4x = \sin 5x \cos x - \cos 5x \sin x \end{cases} \right\} \Rightarrow$

$\Rightarrow \sin 6x + \sin 4x = 2\sin 5x \cos x \Rightarrow \int \sin 5x \cos x\,dx =$

$= \frac{1}{2}\int(\sin 6x + \sin 4x)dx = \frac{1}{2}\left(-\frac{\cos 6x}{6} - \frac{\cos 4x}{4}\right) + C$

d) $\int \cos^2 x\,dx = \frac{1}{2}\int(1+\cos 2x)dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$

e) $\sin^4 x\cdot\cos^2 x = \sin^2 x\cdot\frac{\sin^2 2x}{4} = \frac{1-\cos 2x}{2}\cdot\frac{\sin^2 2x}{4} =$

$= \frac{1}{8}(\sin^2 2x - \sin^2 2x\cos 2x) = \frac{1}{16}(1 - \cos 4x - 2\sin^2 2x \cos 2x)$

$\Rightarrow \int \sin^4 x \cos^2 x\,dx = \frac{1}{16}\int(1-\cos 4x)\,dx - \frac{1}{16}\int 2\sin^2 2x\cos 2x\,dx =$

$$= \frac{1}{16}\left(x - \frac{\sin 4x}{4}\right) - \frac{1}{16}\int \left(\frac{\sin^3 2x}{3}\right)' dx = \frac{1}{16}\left(x - \frac{\sin 4x}{4} - \frac{\sin^3 2x}{3}\right) + C.$$

f) Se Ö. 5.15 b).

Övning 5.34 (Sid. 97)

Lösning

$$\frac{2x^2+6}{(x-1)^2(x^2+2x+5)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+5} = \frac{A(x-1)+B}{(x-1)^2} +$$

$$+ \frac{Cx+D}{x^2+2x+5} = \frac{(A(x-1)+B)(x^2+2x+5) + (x-1)^2(Cx+D)}{(x-1)^2(x^2+2x+5)} \Leftrightarrow$$

$$\Leftrightarrow (A(x-1)+B)(x^2+2x+5) + (x-1)(Cx+D) = 2x^2+6 \;;$$

$$x=1 \Rightarrow 8B = 8 \Leftrightarrow \underline{B=1} \Rightarrow A(x-1)(x^2+2x+5) + (x-1)(Cx+D) =$$

$$= 2x^2+6-x^2-2x-5 = x^2-2x+1 = (x-1)^2 \Leftrightarrow A(x^2+2x+5) +$$

$$+ (Cx+D) = x-1 \Leftrightarrow \underline{A=0} \wedge \underline{Cx+D = x-1}.$$

$$\int \frac{2x^2+6}{(x-1)^2(x^2+2x+5)}dx = \int \left(\frac{1}{(x-1)^2} + \frac{x+1}{x^2+2x+5} - \frac{2}{(x-1)^2+2^2}\right)dx =$$

$$= -\frac{1}{x-1} + \frac{1}{2}\ln(x^2+2x+5) - \arctan\frac{x-1}{2} + C.$$

Övning 5.35 (Sid. 97)

Lösning

$$\int \frac{\sqrt{x^2+2}-x}{\sqrt{x^2+2}+x}dx = \int \frac{(\sqrt{x^2+2}-x)(\sqrt{x^2+2}-x)}{(\sqrt{x^2+2}+x)(\sqrt{x^2+2}-x)}dx = \int \frac{(\sqrt{x^2+2}-x)^2}{2}dx =$$

$$= \int (x^2+1-x\sqrt{x^2+2})dx = \frac{x^3}{3}+x-\int x\sqrt{x^2+2}\,dx \begin{bmatrix} u = x^2+2 \\ du = 2x\,dx \end{bmatrix} =$$

$$= \frac{x^3}{3}+x-\left\{\frac{1}{2}\int\sqrt{u}\,du\right\}_{u=x^2+2} = \frac{x^3}{3}+x-\frac{1}{3}(x^2+2)^{3/2}+C.$$

Övning 5.36 (Sid. 97)

Lösning

$$\frac{2x^2-4x+34}{(x^2+2x+5)(x^2+2x-3)} = \frac{2x^2-4x+34}{(x^2+2x+5)(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{Cx+D}{x^2+2x+5} =$$

$$= \frac{A(x+3)(x^2+2x+5) + B(x-1)(x^2+2x+5) + (x-1)(x+3)(Cx+D)}{(x-1)(x+3)(x^2+2x+5)} \Leftrightarrow$$

$$\Leftrightarrow A(x+3)(x^2+2x+5) + B(x-1)(x^2+2x+5) + (x-1)(x+3)(Cx+D) = 2x^2-4x+34;$$

(i) $x=1 \Rightarrow 32A = 32 \Leftrightarrow A=1 \Rightarrow B(x-1)(x^2+2x+5) + (x-1)(x+3)(Cx+D) =$

$$= 2x^2-4x+34-(x+3)(x^2+2x+5) = (x-1)(-x^2-4x-19) \Leftrightarrow$$

$$\Leftrightarrow \underline{B(x^2+2x+5) + (x+3)(Cx+D) = -x^2-4x-19};$$

(ii) $x=-3 \Rightarrow 8B = -16 \Leftrightarrow \underline{B=-2} \Rightarrow (x+3)(Cx+D) = -x^2-4x-19 +$

$$+ 2(x^2+2x+5) = x^2-9 = (x+3)(x-3) \Leftrightarrow \underline{Cx+D = x-3};$$

$$\int \frac{2x^2-4x+34}{(x^2+2x+5)(x^2+2x-3)}dx = \int\left(\frac{1}{x-1} - \frac{2}{x+3} + \frac{x-3}{x^2+2x+5}\right)dx =$$

$$= \ln|x-1| - 2\ln|x+3| + \int \frac{x+1}{x^2+2x+5}dx - 4\int \frac{1}{(x+1)^2+2^2}dx =$$

$$= \ln|x-1| - 2\ln|x+3| + \frac{1}{2}\ln(x^2+2x+5) - 2\arctan\frac{x+1}{2} + C.$$

Övning 5.37 (Sid. 97)

Lösning

$$u = \tan\frac{x}{2},\quad \cos x = \frac{1-u^2}{1+u^2},\quad dx = \frac{2}{1+u^2}du\;;$$

$$\int \frac{3}{4+5\cos x}dx \left[u = \tan\frac{x}{2}\right] = \left\{3\int \frac{1+u^2}{9-u^2}\cdot\frac{2}{u^2+1}du\right\}_{u=\tan(x/2)} =$$

$$= \left\{6\int \frac{1}{(3-u)(3+u)}du\right\}_{u=\tan\frac{x}{2}} = \left\{\int\left(\frac{1}{u+3} - \frac{1}{u-3}\right)du\right\}_{u=\tan\frac{x}{2}} =$$

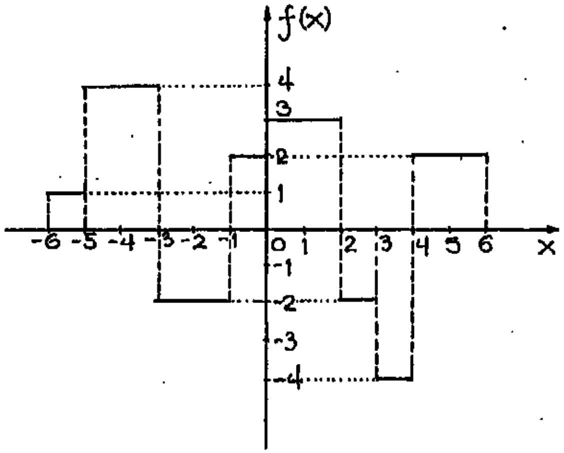
6.

Integralkalkyl

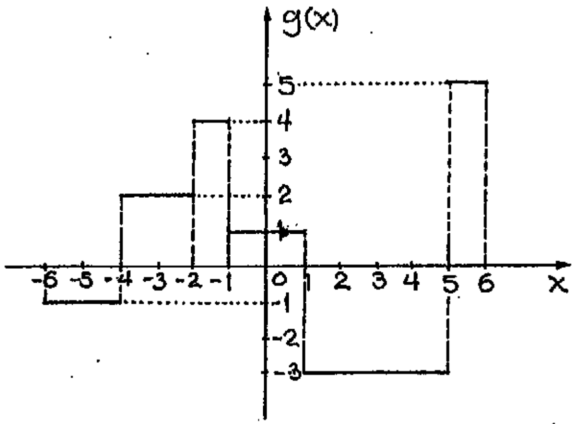
Inledning

Övning 6.1 (Sid. 115)

Lösning



$$f(x)=\begin{cases} 1, & -6\le x\le -5\\ 4, & -5\le x\le -3\\ -2, & -3\le x\le -1\\ 2, & -1\le x< 0\\ 3, & 0\le x< 2\\ -2, & 2\le x< 3\\ -4, & 3\le x< 4\\ 2, & 4\le x\le 6 \end{cases}$$



$$g(x)=\begin{cases} -1, & -6\le x\le -4\\ 2, & -4\le x\le -2\\ 4, & -2\le x\le -1\\ 1, & -1\le x\le 1\\ -3, & 1\le x\le 5\\ 5, & 5\le x\le 6 \end{cases}$$

Jag kommer att referera till (1)–(6), s. 284-286.

a) $I(f)=\int_{-6}^{6} f(x)dx \overset{(6)}{=} \left(\int_{-6}^{-5}+\int_{-5}^{-3}+\int_{-3}^{-1}+\int_{-1}^{0}+\int_{0}^{2}+\int_{2}^{3}+\int_{3}^{4}+\int_{4}^{6}\right) f(x)dx \overset{(\text{def }1)}{=}$

$= 1\cdot(-5-(-6))+4(-3-(-5))+(-2)(-1-(-3))+2\cdot(0-(-1))+3(2-0)+$

$+(-2)\cdot(3-2)+(-4)(4-3)+2(6-4)= 1\cdot 1+4\cdot 2+(-2)\cdot 2+2\cdot 1+3\cdot 2+$

$+(-2)\cdot 1+(-4)\cdot 1+2\cdot 2 = 1+8-4+2+6-2-4+4=\underline{\underline{11}}.$

b) $I(g)=\int_{-6}^{6} g(x)dx \overset{(6)}{=} \left(\int_{-6}^{-4}+\int_{-4}^{-2}+\int_{-2}^{-1}+\int_{-1}^{1}+\int_{1}^{5}+\int_{5}^{6}\right) g(x)dx = (-1)\cdot(-4-(-6))+$

$+2(-2-(-4))+4(-1-(-2))+1\cdot(1-(-1))+(-3)\cdot(5-1)+5(6-5)=$

$=(-1)\cdot 2+2\cdot 2+4\cdot 1+1\cdot 2+(-3)\cdot 4+5\cdot 1=-2+4+4+2-12+5=\underline{\underline{1}}.$

c) $\int_{-6}^{6}(f(x)+g(x))dx = I(f+g) \overset{(4)}{=} I(f)+I(g)=11+1=\underline{\underline{12}}.$

d) $\int_{-6}^{6}(2f(x)-3g(x))dx = I(2f-3g)=2I(f)-3I(g)=2\cdot 11-3\cdot 1=\underline{\underline{19}}.$

e) $\int_{-6}^{6}(f(x))^2dx = I(f^2)=1^2\cdot(-5-(-6))+4^2(-3-(-5))+(-2)^2(-1-(-3))+$

$+2^2(0-(-1))+3^2(2-0)+(-2)^2(3-2)+(-4)^2(4-3)+2^2(6-4)=$

$=1\cdot 1+16\cdot 2+4\cdot 2+4\cdot 1+9\cdot 2+4\cdot 1+16\cdot 1+4\cdot 2=1+32+8+4+18+$

$+4+16+8=\underline{\underline{91}}.$

f) $\int_{-6}^{6}\frac{g(x)}{2+g(x)}dx \overset{(6)}{=} \frac{-1}{2+(-1)}(-4-(-6))+\frac{2}{2+2}(-2-(-4))+$

$+\frac{4}{2+4}(-1-(-2))+\frac{1}{2+1}(1-(-1))+\frac{-3}{2-3}(5-1)+\frac{5}{2+5}(6-5)=$

$=(-1)\cdot 2+\frac{1}{2}\cdot 2+\frac{2}{3}\cdot 1+\frac{1}{3}\cdot 2+3\cdot 4+\frac{5}{7}\cdot 1=\dots=274/21.$

Övning 6.2 (Sid. 115)

Lösning

$I(\Phi) = f(1)\cdot 1 + f(2)\cdot 1 + \cdots + f(10)\cdot 1 = f(1) + f(2) + \cdots + f(10) =$

$= 0{,}3960 + 0{,}3846 + 0{,}3670 + 0{,}3448 + 0{,}32 +$

$+ 0{,}2491 + 0{,}2895 + 0{,}2439 + 0{,}2210 + 0{,}2 = \underline{\underline{3{,}0399}}.$

$I(\Psi) = f(0)\cdot 1 + f(1)\cdot 1 + \cdots + f(9)\cdot 1 = f(0) + f(1) + \cdots + f(9) =$

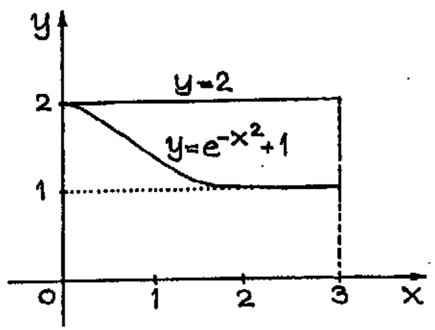
$= 0{,}4 + 2{,}8399 = \underline{\underline{3{,}2399}}.$

$I(\Phi) \le \int_0^{10} f(x)\,dx \le I(\Psi) \Leftrightarrow 3{,}04 \le \cancel{\int_0^{10} f(x)\,dx}^{\;\pi} \le 3{,}24.$

Övning 6.3 (Sid. 116)

Lösning

I samma koordinatsystem uppritas kurvan $y = 1 + e^{-x^2}$ och linjen $y = 2$, för $0 \le x \le 3$.



$y = \Psi(x) = 2$ är en överfunktion till $f(x) = e^{-x^2} + 1$

så (11) i Sats 5 (s. 292) ger

$0 \le x \le 3 \Rightarrow 1 + e^{-x^2} \le 3 \Rightarrow \int_0^3 (1 + e^{-x^2})\,dx \le 2\cdot 3 = 6$ (VSV).

Övning 6.4 (Sid. 116)

Lösning (Se Ö. 6.17a)

$0 \le x \le \frac{1}{2} \Rightarrow \arcsin 0 \le \arcsin x \le \arcsin \frac{1}{2} \Rightarrow$

$\Rightarrow 0 \le \arcsin x \le \frac{\pi}{6} \Leftrightarrow 0 \le (\arcsin x)^2 \le \frac{\pi^2}{36};$

$$\left.\begin{aligned} VL &= \int_0^{1/2} (\arcsin x)^2\,dx = 0{,}0440 > 0 \\ HL &= 1 - \left(\tfrac{\pi}{3}\right)^2 = \frac{9 - \pi^2}{9} = -0{,}0966 < 0 \end{aligned}\right\} \text{Orimligt! } \underline{\underline{\text{OP.}}}$$

Övning 6.5 (Sid. 116)

Lösning

Enligt Sats 7 (integralkalkylens medelvärdessats) på s. 294 i grundboken existerar ξ_n i intervallet $]n, n+1[$, s.a.

$\int_n^{n+1} (1 + \frac{1}{x})^x\,dx = (1 + \frac{1}{\xi_n})^{\xi_n} \cdot (n+1-n) = (1 + \frac{1}{\xi_n})^{\xi_n} \xrightarrow[n\to\infty]{} e.$

Resultat: $\lim_{n\to\infty} \int_n^{n+1} (1 + \frac{1}{x})^x\,dx = e.$

Övning 6.6 (Sid. 116)

Lösning nästa sida.

Lösning

Integralkalkylens medelvärdessats (s. 294) ger

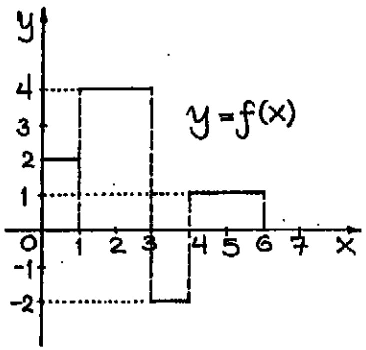
$\int_n^{n+1} x \cdot \sin\frac{1}{x}\,dx = \xi_n \cdot \sin\frac{1}{\xi_n} \cdot (n+1-n) = \xi_n \sin\frac{1}{\xi_n},\ n<\xi_n<n+1.$

$\lim_{n\to\infty} \int_n^{n+1} x \cdot \sin\frac{1}{x}\,dx = \lim_{n\to\infty} \xi_n \cdot \sin\frac{1}{\xi_n} = \lim_{\tau\to 0^+} \frac{\sin\tau}{\tau} = 1.$

Beräkning av integraler

Övning 6.7 (Sid. 116)

Lösning



$$f(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 4, & 1 \leq x < 3 \\ -1, & 3 \leq x < 4 \\ 1, & 4 \leq x \leq 6 \end{cases};$$

$$F(x) = \int f(x)\,dx = \begin{cases} 2x + C_1, & 0 \leq x < 1 \\ 4x + C_2, & 1 \leq x < 3 \\ -2x + C_3, & 3 \leq x < 4 \\ x + C_4, & 4 \leq x \leq 6 \end{cases},$$

f är integrerbar (alla trappfunktioner är det) så S(x) är kontinuerlig i [0, 6]. Det gäller alltså att anpassa konstanterna $C_1 - C_4$.

(i) $S(0) = \int_0^0 f(t)\,dt = 0 = 2 \cdot 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow \underline{S(x) = 2x,\ 0 < x < 1}$

(ii) $\lim_{x\to 1^-} S(x) = \lim_{x\to 1^+} S(x) = S(1) = 2 \Rightarrow 4 \cdot 1 + C_2 = 2 \Leftrightarrow C_2 = -2$

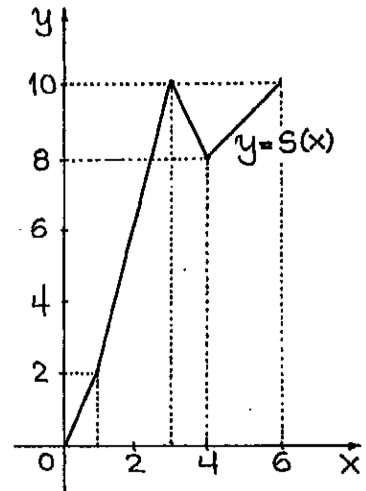
$\Rightarrow \underline{S(x) = 4x - 2,\ 1 < x < 3};$

(iii) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 10 \Rightarrow -6 + C_3 = 10 \Leftrightarrow C_3 = 16$

$\Rightarrow \underline{S(x) = -2x + 16,\ 3 < x < 4}.$

(iv) $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4) \Rightarrow 4 + C_4 = 8 \Leftrightarrow C_4 = 4 \Rightarrow$

$\Rightarrow \underline{S(x) = x + 4,\ 4 < x < 6}.$



$$S(x) = \begin{cases} 2x, & 0 < x < 1 \\ 4x - 2, & 1 < x \leq 3 \\ -2x + 16, & 3 < x \leq 4 \\ x + 4, & 4 < x \leq 6 \end{cases}.$$

På sidan 126 gör författarna processen kort.

Övning 6.8 (Sid. 117)

Lösning

Se nästa sida.

$$f(x)=\begin{cases}2x, & 0\le x<1\\ -x+3, & 1\le x<5\\ -2, & 5\le x\le 6\end{cases} \Rightarrow F(x)=\begin{cases}x^2+C_1, & 0\le x<1\\ -(x-3)^2/2+C_2, & 1\le x<5;\\ -2x+C_3, & 5\le x\le 6\end{cases}$$

$S(x)=\int_0^x f(t)\,dt$ är kontinuerig, ty $f(x)$ är det.

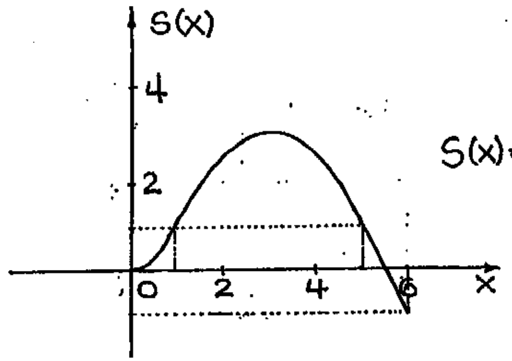
(i) $S(0)=\int_0^0 f(t)\,dt=0=0^2+C_1 \Leftrightarrow C_1=0 \Rightarrow S(x)=x^2,\ 0\le x<1.$

(ii) $\lim_{x\to1^-} S(x)=\lim_{x\to1^+} S(x)=S(1)=1 \Rightarrow -2+C_2=1 \Leftrightarrow C_2=3 \Rightarrow$

$\Rightarrow \underline{S(x)=3-(x-3)^2/2,\ 1\le x<5.}$

(iii) $\lim_{x\to5^-} S(x)=\lim_{x\to5^+} S(x)=S(5) \Rightarrow -10+C_3=1 \Leftrightarrow C_3=11 \Rightarrow$

$\Rightarrow S(x)=-2x+11,\ 5\le x\le 6.$



$$S(x)=\begin{cases}x^2, & 0\le x<1\\ 3-\dfrac{(x-3)^2}{2}, & 1\le x<5.\\ -2x+11, & 5\le x\le 6\end{cases}$$

Övning 6.9 (Sid. 117)

Lösning

$S(x)=\int_0^x f(t)\,dt \Rightarrow S'(x)=f(x),\ 0<x<6.$

x	0		3		6
sgn $S'(x)$		+	0	−	
$S(x)$	$S(0)$	↗	$S(3)$	↘	$S(6)$

Resultat: S blir maximal för $x=3$.

Övning 6.10 (Sid. 117)

Lösning

$f(x)=\int_0^x e^{-t^2}\sin t\,dt \Rightarrow f'(x)=e^{-x^2}\sin x;$

x	0		π		2π
sgn$(f'(x))$		+	0	−	
$f(x)$	0	↗	$f(\pi)$	↘	$f(2\pi)$

Resultat: f antar sitt största värde för $x=\pi$.

Övning 6.11 (Sid. 117)

Lösning

a) $f(x)=\int_1^x \cos t^2\,dt \Rightarrow f'(x)=\cos x^2.$

b) $F(x)=f(\sqrt{x}) \Rightarrow F'(x)=f'(\sqrt{x})\cdot(\sqrt{x})'=\cos x\cdot\frac{1}{2\sqrt{x}}=\frac{\cos x}{2\sqrt{x}}.$

Anm: $\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\,dt=f'(\psi(x))\psi'(x)-f'(\phi(x))\phi'(x).$

Övning 6.12 (Sid. 118)

Lösning

a) $f(u)=\int_1^u \frac{\sin t}{t}\,dt \Rightarrow F(x)=f(\arcsin x)=\int_1^{\arcsin x}\frac{\sin t}{t}\,dt \Rightarrow$

$\Rightarrow F'(x)=f'(\arcsin x)\cdot(\arcsin x)'=f'(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}=$

$$=\frac{\sin(\arcsin x)}{\arcsin x\cdot\sqrt{1-x^2}}=\frac{x}{\sqrt{1-x^2}\cdot\arcsin x},\quad |x|<1.$$

b) $f(u)=\int_{1/2}^{u}\frac{e^{2t}}{t}dt \Rightarrow f'(u)=\frac{e^{2u}}{u}$;

$\mathcal{F}(x)=f(\ln x)\Rightarrow \mathcal{F}'(x)=f'(\ln x)\cdot(\ln x)'=\frac{e^{2\ln x}}{\ln x}\cdot\frac{1}{x}=\frac{e^{\ln x^2}}{x\ln x}=$

$=\frac{x^2}{x\ln x}=\frac{x}{\ln x},\ x>0$

c) $f(u)=\int_{u}^{1}\sqrt{1-t^2}dt=-\int_{1}^{u}\sqrt{1-t^2}dt\Rightarrow f'(u)=-\sqrt{1-u^2}$;

$\mathcal{F}(x)=f(\cos x)\Rightarrow \mathcal{F}'(x)=f'(\cos x)(\cos x)'=-f'(\cos x)\sin x=$

$=+\sqrt{1-\cos^2 x}\cdot\sin x=+\sqrt{\sin^2 x}\cdot\sin x=\sin^2 x,\ 0<x<\pi.$

d) Se Anm i övning 6.11. och följande!

$f(u)=\int_{1/2}^{u}\sqrt{1-t^2}dt\Rightarrow f'(u)=\sqrt{1-u^2}$;

$\mathcal{F}(x)=f(\sin x)-f(\cos x)$, ty $\int_{\cos x}^{\sin x}\sqrt{1-t^2}dt=\int_{1/2}^{\sin x}\sqrt{1-t^2}dt+$

$+\int_{\cos x}^{1/2}\sqrt{1-t^2}dt=\int_{1/2}^{\sin x}\sqrt{1-t^2}dt-\int_{1/2}^{\cos x}\sqrt{1-t^2}dt=f(\sin x)-f(\cos x)$;

$\mathcal{F}'(x)=f'(\sin x)\cos x-f'(\sin x)(-\sin x)=\sqrt{1-\sin^2 x}\cos x+$

$+\sqrt{1-\cos^2 x}\cdot\cos x=\sqrt{\cos^2 x}\cdot\cos x+\sqrt{\sin^2 x}\cdot\sin x=$

$=|\cos x|\cos x+|\sin x|\sin x=(0<x<\frac{\pi}{2})=\cos^2 x+\sin^2 x=1.$

Övning 6.13 (Sid. 118)

Lösning

Jag studerar funktionen

$$f(x)=\int_1^x\frac{\sin t}{t}-x+1,\quad x>1.$$

$f'(x)=\frac{\sin x}{x}-1<0$, ty $\left|\frac{\sin x}{x}\right|\leq\frac{1}{x}<1$; för $x>1$.

f är avtagande, varav följer att $f(x)<f(1)=0$,

dvs. $\int_1^x\frac{\sin t}{t}dx-x+1<0$

Övning 6.14 (Sid. 118)

Lösning

$$\frac{x+1}{x^2+5x+6}=\frac{x+1}{(x+2)(x+3)}=\frac{A}{x+2}+\frac{B}{x+3}=\frac{A(x+3)+B(x+2)}{(x+2)(x+3)}=$$

$$=\frac{(A+B)x+3A+2B}{x^2+5x+6}\Leftrightarrow\begin{cases}A+B=1\\3A+2B=1\end{cases}\Leftrightarrow\begin{cases}A=-1\\B=2\end{cases}\quad(x\neq-2,-3).$$

$$\int_0^1\frac{x+1}{x^2+5x+6}dx=\int_0^1\left(\frac{2}{x+3}-\frac{1}{x+2}\right)dx=\left[2\ln(x+3)-\ln(x+2)\right]_0^1=$$

$$=2\ln 4-\ln 3-2\ln 3+\ln 2=5\ln 2-3\ln 3\approx 0{,}170.$$

Övning 6.15 (Sid. 118)

Lösning

$\int f(x)g(x)dx=\mathcal{F}(x)g(x)-\int\mathcal{F}(x)g'(x)dx$ (Sid. 252).

a) $\int_0^1\ln(1+x^2)dx=\int_0^1 1\cdot\ln(1+x^2)dx=\left[x\cdot\ln(1+x^2)\right]_0^1-$

$-\int_0^1 x\cdot\frac{2x}{x^2+1}dx=\ln 2-2\int_0^1\frac{x^2}{x^2+1}dx=\ln 2-2\int_0^1\left(1-\frac{1}{x^2+1}\right)dx=$

$= \ln 2 - 2\left[x - \arctan x\right]_0^1 = \ln 2 - 2(1 - \arctan 1) - \ln 2 -$

$- 2 + 2\cdot\frac{\pi}{4} = \ln 2 + \frac{\pi}{2} - 2 \approx 0{,}264.$

b) $\int_0^1 e^x \ln(1+e^x)\,dx \overset{!}{=} \left[(e^x+1)\ln(e^x+1)\right]_0^1 - \int_0^1 (e^x+1)\cdot\frac{e^x}{e^x+1}dx =$

$= (e+1)\ln(e+1) - 2\ln 2 - \int_0^1 e^x dx = (e+1)\ln(e+1) - 2\ln 2 -$

$- \left[e^x\right]_0^1 = (e+1)\ln(e+1) - 2\ln 2 - e + 1 \approx 1{,}779.$

Anm $\overset{!}{=}$ underförstås $f(x) = e^x \Rightarrow F(x) = e^x + 1$;

F är (faktiskt) en primitiv till e^x.

c) $\int_1^e (\ln x)^2 dx = \int_1^e 1\cdot(\ln x)^2 dx = \left[x\ln^2 x\right]_1^e - 2\int_1^e \ln x\,dx =$

$= e - 2\left[x\ln x - x\right]_1^e = e - 2(e - e + 1) = e - 2 \approx 0{,}718.$

Att memorera: $\int \ln x\,dx = x\ln x - x$ (Sid. 253).

d) $\int_0^{\pi/4} \frac{x}{\cos^2 x}dx = \int_0^{\pi/4} (\tan x)' x\,dx = \left[x\cdot\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\pi/4} \tan x\,dx =$

$= \frac{\pi}{4} + \int_0^{\pi/4}\left(-\frac{\sin x}{\cos x}\right)dx = \frac{\pi}{4} + \left[\ln\cos x\right]_0^{\pi/4} = \frac{\pi}{4} + \ln\cos\frac{\pi}{4} =$

$= \frac{\pi}{4} + \ln 2^{-1/2} = \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi - 2\ln 2}{4} \approx 0{,}439.$

Övning 6.16 (Sid. 118)

Lösning

Att memorera: $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a}.$

a) $\int_0^{10} \frac{40}{x^2+10^2}dx = \left[4\arctan\frac{x}{10}\right]_0^{10} = 4\arctan 1 = 4\cdot\frac{\pi}{4} = \pi.$

Anm. Man kan även utnyttja substitutionen $x = 10t$; låt oss göra det, då hela övningen handlar om just substitution.

$\int_0^{10} \frac{40}{x^2+100}dx \left[\begin{array}{l|l} x = 10t & x = 10 \Rightarrow t = 1 \\ dx = 10dt & x = 0 \Rightarrow t = 0 \end{array}\right] = 4\int_0^1 \frac{dt}{t^2+1} = \pi.$

b) $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x}+1}{2e^x}$ (Se sid. 121 i boken)

$\int_{-1}^1 \frac{dx}{\cosh x} = \int_{-1}^1 \frac{2e^x}{e^{2x}+1}dx \left[\begin{array}{l|l} t = e^x & x = 1 \Rightarrow t = e \\ dt = e^x dx & x = -1 \Rightarrow t = e^{-1} \end{array}\right] =$

$= \int_{1/e}^e \frac{2}{t^2+1}dt = \left[2\arctan t\right]_{e^{-1}}^e = 2(\arctan e -$

$- \arctan\frac{1}{e}) = 2\arctan\frac{e - e^{-1}}{2} = \underline{2\arctan(\sinh 1)}.$

Anm. $\arctan x - \arctan y = \arctan\frac{x-y}{1+xy} + n\pi.$

c) $\int_0^1 \cos x^{1/3} dx \left[\begin{array}{l|l} x = t^3 & x = 1 \Rightarrow t = 1 \\ dx = 3t^2 dt & x = 0 \Rightarrow t = 0 \end{array}\right] = 3\int_0^1 t^2 \cos t\,dt =$

$= 3\int_0^1 (\cos t)\,t^2 dt = \left[3t^2\sin t\right]_0^1 - 3\int_0^1 (\sin t)\cdot(t^2)' dt =$

$= 3\sin 1 + 6\int_0^1 (-\sin t)\,t\,dt = 3\sin 1 + \left[+6t\cos t\right]_0^1 -$

$- 6\int_0^1 \cos t\,dt = 3\sin 1 + 6\cos 1 - \left[6\sin t\right]_0^1 = 3\sin t +$

$+ 6\cos 1 - 6\sin 1 = \underline{6\cos 1 - 3\sin 1} \approx 0{,}717.$

Anm $D^{-1}f = \int f(x)dx$ (utan konstant).

$D^{-1}(f\cdot g) = (D^{-1}f)g - (D^{-2}f)Dg + (D^{-3}f)D^2 g - \cdots$

d) $\sin 2x = 2\sin x\cos x$ *bör vara bekant.*

$$\int_0^{\pi/2}\sin 2x\, e^{\sin x}dx = 2\int_0^{\pi/2}\sin x\cdot e^{\sin x}\cos x\,dx\left[\begin{array}{l} t=\sin x\\ dt=\cos x\,dx\end{array}\right] =$$

$$= 2\int_0^1 te^t dt = 2\left[te^t\right]_0^1 - 2\int_0^1 e^t dt = 2e-2(e-1) = 2.$$

Anm. $Dxe^x = (x+1)e^x$ men $D^{-1}xe^x = (x-1)e^x$.

Övning 6.17 (Sid. 118)

Lösning

a) $$\int_0^{1/2}(\arcsin x)^2dx\left[\begin{array}{l|l} t=\arcsin x & x=\frac{1}{2}\Rightarrow t=\frac{\pi}{6}\\ x=\sin t & \\ dx=\cos t\,dt & x=0\Rightarrow t=0\end{array}\right] =$$

$$= \int_0^{\pi/6}(\cos t)t^2dt = \left[t^2\sin t\right]_0^{\pi/6} - \int_0^{\pi/6}(\sin t)\cdot 2t\,dt =$$

$$= \left(\frac{\pi}{6}\right)^2\cdot\sin\frac{\pi}{6} + \int_0^{\pi/6}(-\sin t)2t\,dt = \frac{\pi^2}{72} + \left[2t\cos t\right]_0^{\pi/6} -$$

$$-2\int_0^{\pi/6}\cos t\,dt = \frac{\pi^2}{72} + 2\cdot\frac{\pi}{6}\cos\frac{\pi}{6} - 2\left[\sin t\right]_0^{\pi/6} = \frac{\pi^2}{72} +$$

$$+\frac{\pi\sqrt{3}}{6} - 2\sin\frac{\pi}{6} = \underline{\frac{\pi^2}{72} + \frac{\pi\sqrt{3}}{6} - 1} \approx 0{,}044.$$

b) $$\int_0^1\frac{1}{e^x - e^{-2x}+2}dx = \int_0^1\frac{e^{2x}}{e^{3x}+2e^{2x}-1}dx\left[\begin{array}{l|l} t=e^x & 1\to e\\ dt=e^x dx & 0\to 1\end{array}\right] =$$

$$= \int_1^e\frac{t}{t^3+2t^2-1}dt = \int_1^e\frac{t}{(t+1)(t^2+t-1)}dt\,;$$

$$\frac{t}{t^3+2t^2-1} = \frac{t}{(t+1)(t^2+t-1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+t-1} = \frac{A(t^2+t-1)}{t^3+2t^2-1} +$$

$$+\frac{(t+1)(Bt+C)}{t^3+2t^2-1} = \frac{A(t^2+t-1)+(t+1)(Bt+C)}{t^3+2t^2-1} \Leftrightarrow \quad (\textit{forts.})$$

$$\Leftrightarrow A(t^2+t-1)+(t+1)(Bt+C) = t\,; \quad (*)$$

$$t=-1 \Rightarrow -A=-1 \Leftrightarrow A=1 \Rightarrow (t+1)(Bt+C) = t-(t^2+t-1) =$$

$$= -(t^2-1) = -(t+1)(t-1) \Leftrightarrow \underline{Bt+C = -(t-1)}\,;$$

$$\therefore \frac{t}{t^3+2t^2-1} = \frac{1}{t+1} - \frac{t-1}{t^2+t-1} = \frac{1}{t+1} - \frac{1}{2}\,\frac{2t+1-3}{t^2+t-1} = \frac{1}{t+1} -$$

$$-\frac{1}{2}\,\frac{2t+1}{t^2+t-1} + \frac{3}{2}\,\frac{1}{t^2+t-1} = \frac{1}{t+1} - \frac{1}{2}\,\frac{2t+1}{t^2+t-1} + \frac{3}{2}\,\frac{1}{(t-\alpha)(t-\beta)}\,;$$

Att memorera: $\dfrac{1}{(u-a)(u-b)} = \dfrac{1}{a-b}\left(\dfrac{1}{t-a} - \dfrac{1}{t-b}\right),\ a\neq b.$

$$t^2+t-1 = \left(t+\tfrac{1}{2}\right)^2 - \left(\tfrac{\sqrt{5}}{2}\right)^2 = \left(t+\frac{1-\sqrt{5}}{2}\right)\left(t+\frac{1+\sqrt{5}}{2}\right)\,;\ \alpha-\beta = \sqrt{5}\,;$$

$$\int_0^1\frac{1}{e^x-e^{-2x}+2}dx = \int_1^e\left(\frac{1}{t+1} - \frac{1}{2}\,\frac{2t+1}{t^2+t-1} + \frac{3}{2\sqrt{5}}\left(\frac{1}{t-\alpha} - \frac{1}{t-\beta}\right)\right)dt =$$

$$= \left[\ln(t+1) - \tfrac{1}{2}\ln(t^2+t-1) + \frac{3}{2\sqrt{5}}\ln\frac{t-\alpha}{t-\beta}\right]_1^e = \ln(e+1) -$$

$$-\frac{1}{2}\ln(e^2+e-1) + \frac{3}{2\sqrt{5}}\ln\frac{e-\alpha}{e-\beta} - \ln 2 + \frac{1}{2}\ln 1 - \frac{3}{2\sqrt{5}}\ln\frac{1-\alpha}{1-\beta} =$$

$$= \ln\frac{e+1}{2} - \frac{1}{2}\ln(e^2+e-1) + \frac{3}{2\sqrt{5}}\left(\ln\frac{2e+1-\sqrt{5}}{2e+1+\sqrt{5}} - \ln\frac{3-\sqrt{5}}{3+\sqrt{5}}\right).$$

c) $$\int_0^{\pi/4}\frac{\sin^3x}{\cos^5x}dx = \int_0^{\pi/4}(\tan x)^3\frac{dx}{\cos^2x}\left[\begin{array}{l|l} t=\tan x & \frac{\pi}{4}\to 1\\ dt=\frac{dx}{\cos^2x} & 0\to 0\end{array}\right] =$$

$$= \int_0^1 t^3dt = \left[\tfrac{1}{4}t^4\right]_0^1 = \frac{1}{4}.$$

Övning 6.18 (Sid. 118)

Lösning

a) $$\int\frac{x}{(1+x^2)^2}dx\left[t=x^2+1 \Rightarrow \tfrac{1}{2}dt = x\,dx\right] = \frac{1}{2}\int\frac{dt}{t^2} = -\frac{1/2}{x^2+1}\,;$$

$$\int_1^2 \frac{x\ln x}{(x^2+1)^2}dx = \left[-\frac{\ln x}{2(x^2+1)}\right]_1^2 + \frac{1}{2}\int_1^2 \frac{1}{x(x^2+1)}dx = -\frac{\ln 2}{10} +$$
$$+\frac{1}{2}\int_1^2\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx = -\frac{\ln 2}{10}+\frac{1}{2}\left[\ln\frac{x}{\sqrt{x^2+1}}\right]_1^2 = -\frac{\ln 2}{10}+$$
$$+\frac{1}{2}\left(\ln\frac{2}{\sqrt{5}}-\ln\frac{1}{\sqrt{2}}\right) = -\frac{\ln 2}{10}+\frac{1}{2}\ln\left(\frac{8}{5}\right)^{1/2} = \frac{1}{4}\ln\frac{8}{5}-\frac{\ln 2}{10} =$$
$$=\frac{3}{4}\ln 2-\frac{1}{4}\ln 5-\frac{1}{10}\ln 2 = \frac{13}{20}\ln 2-\frac{1}{4}\ln 5.$$

b) $$\int_0^{\pi/4}\frac{1-\tan x}{1+\tan x}dx = \int_0^{\pi/4}\frac{\cos x-\sin x}{\sin x+\cos x}dx = \Big[\ln(\sin x+\cos x)\Big]_0^{\frac{\pi}{2}} =$$
$$=\ln\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)-\ln\cos 1 = \ln\sqrt{2} = \frac{1}{2}\ln 2.$$

c) $$\int x\sqrt{1-x^2}\,dx\,\left[u=1-x^2 \Rightarrow -\frac{1}{2}du = x\,dx\right] = -\frac{1}{2}\int u^{1/2}du =$$
$$=-\frac{1}{2}\cdot\frac{2}{3}u^{3/2} = -\frac{1}{3}u^{3/2} = -\frac{1}{3}(1-x^2)^{3/2};$$
$$\int_0^1 x\sqrt{1-x^2}\arcsin x\,dx = \left[-\frac{1}{3}(1-x^2)^{3/2}\arcsin x\right]_0^1 +$$
$$+\frac{1}{3}\int_0^1(1-x^2)^{3/2}\cdot\frac{1}{\sqrt{1-x^2}}dx = \frac{1}{3}\int_0^1(1-x^2)dx = \frac{1}{3}\left[x-\frac{x^3}{3}\right]_0^1 = \frac{2}{9}.$$

Övning 6.19 (Sid. 118)

Lösning

a) $$\left|\frac{1}{2}-x\right|\cos x = \left|x-\frac{1}{2}\right|\cos x = \begin{cases} -(x-\frac{1}{2})\cos x, & -\frac{1}{2}\le x<\frac{1}{2} \\ (x-\frac{1}{2})\cos x, & \frac{1}{2}\le x\le 1 \end{cases} \Rightarrow$$
$$\Rightarrow \int_{-1/2}^1\left|\frac{1}{2}-x\right|\cos x\,dx = \int_{-1/2}^{1/2}\left(\frac{1}{2}-x\right)\cos x\,dx+\int_{1/2}^1\left(x-\frac{1}{2}\right)\cos x\,dx =$$
$$=\left[\left(\frac{1}{2}-x\right)\sin x\right]_{-1/2}^{1/2}+\int_{-1/2}^{1/2}\sin x\,dx+\left[\left(x-\frac{1}{2}\right)\sin x\right]_{1/2}^1 -$$
$$-\int_{1/2}^1\sin x\,dx = \sin\frac{1}{2}+\frac{1}{2}\sin 1+\cos 1-\cos\frac{1}{2} \approx 0{,}563.$$

b) $$0\le x\le 2\pi \Rightarrow |\sin x| = \begin{cases}\sin x, & 0\le x\le\pi \\ -\sin x, & \pi\le x\le 2\pi\end{cases} \Rightarrow \int_0^{2\pi}e^{-x}|\sin x|dx =$$
$$=\left(\int_0^\pi+\int_\pi^{2\pi}\right)e^{-x}|\sin x|dx = \int_0^\pi e^{-x}\sin x\,dx-\int_\pi^{2\pi}e^{-x}\sin x\,dx;$$
$$\int e^{-x}\sin x\,dx = \int e^{-x}\,\mathrm{Im}\,e^{ix}dx = \int \mathrm{Im}\{e^{(-1+i)x}\}dx =$$
$$=\mathrm{Im}\left\{\int e^{-(1-i)x}dx\right\} = \mathrm{Im}\left\{-\frac{1}{1-i}e^{-(1-i)x}\right\} = -e^{-x}\,\mathrm{Im}\left\{\frac{e^{ix}}{1-i}\right\} =$$
$$=-e^{-x}\,\mathrm{Im}\left\{\frac{1}{2}(1+i)(\cos x+i\sin x)\right\} = -e^{-x}\left(\frac{1}{2}\cos x+\frac{1}{2}\sin x\right);$$
$$\therefore \int_0^{2\pi}e^{-x}|\sin x|dx = \left[-\frac{e^{-x}}{2}(\sin x+\cos x)\right]_0^\pi+\left[\frac{e^{-x}}{2}(\sin x+\cos x)\right]_\pi^{2\pi} =$$
$$=-\frac{1}{2}(-e^{-\pi}-1)+\frac{1}{2}(e^{-2\pi}+e^{-\pi}) = \frac{1}{2}(e^{-2\pi}+2e^{-\pi}+1) = \frac{1}{2}(1+e^{-\pi})^2.$$

Övning 6.20 (Sid. 118)

Lösning

a) Se vad författarna föreslår på sidan 128.

b) $$\frac{\cos x}{1+\cos x} = \frac{2\cos^2\frac{x}{2}-1}{2\cos^2\frac{x}{2}} = 1-\frac{1}{2\cos^2\frac{x}{2}} \Rightarrow \int_0^{\pi/2}\frac{\cos x}{1+\cos x}dx =$$
$$=\int_0^{\pi/2}\left(1-\frac{1/2}{\cos^2\frac{x}{2}}\right)dx = \frac{\pi}{2}-\int_0^{\pi/2}\frac{1}{\cos^2\frac{x}{2}}d\left(\frac{x}{2}\right) = \frac{\pi}{2}-\tan\frac{\pi}{4} = \frac{\pi}{2}-1.$$

Anm. $\int f(u)\,du = F(u) \Rightarrow \int f(g(x))\,dg(x) = F(g(x))$.

$dg(x) = g'(x)dx$ (differentialen).

c) $$\tan^3x+\tan^4x = \tan^2x(\tan x+\tan^2x) = \frac{\sin^2x}{\cos^2x}(\tan x+$$
$$+\tan^2x) = \frac{1-\cos^2x}{\cos^2x}(\tan x+\tan^2x) = \frac{\tan x+\tan^2x}{\cos^2x} -$$

$$-\left(\frac{\sin x}{\cos x}+\frac{1}{\cos^2 x}-1\right) \Rightarrow \int_0^{\pi/4}(\tan^3 x+\tan^4 x)\,dx =$$

$$= \int_0^{\pi/4}(\tan x+\tan^2 x)\,d(\tan x) + \int_0^{\pi/4}\left(-\frac{\sin x}{\cos x}+1-\frac{1}{\cos^2 x}\right)dx =$$

$$= \left[\frac{1}{2}\tan^2 x+\frac{1}{3}\tan^3 x\right]_0^{\pi/4} + \left[\ln\cos x + x - \tan x\right]_0^{\pi/4} =$$

$$= \frac{1}{2}+\frac{1}{3}+\ln\frac{1}{\sqrt{2}}+\frac{\pi}{4}-1 = \frac{\pi}{4}-\frac{1}{6}-\frac{1}{2}\ln 2 \approx 0{,}272.$$

Övning 6.21 (Sid. 118)

Lösning

a) $$\int_{-1}^{3}\frac{x+3}{\sqrt{x^2+2x+10}}dx = \int_{-1}^{3}\frac{(x+1)+2}{\sqrt{(x+1)^2+3^2}}dx \left[\begin{array}{l|l} x+1=3u & 3\to 4/3 \\ dx=3du & -1\to 0\end{array}\right] =$$

$$= \int_0^{4/3}\frac{u+2}{\sqrt{9u^2+9}}3\,du = \int_0^{4/3}\left(\frac{u}{\sqrt{u^2+1}}+\frac{2}{\sqrt{u^2+1}}\right)du = \Big[\sqrt{u^2+1} +$$

$$+ 2\ln(u+\sqrt{u^2+1})\Big]_0^{\frac{4}{3}} = \frac{5}{3}+2\ln\left(\frac{4}{3}+\frac{5}{3}\right)-1 = \frac{2}{3}+2\ln 3.$$

b) $$\int_0^{\pi}\frac{\sin x\,dx}{\sqrt{\cos^2 x+2\cos x+3}} = \int_0^{\pi}\frac{\sin x\,dx}{\sqrt{(1+\cos x)^2+2}} \left[\begin{array}{l} t=1+\cos x \\ dt=-\sin x\,dx \\ \pi\to 0;\ 0\to 2\end{array}\right] =$$

$$= \int_2^0\frac{-1}{\sqrt{t^2+2}}dt = \int_0^2\frac{dt}{\sqrt{t^2+2}} = \left[\ln(t+\sqrt{t^2+2})\right]_0^2 = \ln\frac{2+\sqrt{6}}{\sqrt{2}}.$$

Övning 6.22 (Sid. 119)

Lösning Se sidan 128.

Övning 6.23 (Sid. 119)

Lösning

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\cdot\ln\left(1+\frac{k}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}\ln\left(1+\frac{k}{n}\right)\cdot\frac{1}{n} =$$

$$= \int_0^1 x\cdot\ln(1+x)\,dx = \left[\frac{1}{2}x^2\ln(1+x)\right]_0^1 - \frac{1}{2}\int_0^1\frac{x^2}{x+1}dx =$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\int_0^1\left(x-1+\frac{1}{x+1}\right)dx = \frac{1}{2}\ln 2 - \frac{1}{2}\Big[\frac{1}{2}(x-1)^2 +$$

$$+ \ln(1+x)\Big]_0^1 = \frac{1}{2}\ln 2 - \frac{1}{2}\left(0+\ln 2-\frac{1}{2}\right) = \frac{1}{4}.$$

Generaliserade integraler

Övning 6.24 (Sid. 119)

Lösning Löst på sidan 129.

Övning 6.25 (Sid. 119)

Lösning

a) $$\int_0^{\infty}\frac{dx}{x^2+1} = \lim_{R\to\infty}\int_0^R\frac{dx}{x^2+1} = \lim_{R\to\infty}[\arctan x]_0^R = \lim_{R\to\infty}\arctan R -$$

$$-\arctan 0 = \frac{\pi}{2}-0 = \frac{\pi}{2}.$$

b) $$\int_0^{\infty}\frac{x}{x^2+1}dx = \lim_{R\to\infty}\int_0^R\frac{x}{x^2+1}dx = \lim_{R\to\infty}\left[\ln\sqrt{x^2+1}\right]_0^R =$$

$$= \lim_{R\to\infty}\ln\sqrt{R^2+1} - \ln 1 = \infty;$$ integralen divergent.

c) $$\int_0^{\infty}xe^{-x}dx = \lim_{R\to\infty}\int_0^R xe^{-x}dx = \lim_{R\to\infty}\left[-(x+1)e^{-x}\right]_0^R =$$

$$= -\lim_{R\to\infty}(R+1)e^{-R}+1 = 0+1 = 1.$$

d) $$\int\frac{\ln(2x-1)}{x^2}dx = \int\frac{1}{x^2}\ln(2x-1)\,dx = -\frac{1}{x}\ln(2x-1) +$$

$$+ \int\frac{1}{x}\cdot\frac{2}{2x-1}dx = -\frac{1}{x}\ln(2x-1) + \int\left(\frac{2}{2x-1}-\frac{1}{x}\right)dx = -\frac{\ln(2x-1)}{x} -$$

$$+\ln(2x-1)-\ln x+C=\ln\left(2-\frac{1}{x}\right)-\frac{1}{x}\ln(2x-1)+C \Rightarrow$$

$$\int_1^{\infty}\frac{\ln(2x-1)}{x^2}dx=\lim_{R\to\infty}\left(\ln\left(2-\frac{1}{R}\right)-\frac{1}{R}\ln(2R-1)\right)=\ln 2.$$

e) $\alpha\neq\beta \Rightarrow \dfrac{1}{(x-\alpha)(x-\beta)}=\dfrac{1}{\alpha-\beta}\left(\dfrac{1}{x-\alpha}-\dfrac{1}{x-\beta}\right)$ (att memorera).

$$\frac{x}{x^4-1}=\frac{x}{(x^2-1)(x^2+1)}=\frac{1}{2}\left(\frac{x}{x^2-1}-\frac{x}{x^2+1}\right);$$

$$\int_2^{\infty}\frac{x}{x^4-1}dx=\frac{1}{2}\lim_{R\to\infty}\int_2^R\left(\frac{x}{x^2-1}-\frac{x}{x^2+1}\right)dx=\frac{1}{2}\lim_{R\to\infty}\left[\ln\sqrt{\frac{x^2-1}{x^2+1}}\right]_2^R=$$

$$=\frac{1}{2}\lim_{R\to\infty}\frac{1}{2}\ln\frac{R^2-1}{R^2+1}-\frac{1}{4}\ln\frac{3}{5}=0+\frac{1}{4}\ln\frac{5}{3}=\frac{1}{4}\ln\frac{5}{3}.$$

Övning 6.26 (Sid. 119)

Lösning

a) $\displaystyle\int_0^1\frac{\ln x}{\sqrt{x}}dx=\lim_{\varepsilon\to0^+}\int_\varepsilon^1\frac{1}{\sqrt{x}}\ln x\,dx=\lim_{\varepsilon\to0^+}\left[2x^{1/2}\ln x\right]_\varepsilon^1-$

$$-\lim_{\varepsilon\to0^+}2\int_\varepsilon^1\frac{1}{\sqrt{x}}dx=-2\lim_{\varepsilon\to0^+}\varepsilon^{1/2}\ln\varepsilon-\lim_{\varepsilon\to0}\left[4\sqrt{x}\right]_\varepsilon^1=$$

$=0-4+4\lim\limits_{\varepsilon\to0^+}\sqrt{\varepsilon}=-4.$ (Se (9) sidan 140).

b) $\displaystyle\int_{1/2}^1\frac{dx}{x\ln x}=\lim_{\varepsilon\to0^+}\int_{1/2}^{1-\varepsilon}\frac{dx}{x\ln x}=\lim_{\varepsilon\to0^+}\left[\ln|\ln x|\right]_{1/2}^{1-\varepsilon}=$

$=\lim\limits_{\varepsilon\to0^+}\ln|\ln(1-\varepsilon)|-\ln\left|\ln\frac{1}{2}\right|=-\infty$; divergent.

Övning 6.27 (Sid. 119)

Lösning

a) $\displaystyle\int_0^1\ln x\,dx=\lim_{\varepsilon\to0^+}\int_\varepsilon^1\ln x\,dx=\lim_{\varepsilon\to0^+}\left[x\ln x-x\right]_\varepsilon^1=0-1$

$-\lim\limits_{\varepsilon\to0^+}(\varepsilon\ln\varepsilon-\varepsilon)=0-1-0=-1.$

Anm. $\lim\limits_{\varepsilon\to0^+}\varepsilon\ln\varepsilon=0$, enl. (25) på sid. 155.

b) $\displaystyle\int_1^2\frac{dx}{x^2-1}=\lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^2\frac{1}{x^2-1}dx=\frac{1}{2}\lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^2\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=$

$$=\frac{1}{2}\lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^2\frac{dx}{x-1}-\frac{1}{2}\int_1^2\frac{dx}{x+1}=\frac{1}{2}\lim_{\varepsilon\to0^+}\left[\ln(x-1)\right]_{1+\varepsilon}^2-$$

$$-\left[\frac{1}{2}\ln(1+x)\right]_1^2=\frac{1}{2}\ln 1-\frac{1}{2}\lim_{\varepsilon\to0^+}\varepsilon-\frac{1}{2}\ln\frac{3}{2}=\infty.$$

Anm. $\int_a^b f(x)dx=\infty$ skrivs endast om $a<b$, $f(x)\geq0$.

c) $\displaystyle\int_0^1\arctan\sqrt{x}\cdot\frac{dx}{\sqrt{x}}=\lim_{\varepsilon\to0^+}\int_\varepsilon^1 2\arctan\sqrt{x}\cdot d(\sqrt{x})=[t=\sqrt{x}]=$

$$=\lim_{\varepsilon\to0^+}\int_{\sqrt{\varepsilon}}^1 2\arctan t\,dt=\lim_{\varepsilon\to0^+}\left[2t\cdot\arctan t\right]_{\sqrt{\varepsilon}}^1-$$

$$-\lim_{\varepsilon\to0^+}\int_{\sqrt{\varepsilon}}^1\frac{2t}{t^2+1}dt=2\arctan 1-2\lim_{\varepsilon\to0^+}\sqrt{\varepsilon}\arctan\sqrt{\varepsilon}-$$

$$-\lim_{\varepsilon\to0}\left[\ln(t^2+1)\right]_{\sqrt{\varepsilon}}^1=2\cdot\frac{\pi}{4}-0-\ln 2+\lim_{\varepsilon\to0^+}\ln(1+\varepsilon)=\pi$$

$$=\frac{\pi}{2}-\ln 2+\ln 1=\pi/2-\ln 2.$$

Anm. $\lim\limits_{x\to0^+}\dfrac{\arctan\sqrt{x}}{\sqrt{x}}\ [t=\sqrt{x}]=\lim\limits_{t\to0^+}\dfrac{\arctan t}{t}=1$,

enligt 2.14 a), så integralen är ordinär.

d) $\displaystyle\int_1^2\frac{dx}{\sqrt{x^2-1}}=\lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^2\frac{dx}{\sqrt{x^2-1}}=\lim_{\varepsilon\to0^+}\left[\ln(x+\sqrt{x^2-1})\right]_{1+\varepsilon}^2=$

$$=\ln(2+\sqrt{3})-\lim_{\varepsilon\to0^+}\ln(1+\varepsilon+\sqrt{\varepsilon^2+2\varepsilon})=\underline{\ln(2+\sqrt{3})}.$$

Övning 6.28 (Sid. 119)

Lösning

$\int \frac{1}{x+1}\frac{dx}{\sqrt{x}} \left[t=\sqrt{x} \Rightarrow dt = \frac{dx}{2\sqrt{x}} \right] = 2\int \frac{dt}{t^2+1} = 2\arctan\sqrt{x}$;

$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \lim_{R\to\infty} \left[2\arctan\sqrt{x}\right]_0^R = \lim_{R\to\infty} 2\arctan\sqrt{R} = \pi$.

Det är tillåtet att resonera så! Läs även facit!

Övning 6.29 (Sid. 119)

Lösning

a) $\int_R^S \frac{dx}{x^2+1} = [\arctan x]_R^S = \arctan S - \arctan R \xrightarrow[R\to-\infty]{S\to\infty} \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}+\frac{\pi}{2} = \pi$.

b) $\int_1^\infty \frac{dx}{x^2-1} = \lim_{\varepsilon\to 0^+}\int_{1+\varepsilon}^2 \frac{1}{x^2-1}dx + \lim_{R\to\infty}\int_2^R \frac{1}{x^2-1}dx = I_1 + I_2$;

$I_1 = \lim_{\varepsilon\to 0^+}\left[\frac{1}{2}\ln\frac{x-1}{x+1}\right]_{1+\varepsilon}^2 = \frac{1}{2}\ln\frac{1}{3} + \lim_{\varepsilon\to 0^+}\frac{1}{2}\ln\frac{2+\varepsilon}{\varepsilon} = \infty$;

$I_2 = \lim_{R\to\infty}\frac{1}{2}\ln\frac{R-1}{R+1} + \frac{1}{2}\ln 3 = \frac{1}{2}\ln 3$;

Resultat: Den givna integralen är divergent.

c) $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\arcsin(2x) \Rightarrow \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-4x^2}}dx = \arcsin 1 = \frac{\pi}{2}$.

d) $\int \frac{dx}{\sqrt{x(1-x)}} \left[\begin{array}{l} x(1-x) = (1-(2x-1)^2)/4 \\ t=(2x-1) \Rightarrow dx = \frac{1}{2}dt \end{array} \right] = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin(2x-1)$;

$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \left[\arcsin(2x-1)\right]_0^1 = 2\arcsin 1 = 2\cdot\frac{\pi}{2} = \pi$.

Övning 6.30 (Sid. 119)

Lösning

a) $x>1 \Rightarrow \ln x > 0 \Rightarrow x^2+\ln x > x^2 \Leftrightarrow 0 < \frac{1}{x^2+\ln x} < \frac{1}{x^2} \Rightarrow$

$\Rightarrow 0 < \int_1^R \frac{dx}{x^2+\ln x} < \int_1^R \frac{dx}{x^2} = 1 - \frac{1}{R} \xrightarrow[R\to\infty]{} 1 \Rightarrow \int_1^\infty \frac{dx}{x^2+\ln x} < \infty$,

enligt Sats 11 på sidan 306 i grundboken.

b) $x>1 \Leftrightarrow \ln x > 0 \Leftrightarrow -\ln x < 0 \Leftrightarrow x - \ln x < x \Leftrightarrow \frac{1}{x-\ln x} > \frac{1}{x} \Rightarrow \int_1^R \frac{dx}{x-\ln x} > \int_1^R \frac{dx}{x} = \ln R \xrightarrow[R\to\infty]{} \infty \Rightarrow \int_1^\infty \frac{dx}{x-\ln x} = \infty$,

enligt Sats 11 på sidan 306 i grundboken.

Jag kommer att visa ett "konvergenskriterium i gränsvärdesform", som författarna har missat.

Sats: Om f och g är positiva funktioner, som för varje $R>a$ är integrerbara i $a \le x \le R$ och som är sådana att $\lim_{x\to\infty}\frac{f(x)}{g(x)} = A$, $A>0$, så är integralerna $\int_a^\infty f(x)dx$ och $\int_a^\infty g(x)dx$ antingen båda konvergenta eller båda divergenta.

Bevis: Förutsättningen medför, att till varje $\varepsilon>0$ finns ett ω, sådant att

$$g(x)(A-\varepsilon) < f(x) < (A+\varepsilon)g(x), \quad x>\omega.$$

forts.

Om vi här väljer $\varepsilon < A$ finner vi, att satsen följer ur Sats 11 (s. 306) i läroboken.

c) $f(x)=\dfrac{1}{x^2-\ln x}$, $x\geq 1$; $g(x)=\dfrac{1}{x^2}$, $x\geq 1$; $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=1$;
$\int_1^\infty \dfrac{dx}{x^2}<\infty$, så även $\int_1^\infty \dfrac{dx}{x^2-\ln x}<\infty$, enligt Satsen.

d) $f(x)=\dfrac{1}{x+\ln x}$, $x>1$; $g(x)=\dfrac{1}{x}$, $x>1$; $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=1$;
$\int_1^\infty \dfrac{dx}{x}=\infty$, så även $\int_1^\infty \dfrac{dx}{x+\ln x}=\infty$, enl. kriteriet.

Övning 6.31 (Sid. 120)

Lösning

a) $x>2 \Rightarrow x^3+1>x^3 \Leftrightarrow \sqrt{x^3+1}>\sqrt{x^3} \Leftrightarrow \dfrac{1}{\sqrt{x^3+1}}<\dfrac{1}{x^{3/2}} \Rightarrow$
$\Rightarrow 0<\int_2^R \dfrac{dx}{\sqrt{x^3+1}}<\int_2^R \dfrac{dx}{x^{3/2}}=\left[-2x^{-1/2}\right]_2^R=\sqrt{2}-\dfrac{2}{\sqrt{R}}\xrightarrow[R\to\infty]{}\sqrt{2}$
$\Rightarrow \int_2^\infty \dfrac{dx}{\sqrt{x^3+1}}<\infty$, enligt Sats 11.

b) $\int_2^\infty \dfrac{dx}{\sqrt{x-1}}=\lim\limits_{R\to\infty}\int_2^R \dfrac{dx}{\sqrt{x-1}}=\lim\limits_{R\to\infty}\left[2\sqrt{x-1}\right]_2^R=\lim\limits_{R\to\infty}2\sqrt{R-1}=\infty.$

c) $f(x)=\dfrac{1}{\sqrt{x^3-1}}$, $x>1$; f är singulär i $x=1$.
$\int_1^\infty \dfrac{dx}{\sqrt{x^3-1}}=\int_1^2 \dfrac{1}{\sqrt{x^3-1}}dx+\int_2^\infty \dfrac{dx}{\sqrt{x^3-1}}=I_1+I_2$;
$1\leq x\leq 2 \Rightarrow x^3\geq x \Leftrightarrow x^3-1\geq x-1 \Leftrightarrow \sqrt{x^3-1}\geq\sqrt{x-1} \Leftrightarrow$
$\Leftrightarrow 0<\dfrac{1}{\sqrt{x^3-1}}\leq\dfrac{1}{\sqrt{x-1}} \Rightarrow \int_{1+\varepsilon}^2 \dfrac{dx}{\sqrt{x^3-1}}=\lim\limits_{\varepsilon\to 0^+}\int_{1+\varepsilon}^2 \dfrac{dx}{\sqrt{x^3-1}}\leq$
$\leq \lim\limits_{\varepsilon\to 0^+}\int_{1+\varepsilon}^2 \dfrac{dx}{\sqrt{x-1}}=\lim\limits_{\varepsilon\to 0^+}\left[2\sqrt{x-1}\right]_{1+\varepsilon}^2=2 \Rightarrow I_1<\infty$; (Ex. 13!)

För stora x har vi $x^3-1\approx x^3$, dvs. $g(x)=\dfrac{1}{x^{3/2}}$ kan jämföras med $f(x)=\dfrac{1}{\sqrt{x^3-1}}$, $x\geq 2$.
$\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=1>0$; $\int_2^\infty \dfrac{dx}{x^{3/2}}<\infty$ (Ex. 12, s. 303); $I_2=$
$=\int_2^\infty \dfrac{dx}{\sqrt{x^3-1}}<\infty$, enligt jämförelsekriteriet i gränsvärdesform.

Integralen är således konvergent.

Övning 6.32 (Sid. 120)

Lösning

a) För små x är $x^5\approx 0$, så att $x+x^5\approx x \Rightarrow \dfrac{1}{\sqrt{x+x^5}}\approx$
$\approx\dfrac{1}{\sqrt{x}}$; vi jämför $f(x)=\dfrac{1}{\sqrt{x+x^5}}$, $0<x\leq 1$, med $g(x)=$
$=\dfrac{1}{\sqrt{x}}$, $0<x\leq 1$; sats 11 ger omedelbart
$\int_0^1 \dfrac{dx}{\sqrt{x+x^5}}\leq\int_0^1 \dfrac{dx}{x^{1/2}}<\infty$, enl. Ex. 15, s. 305, så den givna integralen är konvergent.

b) $f(x)=\dfrac{1}{\sin x}$, $0<x\leq 1$, jämförs med $g(x)=\dfrac{1}{x}$, $0<x\leq 1$
$0<x\leq 1 \Rightarrow \sin x\leq x \Rightarrow \dfrac{1}{\sin x}\geq\dfrac{1}{x} \Rightarrow \int_\varepsilon^1 \dfrac{dx}{\sin x}>\int_\varepsilon^1 \dfrac{dx}{x}=$
$=-\ln\varepsilon \xrightarrow[\varepsilon\to 0^+]{}\infty \Rightarrow \int_0^1 \dfrac{dx}{\sin x}=\infty$, divergent alltså.

c) $\int_0^\infty \dfrac{dx}{\sqrt{x}(e^x+1)}=\left(\int_0^1+\int_1^\infty\right)\dfrac{dx}{\sqrt{x}(e^x+1)}=I_1+I_2$;
$0\leq x\leq 1 \Rightarrow e^x>x \Leftrightarrow e^x+1>x+1 \Leftrightarrow \sqrt{x}(e^x+1)>\sqrt{x}(x+1) \Rightarrow$

$$\Rightarrow 0<\int_{\varepsilon}^{1}\frac{dx}{\sqrt{x}(e^x+1)}<\int_{\varepsilon}^{1}\frac{dx}{\sqrt{x}(x+1)}\left[\begin{array}{l|l} x=t^2 & 1\to 1\\ dx=2t\,dt & \varepsilon\to\sqrt{\varepsilon}\end{array}\right]=2\int_{\sqrt{\varepsilon}}^{1}\frac{dt}{1+t^2}=$$

$= 2\arctan 1-2\arctan\sqrt{\varepsilon}\xrightarrow[\varepsilon\to 0^+]{}\pi/2$, dvs. $I_1<\infty$.

$x\geq 1\Rightarrow \sqrt{x}(e^x+1)>\sqrt{x}e^x \Leftrightarrow 0<\frac{1}{\sqrt{x}(e^x+1)}<\frac{1}{\sqrt{x}e^x}\Rightarrow$

$\Rightarrow 0<\int_1^{\infty}\frac{dx}{\sqrt{x}(e^x+1)}<\int_1^{\infty}\frac{dx}{xe^x}<\infty$, enl. Ex. 17 i boken.

Alltså är $\int_0^{\infty}\frac{dx}{\sqrt{x}(e^x+1)}$ konvergent.

Övning 6.33 (Sid. 120)

Lösning

$x>0\Rightarrow \sin^2 x\geq 0\Rightarrow e^x+\sin^2 x\geq e^x\Rightarrow \frac{1}{e^x+\sin^2 x}\leq e^{-x}$

$\Rightarrow 0<\int_0^R\frac{dx}{e^x+\sin^2 x}\leq\int_0^R e^{-x}dx=1-e^{-R}\xrightarrow[R\to\infty]{}1.$

Jag har visat att $\int_0^{\infty}\frac{dx}{\sin^2 x+e^x}\leq 1$. (Se även facit.)

Övning 6.34 (Sid. 120)

Lösning

$x>1\Rightarrow \ln x>0\Rightarrow x^3+\ln x>x^3\Leftrightarrow \frac{1}{x^3+\ln x}<\frac{1}{x^3}\Leftrightarrow$

$\Leftrightarrow \frac{x}{x^3+\ln x}<\frac{1}{x^2}\Rightarrow \int_1^R\frac{x}{x^3+\ln x}dx<\int_1^R\frac{dx}{x^2}<1\Rightarrow\int_1^{\infty}\frac{x\,dx}{x^3+\ln x}\leq 1.$

Att memorera: Skriv aldrig i t.ex. en tentamen:

$0\leq f(x)\leq g(x)\Rightarrow 0\leq\int_a^{\infty}f(x)dx\leq\int_a^{\infty}g(x)dx$ (fel);

skriv i stället $0\leq f(x)\leq g(x)\Rightarrow\int_a^R f(x)dx\leq\int_a^R g(x)dx\ldots$

Blandade uppgifter

Övning 6.35 (Sid. 120)

Lösning

$$\int_1^{\infty}\frac{dx}{x^2+3x+2}=\lim_{R\to\infty}\int_1^R\frac{dx}{(x+1)(x+2)}=\lim_{R\to\infty}\int_1^R\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx=$$

$$=\lim_{R\to\infty}\left[\ln\frac{x+1}{x+2}\right]_1^R=\lim_{R\to\infty}\ln\frac{R+1}{R+2}-\ln\frac{2}{3}=\ln\frac{3}{2}.$$

Övning 6.36 (Sid. 120)

Lösning

$g(u)=\int_1^u\frac{t^3}{e^t-1}dt\Rightarrow g'(u)=\frac{u^3}{e^u-1}$;

$f(x)=g(\ln x)=\int_1^{\ln x}\frac{t^3}{e^t-1}dt\Rightarrow f'(x)=g'(\ln x)\cdot(\ln x)'=$

$=\frac{\ln^3 x}{e^{\ln x}-1}\cdot\frac{1}{x}=\frac{\ln^3 x}{x(x-1)}>0$ för $x>1$, dvs. f är växande.

Övning 6.37 (Sid. 120)

Lösning

$$\frac{\sin 2x}{3+2\sin x-\cos^2 x}=\frac{2\sin x\cos x}{3+2\sin x-1+\sin^2 x}=\frac{2\sin x\cos x}{\sin^2 x+2\sin x+2}\Rightarrow$$

$$\Rightarrow\int_0^{\pi/2}\frac{\sin 2x}{3+2\sin x-\cos^2 x}dx=\int_0^{\pi/2}\frac{2\sin x}{(1+\sin x)^2+1}\cos x\,dx\left[\begin{array}{l} t=1+\sin x\\ dt=\cos x\,dx\end{array}\right]=$$

$$=\int_1^2\frac{2(t-1)}{t^2+1}dt=\int_1^2\left(\frac{2t}{t^2+1}-\frac{2}{t^2+1}\right)dt=\left[\ln(t^2+1)-\arctan t\right]_1^2=$$

$$=\ln 5-\ln 2-2(\arctan 2-\arctan 1)=\ln\frac{5}{2}+\frac{\pi}{2}-2\arctan 2.$$

Övning 6.38 (Sid. 120)

Lösning

$$\frac{1}{x^4-1}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{2}\left(\frac{1}{x^2-1}-\frac{1}{x^2+1}\right);$$

$$\frac{x^2}{x^4-1}=\frac{x^2-1+1}{x^4-1}=\frac{1}{x^2+1}+\frac{1}{x^4-1}=\frac{1}{2}\left(\frac{1}{x^2+1}+\frac{1}{x^2-1}\right)=\frac{1}{2}\frac{1}{x^2+1}+$$

$$+\frac{1}{2}\frac{1}{(x-1)(x+1)}=\frac{1}{2}\frac{1}{x^2+1}+\frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+1}\right);$$

$$\int_{\sqrt{3}}^{\infty}\frac{x^2}{x^4-1}dx=\frac{1}{4}\lim_{R\to\infty}\int_{\sqrt{3}}^{R}\left(\frac{2}{x^2+1}+\frac{1}{x-1}-\frac{1}{x+1}\right)dx=$$

$$=\frac{1}{4}\lim_{R\to\infty}\left[2\arctan x+\ln\frac{x-1}{x+1}\right]_{\sqrt{3}}^{R}=$$

$$=\frac{1}{4}\lim_{R\to\infty}\left(2\arctan R+\ln\frac{R-1}{R+1}-2\cdot\frac{\pi}{3}-\ln\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)=$$

$$=\frac{1}{4}\left(2\cdot\frac{\pi}{2}-\frac{2\pi}{3}-\ln\frac{(\sqrt{3}-1)^2}{2}\right)=\frac{\pi}{12}-\frac{1}{4}\ln(2-\sqrt{3}).$$

Övning 6.39 (Sid. 120)

Lösning

$$\int_0^{2\pi}\frac{dx}{2+\sin x}=\int_0^{\pi}\frac{dx}{2+\sin x}+\int_{\pi}^{2\pi}\frac{dx}{2+\sin x}\left[\begin{array}{c|c} t=x-\pi & 2\pi\to\pi \\ dx=dt & \pi\to 0\end{array}\right]=$$

$$=\int_0^{\pi}\frac{dx}{2+\sin x}+\int_0^{\pi}\frac{dt}{2+\sin t}=2\int_0^{\pi}\frac{dx}{2+\sin x}=(\text{Se Ö.5.31 b}))=$$

$$=\left[\frac{4}{\sqrt{3}}\arctan\frac{2\tan(x/2)+1}{\sqrt{3}}\right]_0^{\pi}=\frac{4}{\sqrt{3}}\lim_{x\to\pi}\arctan\frac{2\tan(x/2)+1}{\sqrt{3}}-$$

$$-\frac{4}{\sqrt{3}}\arctan\frac{1}{\sqrt{3}}=\frac{4}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{4}{\sqrt{3}}\frac{\pi}{6}=\frac{2\pi}{3\sqrt{3}}.$$

Ovanstående integral var generaliserad (Se $\pm$).

Övning 6.40 (Sid. 121)

Lösning

$$x>2\Rightarrow\frac{2x-4}{(2x+1)(x^2+1)}=\frac{2x-4}{2x+1}\cdot\frac{1}{x^2+1}\leq\frac{1}{x^2+1};$$

$$0\leq\int_2^R\frac{2x-4}{(2x+1)(x^2+1)}dx<\int_2^R\frac{dx}{x^2+1}=\arctan R-\arctan 2<\frac{\pi}{2}$$

$$\Rightarrow\int_2^{\infty}\frac{2x-4}{(2x+1)(x^2+1)}dx<\infty;$$

$$\frac{2x-4}{(2x+1)(x^2+1)}=\frac{A}{2x+1}-\frac{Bx+C}{x^2+1}=\frac{A(x^2+1)+(2x+1)(Bx+C)}{(2x+1)(x^2+1)}\Leftrightarrow$$

$$\Leftrightarrow A(x^2+1)+(2x+1)(Bx+C)=(A+2B)x^2+(B+2C)x+A+C=2x-4$$

$$\Leftrightarrow\begin{cases}A+2B=0 & (-1)\\ B+2C=2\\ A+C=-4\end{cases}\Leftrightarrow\begin{cases}A+2B=0\\ B+2C=2 & (2)\\ -2B+C=-4\end{cases}\Leftrightarrow\begin{cases}A=-4\\ B=2\\ C=0\end{cases}\Rightarrow$$

$$\Rightarrow\int_2^R\frac{2x-4}{(2x+1)(x^2+1)}dx=\int_2^R\left(\frac{2x}{x^2+1}-\frac{4}{2x+1}\right)dx=\left[\ln\frac{x^2+1}{(2x+1)^2}\right]_2^R=$$

$$=\ln\frac{R^2+1}{(2R+1)^2}-\ln\frac{5}{25}\Rightarrow\int_2^{\infty}\frac{2x-4}{(2x+1)(x^2+1)}dx=\lim_{R\to\infty}\ln\frac{R^2+1}{(2R+1)^2}+$$

$$=\lim_{R\to\infty}\ln\frac{1+1/R^2}{(2+1/R)^2}+\ln 5=\ln\frac{1}{4}+\ln 5=\ln\frac{5}{4}.$$

Övning 6.41 (Sid. 121)

Lösning

$$\int_0^{\pi/4}\frac{\cos x}{\sin^3 x}dx=\int_0^{\pi/4}\cot x\,\frac{dx}{\sin^2 x}=-\int_0^{\pi/4}\cot x\,d(\cot x)=$$

$$=\lim_{\varepsilon\to 0^+}\left[-\frac{1}{2}(\cot x)^2\right]_{\varepsilon}^{\pi/4}=\lim_{\varepsilon\to 0^+}\frac{1}{2}\cot^2 x-\frac{1}{2}=+\infty,\text{ divergent.}$$

Anm. $\int f(x)dx=F(x)\Rightarrow\int f(g(x))d(g(x))=F(g(x)).$

Övning 6.42 (Sid. 121)

Lösning

Sats 6 på sidan 294 konsulteras.

$$R>1 \Rightarrow \left|\int_1^R \frac{2+\sin x}{1+x^2}dx\right| \leq \int_1^R \frac{|2+\sin x|}{x^2+1}dx \leq \int_1^R \frac{2+|\sin x|}{x^2+1}dx$$

$$\leq \int_1^R \frac{3}{x^2+1}dx = \left[3\arctan x\right]_1^R = 3\left(\arctan R - \frac{\pi}{4}\right) \xrightarrow[R\to\infty]{} \frac{3\pi}{4};$$

Resultat: $\int_1^\infty \frac{2+\sin x}{x^2+1}dx < \infty.$

Övning 6.43 (Sid. 121)

Lösning

$$F(x) = \int_0^x \frac{(1-t)dt}{(1+t^2)(1+t)} \Rightarrow F'(x) = \frac{1-x}{(1+x^2)(1+x)} = g(x)\cdot(1-x),\ g(x)>0;$$

$$\left.\begin{cases} 0<x<1 \Rightarrow F'(x)>0 \Rightarrow F \text{ växande} \\ 1<x<2 \Rightarrow F'(x)<0 \Rightarrow F \text{ avtagande}\end{cases}\right\} \Rightarrow F(x) \leq F(1) =$$

$$= \int_0^1 \frac{1-t}{(1+t^2)(1+t)}dt = \int_0^1 \left(\frac{1}{t+1} - \frac{t}{t^2+1}\right)dt = \left[\ln\frac{t+1}{\sqrt{t^2+1}}\right]_0^1 = \frac{1}{2}\ln 2.$$

Övning 6.44 (Sid. 121)

Lösning

Jag utnyttjar primitiven från Ö. 5.13 och får

$$\int_0^1 \frac{dx}{e^x+e^{-x}} = \left[\arctan e^x\right]_0^1 = \arctan(e) - \arctan 1 =$$

$$= \arctan(e) - \frac{\pi}{4} \approx 0{,}433.$$

Övning 6.45 (Sid. 121)

Lösning

$$\int_0^{\pi/2} \cos t\sqrt{1+\sin^2 t}\,dt \left[\begin{array}{l|l} u=\sin t & \pi/2 \to 1 \\ du=\cos t\,dt & 0\to 0\end{array}\right] = \int_0^1 \sqrt{1+u^2}\,du =$$

$$= \left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln(u+\sqrt{u^2+1})\right)\right]_0^1 = \frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right).$$

Övning 6.46 (Sid. 121)

Lösning

$$A(x) = \arctan x^2 = \int_0^{x^2}\frac{dt}{t^2+1} \Rightarrow A'(x) = \frac{2x}{1+x^4} = f(x),\ x>0.$$

Övning 6.47 (Sid. 121)

Lösning

$$\int_1^2 (x-a)\ln x\,dx = \left[\frac{1}{2}(x-a)^2\ln x\right]_1^2 - \frac{1}{2}\int_1^2 (x-a)^2\cdot\frac{1}{x}dx$$

$$= \frac{1}{2}(2-a)^2\ln 2 - \frac{1}{2}\int_1^2\left(x-2a+\frac{a^2}{x}\right)dx = \frac{1}{2}(a-2)^2\ln 2 -$$

$$-\left[\frac{1}{2}\left(\frac{x^2}{2} - 2ax + a^2\ln x\right)\right]_1^2 = \frac{1}{2}(a-2)^2\ln 2 - \frac{1}{2}(2-4a+$$

$$+a^2\ln 2 - \frac{1}{2} + 2a) = \frac{1}{2}\cancel{a^2\ln 2} - 2a\ln 2 + 2\ln 2 - 1 + 2a +$$

$$-\frac{1}{2}\cancel{a^2\ln 2} + \frac{1}{4} - a \ = (2\ln 2)(1-a) + a - \frac{3}{4} = 0 \Leftrightarrow 4a - 3 +$$

$$+(8\ln 2)(1-a) = (4-8\ln 2)a - 3 + 8\ln 2 \Leftrightarrow (8\ln 2 - 4)a =$$

$$= 8\ln 2 - 3 \Leftrightarrow a = \frac{8\ln 2 - 3}{8\ln 2 - 4} \approx 1{,}647.$$

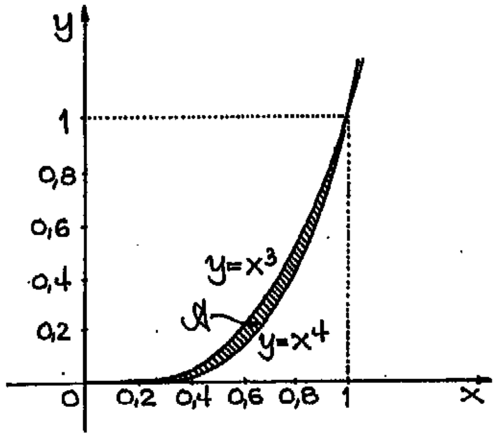
7. Användningar av integraler

Areabestämningar

Övning 7.1 (Sid. 132)

Lösning

x	0,2	0,4	0,5	0,6	0,8	0,9
x^3	0,01	0,06	0,13	0,22	0,51	0,73
x^4	0	0,03	0,06	0,13	0,41	0,66



$$\mathcal{A}=\int_0^1 (x^3-x^4)\,dx=\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1=\frac{1}{4}-\frac{1}{5}=\frac{5-4}{20}=\frac{1}{20}=0{,}05\text{ ae.}$$

Övning 7.2 (Sid. 132)

Lösning

Integrationsgränserna bestäms genom att sätta y-koordinaterna lika: $\frac{2}{x^2+1}=\frac{1}{\sqrt{x^2+1}} \Leftrightarrow x^2+1=2\sqrt{x^2+1}$

$\Leftrightarrow \sqrt{x^2+1}=2 \Leftrightarrow x^2+1=4 \Leftrightarrow x^2=3 \Leftrightarrow x=\pm\sqrt{3}.$

$0\le x\le\sqrt{3} \Rightarrow \frac{2}{x^2+1}\ge\frac{1}{\sqrt{x^2+1}} \Rightarrow \mathcal{A}=\int_0^{\sqrt{3}}\left(\frac{2}{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)dx=$

$=\left[2\arctan x-\ln(x+\sqrt{x^2+1})\right]_0^{\sqrt{3}}=2\arctan\sqrt{3}-\ln(2+\sqrt{3})=$

$=2\cdot\frac{\pi}{3}-\ln(2+\sqrt{3})=\frac{2\pi}{3}-\ln(2+\sqrt{3})\approx 0{,}777$ ae.

Övning 7.3 (Sid. 132)

Lösning

$\mathcal{A}=\int_0^{\pi/2}(\sin x-\sin^2 x)\,dx=\int_0^{\pi/2}\left(\sin x-\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx=$

$=\left[-\cos x-\frac{x}{2}+\frac{1}{4}\sin 2x\right]_0^{\pi/2}=-\cos\frac{\pi}{2}-\frac{\pi}{4}+\frac{1}{4}\sin\pi+\cos 0=$

$=1-\frac{\pi}{4}\approx 0{,}215$ ae.

Diverse fysikaliska tillämpningar

Övning 7.4 (Sid. 132)

Lösning

$\frac{ds}{dt}=1600(t-4t^2)dt \Rightarrow ds=1600(t-4t^2)dt \Rightarrow s(\tfrac{1}{4})=$

$=\int_0^{1/4}1600(t-4t^2)dt=\left[1600\left(\frac{t^2}{2}-\frac{4t^3}{3}\right)\right]_0^{1/4}=1600\left(\frac{1}{32}-\frac{1}{48}\right)=$

$=\frac{1600}{8}\left(\frac{1}{4}-\frac{1}{6}\right)=200\cdot\frac{1}{12}=\frac{100}{6}\approx 16{,}7$ km.

Svar: Bilens förflyttning under den första kvarten blev 16,7 km.

Övning 7.5 (Sid. 132)

Lösning

$\frac{dv}{dt} = 100\cos t \Leftrightarrow dv = 100\cos t\,dt \Rightarrow v(3{,}0) = \int_0^3 \frac{dv}{dt}dt =$

$= \int_0^{3,0} 100\cos t\,dt = [100\sin t]_0^{3,0} = 100\sin 3{,}0 \approx 14{,}11.$

Svar: Vid tiden 3,0 s är partikelns fart 14 m/s.

Övning 7.6 (Sid. 132)

Lösning

Perioden till $\sin\frac{\pi t}{12}$ är $= \frac{2\pi}{\pi/12} = 24\,h = 1$ dygn.

$|\sin\frac{\pi t}{12}| = \begin{cases} \sin\frac{\pi t}{12}, & 0 \le t \le 12 \\ -\sin\frac{\pi t}{12}, & 12 \le t \le 24 \end{cases} \Rightarrow E = \left(\int_0^{12} + \int_{12}^{24}\right)P(t)dt =$

$= \int_0^{12}(1+\pi\cdot\sin\frac{\pi t}{12})dt + \int_{12}^{24}(1-\pi\cdot\sin\frac{\pi t}{12})dt =$

$= [t - 12\cos\frac{\pi t}{12}]_0^{12} + [t + 12\cos\frac{\pi t}{12}]_{12}^{24} = 12 - 12\cos\pi + 12 + 24 +$

$+ 12\cos 2\pi - 12 - 12\cos\pi = 12+12+12+24+12-12+12 = 72$ W.

Resultat: Under ett dygn förbrukar verksamheten 72 kWh.

Övning 7.7 (Sid. 133)

Lösning

a) $W = \int_{x_1}^{x_2} F(x)dx = \int_1^3 2dx = [2x]_1^3 = 2(3-1) = 2^2 = \underline{4\text{ Nm.}}$

b) $W = \int_{x_1}^{x_2} F(x)dx = \int_0^2 3x\,dx = [\frac{3}{2}x^2]_0^2 = 3\cdot 2 = \underline{6\text{ Nm.}}$

c) $W = \int_{x_1}^{x_2} F(x)dx = \int_0^L kx\,dx = [\frac{k}{2}x^2]_0^L = \underline{\frac{kL^2}{2}\text{ Nm.}}$

Övning 7.8 (Sid. 133)

Lösning

a) $Q = \int_{t_1}^{t_2} i(t)dt = \int_1^3 2\,dt = 2[t]_1^3 = 2\cdot(3-1) = 2^2 = \underline{4\text{ C.}}$

b) $Q = \int_{t_1}^{t_2} i(t)\,dt = \int_0^{0,01} 6\sin(100\pi t)dt = \left[-\frac{3\cos 100\pi t}{50\pi}\right]_0^{0,01} =$

$= \frac{3}{50\pi}(1-\cos\pi) = \frac{3\cdot 2}{50\pi} = \frac{3}{25\pi} \approx 38\cdot 10^{-3}\text{ As} = \underline{38\text{ mC.}}$

Massa

Övning 7.9 (Sid. 133)

Lösning

$dm = \rho(x)dx \Rightarrow m = \int_1^2 x^2 dx = \frac{1}{3}(2^3 - 1^3) = \frac{7}{3} \approx 2{,}3$ kg.

Övning 7.10 (Sid. 133)

Lösning

Se nästföljande sida.

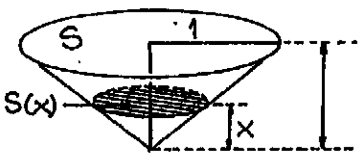
$dm = \rho(x)dx = \ln(1+x)dx \Rightarrow m = \int_0^2 \ln(1+x)dx =$

$= [(x+1)\ln(x+1) - x]_0^2 = 3\ln 3 - 2 \approx 1,3.$

Svar: Trådens massa är 1,3 kg.

Övning 7.11 (Sid. 133)

Lösning



$dV = S(x)dx \Rightarrow dm = \rho(x)dV = (10-x^2)\pi x^2 dx =$

$= \pi(10x^2 - x^4)dx \Rightarrow m = \int_0^1 \pi(10x^2 - x^4)dx = \pi\left[\frac{10x^3}{3} - \frac{x^5}{5}\right]_0^1 =$

$= \pi\left(\frac{10}{3} - \frac{1}{5}\right) = \frac{47\pi}{15} \approx 9,84$ kg.

Övning 7.12 (Sid. 133)

Lösning

$dm = \rho(x)dx = kx(L-x)dx = k(Lx - x^2)dx,\ 0 \leq x \leq L;$

$m = \int_0^L k(Lx - x^2)dx = k\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^L = kL^3\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{kL^3}{6}.$

Övning 7.13 (Sid. 133)

Lösning

Se nästa sida.

Full silo $\Rightarrow H = 20;$

$dm = \rho dV = \ln(25-h)4^2\pi dh \Rightarrow m = 16\pi\int_0^{20} \ln(5-h)dh =$

$= 16\pi[-(25-h)\ln(25-h) - h]_0^{20} = 16\pi(25\ln 25 - 5\ln 5 - 20) =$

$= 16\pi(45\ln 5 - 20) \approx 2635.$

Svar: Massan i en full silo är 2,6 ton.

Volym

Övning 7.14 (Sid. 133)

Lösning

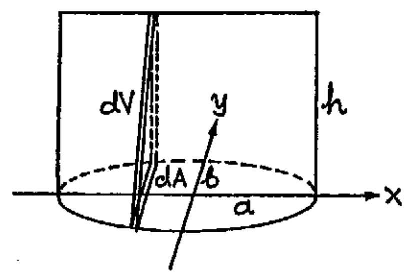
$dV = S(x)dx = (\sqrt{4-x^2})^2 dx = (4-x^2)dx;$ (skivformeln).

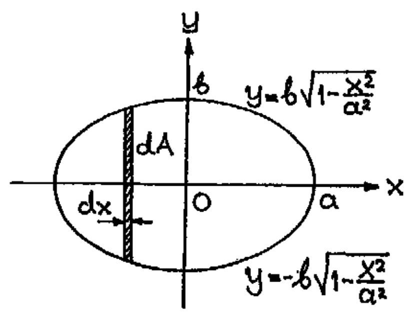
a) $V = \int_0^2 (4-x^2)dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3} = 5\frac{1}{3}$ ve.

b) $V = \int_0^1 (4-x^2)dx = \left[4x - \frac{x^3}{3}\right]_0^1 = 4 - \frac{1}{3} = \frac{11}{3} = 3\frac{2}{3}$ ve.

Övning 7.15 (Sid. 134)

Lösning





Ekvationen för basen är $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$ (ellipsskiva).

$dV = \frac{h}{2}\cdot dA = \frac{h}{2}\cdot(b\sqrt{1-x^2/a^2}-(-b\sqrt{1-x^2/a^2})dx = bh\sqrt{1-\frac{x^2}{a^2}}dx$

$\Rightarrow V = bh\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}dx \left[\begin{array}{l|l} x = a\sin t & x=a \Rightarrow t=\frac{\pi}{2} \\ dx = a\cos t dt & x=-a \Rightarrow t=-\frac{\pi}{2}\end{array}\right] =$

$= abh\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt = abh\cdot 2\int_0^{\pi/2}\cos^2 t\,dt = abh\frac{\pi}{2} = \frac{abh\pi}{2}.$

Anm. Ovanstående kropp är en konoid.

Övning 7.16 (Sid. 134)

Lösning

$dV = \pi y^2 dx = \pi(xe^x)^2 dx = \pi x^2 e^{2x}dx,\ 0\leq x\leq 1;$

$V = \int_0^1 \pi x^2 e^{2x}dx = [\frac{\pi}{2}x^2 e^{2x}]_0^1 - \pi\int_0^1 xe^{2x}dx = \frac{\pi e^2}{2} -$

$-[\frac{\pi}{2}xe^{2x}]_0^1 + \frac{\pi}{2}\int_0^1 e^{2x}dx = \frac{\pi e^2}{2} - \frac{\pi e^2}{2} + [\frac{\pi}{4}e^{2x}]_0^1 = \frac{\pi(e^2-1)}{4}.$

Övning 7.17 (Sid. 134)

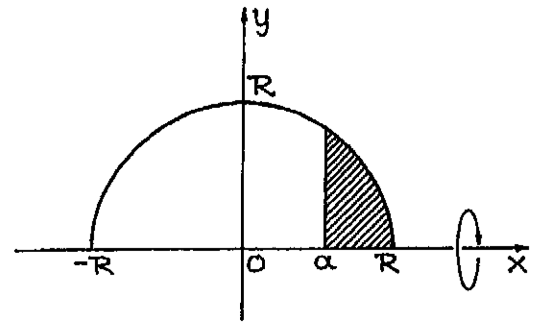
Lösning

$dV = \pi y^2 dx = \pi(x^3)^2 dx = \pi x^6 dx \Rightarrow V = \int_0^1 \pi x^6 dx = \frac{\pi}{7}$ ve.

Övning 7.18 (Sid. 134)

Lösning

När det skuggade ytstycket (se fig.) roterar ett varv kring x-axeln alstras en sfärisk kalott;



$dV = \pi y^2 dx = \pi(R^2-x^2)dx \Rightarrow V_1 = \int_a^R \pi(R^2-x^2)dx =$

$= \pi[R^2x-\frac{x^3}{3}]_a^R = \pi(\frac{2R^3}{3}-aR^2+\frac{a^3}{3}) = \frac{\pi}{3}(2R^3-3aR^2+a^3);$

Hela klotet har volymen $V = \frac{4\pi R^3}{3}$, så det stympade klotet har volymen $V_2 = V - V_1$ eller

$V_2 = \frac{\pi}{3}(2R^3+3aR^2-a^3).$

Övning 7.19 (Sid. 134)

Lösning

$dV = \pi y^2 dx = \pi(\sin x + 2\cos x)^2 dx = \pi(\sin^2 x + 4\cos^2 x +$

$+4\sin x\cos x)dx = \pi(1+3\cos^2 x + 2\sin 2x)dx,\ 0\leq x\leq\frac{\pi}{2}.$

$V = \int_0^{\pi/2}\pi(1+3\cos^2 x+2\sin 2x)dx = \pi\int_0^{\pi/2}(2\sin 2x+\frac{3}{2}\cos 2x+$

$+5/2)dx = \pi[-\cos 2x+\frac{3}{4}\sin 2x+\frac{5x}{2}]_0^{\pi/2} = \pi(-\cos\pi+\frac{5\pi}{4}+$

$+\cos 0) = \pi(2+\frac{5\pi}{4}) = 2\pi+5\pi^2/4 \approx 18{,}620$ ve.

Övning 7.20 (Sid. 134)

Lösning

$dV = \pi y^2 dx + \pi \frac{1}{x(x^2+1)} dx = \pi\left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx\,,\quad x \geq 1\,;$

$V = \pi \int_1^\infty \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \pi\left[\ln \frac{x}{\sqrt{x^2+1}}\right]_1^\infty = \lim_{R\to\infty} \ln \frac{\pi R}{\sqrt{R^2+1}} -$

$- \pi \ln \frac{1}{\sqrt{2}} = 0 - \pi \ln 2^{-1/2} = \frac{\pi}{2} \ln 2 \approx 1{,}090$ ve.

Övning 7.21 (Sid. 134)

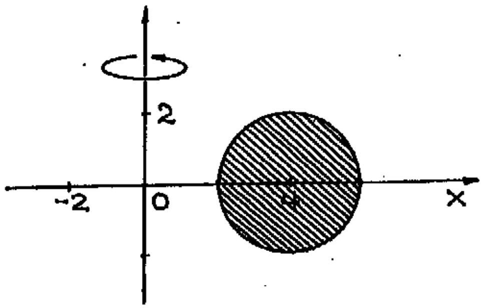
Lösning

$dV = 2\pi x y dx = 2\pi x e^{-x^2} = \pi(-e^{-x^2})' dx\,,\quad 0 < x < \infty\,;$

$V = \int_0^\infty \pi(-e^{-x^2})' dx = \lim_{R\to\infty} \pi\left[-e^{-x^2}\right]_0^R = \pi(1 - \lim_{R\to\infty} e^{-R^2}) = \pi.$

Övning 7.22 (Sid. 134)

Lösning



Guldins andra regel (Se grundboken s. 339) ger omedelbart

$V = 2\pi \cdot 4 \cdot \pi \cdot 2^2 = 32\pi^2 \approx 315{,}8$ ve.

Längd av kurvor

Övning 7.23 (Sid. 134)

Lösning

P.g.a. symmetrin betraktas endast den del av kurvan som finns i den 1:a kvadranten.

$\begin{cases} x = \cos^3 t \Rightarrow \frac{dx}{dt} = -3\cos^2 t \sin t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9\cos^4 t \sin^2 t \\ y = \sin^3 t \Rightarrow \frac{dy}{dt} = 3\sin^2 t \cos t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 9\sin^4 t \cos^2 t \end{cases} \Rightarrow$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^2 t \sin^2 t\,(\underbrace{\cos^2 t + \sin^2 t}_{=1}) \Rightarrow ds =$

$= 3\sin t \cos t\, dt = \frac{3}{2} \sin 2t\, dt\,;$

$l(C) = 4 \cdot \frac{3}{2} \int_0^{\pi/2} \sin 2t\, dt = 3\left[-\cos 2t\right]_0^{\pi/2} = 3(\cos 0 - \cos \pi) = 6.$

Svar. Asteroidens längd är 6 längdenheter.

Övning 7.24 (Sid. 135)

Lösning

$\begin{cases} x = e^{-t/6} \cos t \Rightarrow \frac{dx}{dt} = -\frac{e^{-t/6}}{6}(\cos t + 6\sin t) \\ y = e^{-t/6} \sin t \Rightarrow \frac{dy}{dt} = -\frac{e^{-t/6}}{6}(\sin t - 6\cos t) \end{cases} \Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 =$

$= \frac{e^{-t/3}}{36}(\cos^2 t + 36\sin^2 t + \sin^2 t + 6\cos^2 t) = \frac{37}{36} e^{-t/3} \Rightarrow$

$\Rightarrow ds = \frac{\sqrt{37}}{6} e^{-t/6} dt \Rightarrow s = \sqrt{37} \int_0^\infty \frac{1}{6} e^{-t/6} dt = \underline{\sqrt{37}\ \text{le}}.$

Övning 7.25 (Sid. 135)

Lösning

$$y=\ln(1-x^2) \Rightarrow y'=\frac{-2x}{1-x^2} \Rightarrow y'^2=\frac{4x^2}{(1-x^2)^2} \Rightarrow y'^2+1=$$

$$=\frac{4x^2}{(1-x^2)^2}+1=\frac{4x^2+(1-x^2)^2}{(1-x^2)^2}=\frac{(1+x^2)^2}{(1-x^2)^2} \Rightarrow ds=\frac{1+x^2}{1-x^2}dx$$

$$\Rightarrow s=\int_0^{1/2}\left(\frac{2}{1-x^2}-1\right)dx=\int_0^{1/2}\left(\frac{1}{1+x}+\frac{1}{1-x}-1\right)dx=$$

$$=\left[\ln(1+x)-\ln(1-x)-x\right]_0^{1/2}=\ln\frac{3}{2}-\ln\frac{1}{2}-\frac{1}{2}=\underline{\ln 3-\frac{1}{2}}.$$

Övning 7.26 (Sid. 135)

Lösning

$$y=x^2-1\leq 0 \Leftrightarrow x^2\leq 1 \Leftrightarrow \sqrt{x^2}\leq 1 \Leftrightarrow |x|\leq 1 \Leftrightarrow \underline{-1\leq x\leq 1}.$$

$$y'=2x \Rightarrow ds=\sqrt{y'^2+1}\,dx=\sqrt{(2x)^2+1}\,dx\ ,\ -1\leq x\leq 1\ ;$$

$$s=\int_{-1}^{1}\sqrt{(2x)^2+1}\,dx\left[\begin{array}{l|l} t=2x & x=1\Rightarrow t\ \ 2 \\ dt=2dx & x=-1\Rightarrow t=-2\end{array}\right]=\frac{1}{2}\int_{-2}^{2}\sqrt{t^2+1}\,dt$$

$$=\int_0^2\sqrt{t^2+1}\,dt=\left[\frac{1}{2}t\sqrt{t^2+1}+\frac{1}{2}\ln(t+\sqrt{t^2+1})\right]_0^2=\sqrt{5}+\frac{1}{2}\ln(2+\sqrt{5}).$$

Svar: Kurvbågens längd är $\sqrt{5}+\frac{1}{2}\ln(2+\sqrt{5})\approx 2{,}96$.

Övning 7.27 (Sid. 135)

Lösning

$$r=1+\cos\theta \Rightarrow \frac{dr}{d\theta}=-\sin\theta \Rightarrow \left(\frac{dr}{d\theta}\right)^2+r^2=\sin^2\theta+(1+\cos\theta)^2=$$

$$=\sin^2\theta+\cos^2\theta+2\cos\theta+1=1+2\cos\theta+1=4\cdot\frac{1+\cos\theta}{2}=$$

$$=4\cos^2\frac{\theta}{2}=\left(2\cos\frac{\theta}{2}\right)^2 \Rightarrow ds=2\cos\frac{\theta}{2}d\theta \Rightarrow s=2\int_{-\pi}^{\pi}\cos\frac{\theta}{2}d\theta=$$

$$=\left[4\sin\frac{\theta}{2}\right]_{-\pi}^{\pi}=4\cdot 2\sin\frac{\pi}{2}=\underline{8\ le.}$$

Övning 7.28 (Sid. 135)

Lösning

$$r=2\theta^2 \Rightarrow \frac{dr}{d\theta}=4\theta \Rightarrow \left(\frac{dr}{d\theta}\right)^2+r^2=16\theta^2+4\theta^4=4\theta^2(\theta^2+4)$$

$$\Rightarrow ds=2\theta\sqrt{\theta^2+4}\,d\theta \Rightarrow s=\int_0^{\sqrt{5}}\sqrt{\theta^2+4}\cdot 2\theta d\theta\ [v=\theta^2]=$$

$$=\int_0^5\sqrt{v+4}\,dv=\left[\frac{2}{3}(v+4)^{3/2}\right]_0^5=\frac{2}{3}\cdot(27-8)=\frac{38}{3}\approx 12\tfrac{2}{3}\,le.$$

Övning 7.29 (Sid. 135)

Lösning

$$r=\theta \Rightarrow ds=\sqrt{\dot r^2+r^2}\,d\theta=\sqrt{\theta^2+1}\,d\theta \Rightarrow dm=\rho\cdot ds=$$

$$=\theta\sqrt{\theta^2+1}\,d\theta \Rightarrow m=\int_0^3\sqrt{\theta^2+1}\,\theta d\theta\left[t=\theta^2\Rightarrow \theta d\theta=\frac{dt}{2}\right]=$$

$$=\frac{1}{2}\int_0^9\sqrt{t+1}\,dt=\left[\frac{1}{3}(t+1)^{3/2}\right]_0^9=\frac{10\sqrt{10}-1}{3}\approx 10{,}2\ le.$$

Övning 7.30 (Sid. 135)

Lösning

$$\begin{cases} x=20u^3 \Rightarrow \dfrac{dx}{du}=60u^2 \\ y=20(1-(1-u^2)^{3/2}) \Rightarrow \dfrac{dy}{du}=60u\sqrt{1-u^2}\end{cases} \Rightarrow \left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2=$$

$= 60^2u^4 + 60^2u^2 - 60^2u^4 = 60^2u^2 = (60u)^2 \Rightarrow ds = 60u\,du \Rightarrow$

$\Rightarrow s(u) = \int_0^u 60u\,du = 30u^2 \Rightarrow s(1) = 30$;

$dt = \frac{dt}{ds}ds = \frac{ds}{ds/dt} = \frac{ds}{v} = \frac{30+s}{600}ds = \frac{30+30u^2}{600} \cdot 60u\,du =$

$= 3(1+u^2)u\,du = 3(u+u^3)du \Rightarrow t = 3\int_0^1 (u+u^3)du =$

$= 3\left[\frac{1}{2}u^2 + \frac{1}{4}u^4\right]_0^1 = 3 \cdot \frac{3}{4} = \frac{9}{4} = 2{,}25.$

Resultat: Vägen är 30 km lång. Turen tar 2 timmar och en kvart.

Rotationsytor

Övning 7.31 (Sid. 136)

Lösning

$y = x^3 \Rightarrow y' = 3x^2 \Rightarrow ds = \sqrt{y'^2+1}\,dx = \sqrt{(3x^2)^2+1}\,dx \Rightarrow$

$\Rightarrow d\sigma = 2\pi x^3\sqrt{(3x^2)^2+1}\,dx \Rightarrow S = 2\pi\int_0^1 x^3\sqrt{(3x^2)^2+1}\,dx =$

$= \left[\begin{array}{l|l} u = 9x^4+1 & x=1 \Rightarrow u=10 \\ x^3dx = \frac{1}{36}du & x=0 \Rightarrow u=1 \end{array}\right] = \frac{\pi}{18}\int_1^{10}\sqrt{u}\,du = \left[\frac{\pi}{27}u^{3/2}\right]_1^{10} =$

$= \frac{\pi}{27}(10\sqrt{10}-1) \approx 3{,}563$ ae.

Övning 7.32 (Sid. 136)

Lösning

$y = \cosh x \Rightarrow y' = \sinh x \Rightarrow y'^2+1 = \sinh^2 x + 1 = \cosh^2 x$

$\Rightarrow ds = \sqrt{y'^2+1}\,dx = \cosh x\,dx \Rightarrow d\sigma = 2\pi\cosh^2 x\,dx \Rightarrow$

$\Rightarrow S = \pi\int_{-1}^{1}(1+\cosh 2x)dx = \pi\left[x + \frac{1}{2}\sinh 2x\right]_{-1}^{1} = \pi(1 + \frac{1}{2}\sinh 2 + 1 + \frac{1}{2}\sinh 2) = \pi(2+\sinh 2) \approx 17{,}677$ ae.

Övning 7.33 (Sid. 136)

Lösning

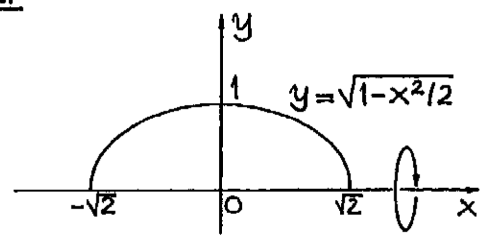
$y = 2\sqrt{x} \Rightarrow y' = \frac{1}{\sqrt{x}} \Rightarrow y'^2+1 = \frac{1}{x}+1 = \frac{x+1}{x} \Rightarrow ds = \sqrt{y'^2+1}\,dx$

$= \frac{\sqrt{x+1}}{\sqrt{x}}dx \Rightarrow d\sigma = 2\pi \cdot 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}}dx = 4\pi\sqrt{x+1}\,dx \Rightarrow$

$\Rightarrow S = 4\pi\int_0^3\sqrt{x+1}\,dx = 4\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = \frac{8\pi}{3} \cdot 7 = \frac{56\pi}{3}$ ae.

Övning 7.34 (Sid. 136)

Lösningar



$y^2 = 1 - x^2/2 \Rightarrow 2yy' = -x \Leftrightarrow y' = \frac{x}{2y} \Rightarrow y'^2+1 = \frac{x^2}{4y^2}+1 =$

$= \frac{x^2+4y^2}{4y^2} = \frac{4-x^2}{4y^2} \Rightarrow ds = \frac{\sqrt{4-x^2}}{2y}dx \Rightarrow d\sigma = 2\pi y \cdot \frac{\sqrt{4-x^2}}{2y}dx =$

$= \pi\sqrt{4-x^2}\,dx \Rightarrow S = \pi\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{4-x^2}\,dx \left[\begin{array}{l|l} x = 2\sin t & \sqrt{2} \to \pi/4 \\ dx = 2\cos t\,dt & -\sqrt{2} \to -\pi/4 \end{array}\right]$

$$= \pi \int_{-\pi/4}^{\pi/4} 4\cos^2 t\,dt = 2\pi \int_{-\pi/4}^{\pi/4} (1+\cos 2t)\,dt = 4\pi \int_0^{\pi/4} (1+\cos 2t)\,dt =$$
$$= 2\pi \Big[2t + \sin 2t\Big]_0^{\pi/4} = 2\pi\left(2\cdot\frac{\pi}{4} + \sin\frac{\pi}{2}\right) = \pi^2 + 2\pi \approx 16{,}15 \text{ ae}.$$

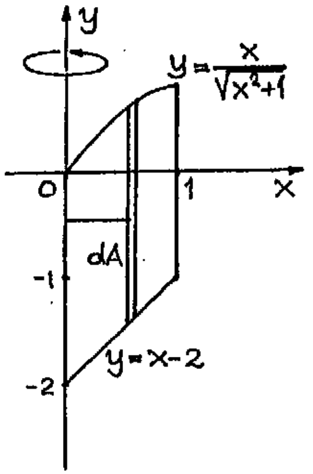
Tröghetsmoment

Övning 7.35 (Sid. 136)

Lösning (Sid. 153-154 i boken.)

Övning 7.36 (Sid. 136)

Lösning



$$A = \{(x,y) : 2-x \le y \le \frac{x}{\sqrt{x^2+1}},\ 0 \le x \le 1\},\quad \sigma = \frac{m}{\mu(A)};$$
$$\mu(A) = \int_0^1 \left(\frac{x}{\sqrt{x^2+1}} - (x-2)\right)dx = \left[\sqrt{x^2+1} - \frac{1}{2}(x-2)^2\right]_0^1 = \sqrt{2} + \frac{1}{2};$$
$$J_x = \int_K r^2\,dm = \int_K r^2 \sigma\,dA = \sigma \int_0^1 x^2\left(\frac{x}{\sqrt{x^2+1}} + 2 - x\right)dx =$$
$$= \sigma \int_0^1 x^2 \cdot \frac{x}{\sqrt{x^2+1}}\,dx + \sigma \int_0^1 (2x^2 - x^3)\,dx = \sigma\left[x^2\sqrt{x^2+1}\right]_0^1 -$$
$$- \sigma \int_0^1 \sqrt{x^2+1} \cdot 2x\,dx + \sigma\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \sqrt{2}\sigma - \sigma\left[\frac{2}{3}(x^2+1)^{3/2}\right]_0^1 +$$
$$+ \sigma\left(\frac{2}{3} - \frac{1}{4}\right) = \sqrt{2}\sigma - \left(\frac{4\sqrt{2}}{3} - \frac{2}{3}\right)\sigma + \frac{5}{12}\sigma = \left(\frac{4-\sqrt{2}}{3} - \frac{1}{4}\right)\cdot\frac{2m}{2\sqrt{2}+1} =$$
$$= \frac{13-4\sqrt{2}}{12} \cdot \frac{2(\sqrt{8}-1)}{(\sqrt{8}+1)(\sqrt{8}-1)}\,m = \frac{(13-4\sqrt{2})(\sqrt{8}-1)}{42}\,m = \frac{30\sqrt{2}-29}{42}\,m.$$

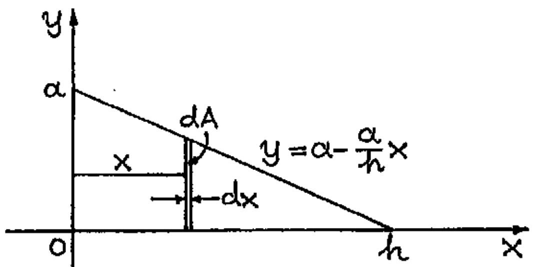
Resultat: Det sökta tröghetsmomentet är

$J_y = \frac{30\sqrt{2}-29}{42}\,m \approx 0{,}320\ m$. ($m$ står för "massa".)

Övning 7.37 (Sid. 137)

Lösning

Alla trianglar med samma bas och samma höjd ger samma resultat, så jag tar den rätvinkliga; räkningarna blir färre.



$$J_y = \int_K r^2\,dm = \int_K r^2 \sigma\,dA = \sigma \int_0^1 x^2 \left(a - \frac{a}{h}x\right)dx = \left(\sigma = \frac{2m}{ah}\right) =$$
$$= \frac{2m}{h}\int_0^1 \left(x^2 - \frac{x^3}{h}\right)dx = \frac{2m}{h^2}\left[\frac{hx^3}{3} - \frac{x^4}{4}\right]_0^h = \frac{2mh^4}{12h^2} = \frac{1}{6}mh^2.$$

Resultat: Det sökta trögetsmomentet är $\frac{mh^2}{6}$.

Masscentrum, tyngdpunkt, tryckcentrum.

Övning 7.38 (Sid. 137)

Lösning (Se sid. 154-155 i boken).

Övning 7.39 (Sid. 137)

Lösning

$dm = \tau\, dx = x^2 dx \Rightarrow M = \int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8-1}{3} = \frac{7}{3}$

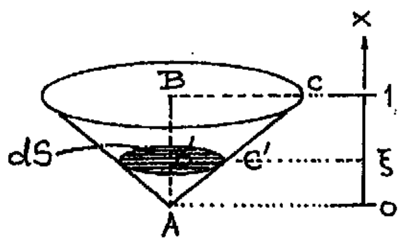
$M x_{mc} = \int_1^2 x \cdot x^2 dx = \int_1^2 x^3 dx = \frac{2^4-1}{4} = \frac{15}{4} \Rightarrow x_{mc} = \frac{45}{28}$.

Anm Angående beteckningar i fysiken: τ är reserverat för linjär densitet (trådtäthet), σ för yttäthet och ρ för volymdensitet.

Resultat: Tråden tyngdpunkt faller 60 cm från dess vänstra ända.

Övning 7.40 (Sid. 137)

Lösning



forts.

$\Delta ABC \sim \Delta AB'C' \Rightarrow \frac{B'C'}{BC} = \frac{AB'}{AB} \Rightarrow B'C' = \xi \Rightarrow dS = \pi\xi^2$

$\Rightarrow dV = \pi\xi^2 d\xi \Rightarrow dm = \rho dV = (10-\xi^2)\pi\xi^2 d\xi \Rightarrow$

$\Rightarrow m = \int_0^1 \pi(10\xi^2 - \xi^4) d\xi = \pi\left[\frac{10\xi^3}{3} - \frac{\xi^5}{5}\right]_0^1 = \pi\left(\frac{10}{3} - \frac{1}{5}\right) = \frac{47\pi}{15}$

$\Rightarrow x_{mc} = \frac{1}{m}\int_0^1 \xi\, dm = \frac{1}{m}\int_0^1 \xi(10-\xi^2)\pi\xi^2 d\xi = \frac{15}{47}\left[\frac{5\xi^4}{2} - \frac{\xi^6}{6}\right]_0^1 =$

$= \frac{15}{47}\left(\frac{5}{2} - \frac{1}{6}\right) = \frac{15 \cdot 14}{47 \cdot 6} = \frac{35}{47} \approx 0{,}74.$

Resultat: Tyngdpunkten ligger på symmetriaxeln och på 74 cm avstånd från spetsen.

Övning 7.41 (Sid. 137)

Lösning

$V = \int_0^{1/2} \pi e^{2x} dx = \frac{\pi}{2}(e-1) \Rightarrow x_{mc} = \frac{\pi}{V}\int_0^1 x e^{2x} dx =$

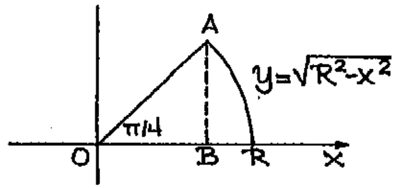
$= \frac{1}{e-1}\int_0^{1/2} 2x e^{2x} dx \; [u = 2x] = \frac{1}{2(e-1)}\int_0^1 u e^u du =$

$= \frac{1}{2(e-1)}\left[(u-1)e^u\right]_0^1 = \frac{1}{2(e-1)} \approx 0{,}29.$

Anm. Kroppen är homogen, dvs. $\rho = \rho_0$, som kan förkortas bort, alt. sättas lika med 1.

Övning 7.42 (Sid. 137)

Lösning



När cirkelsektorn roterar ett varv kring x-axeln alstras en klotsektor bestående av en rät cirkulär kon och en sfärisk kalott:

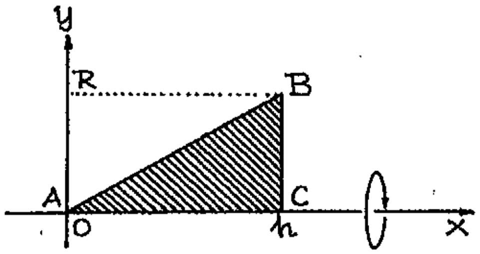
$$V = V_1 + V_2 = \pi \int_0^{R/\sqrt{2}} x^2 dx + \pi \int_{R/\sqrt{2}}^{R} (R^2 - x^2)dx = \left[\frac{\pi}{3}x^3\right]_0^{R/\sqrt{2}} +$$
$$+\pi\left[R^2x - \frac{x^3}{3}\right]_{R/\sqrt{2}}^{R} = \pi\cdot\frac{R^3}{6\sqrt{2}} + \pi\left(\frac{2R^3}{3} - \frac{R^3}{\sqrt{2}} + \frac{R^3}{6\sqrt{2}}\right) = \frac{\pi\sqrt{2}}{12}R^3 +$$
$$+\pi\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right)R^3 = \frac{\pi\sqrt{2}}{12}R^3 + \frac{\pi(8-5\sqrt{2})}{12}R^3 = \frac{\pi(2-\sqrt{2})}{12}R^3.$$

Tyngdpunkten bestäms enligt följande:

$$\frac{\pi\sqrt{2}}{12}R^3\cdot\bar{x}_1 = \int_0^{R/\sqrt{2}} x\cdot\pi x^2 dx = \pi\left[\frac{x^4}{4}\right]_0^{R/\sqrt{2}} = \frac{\pi R^4}{16}; \quad (1)$$
$$\frac{\pi(8-5\sqrt{2})}{2}\bar{x}_2 = \int_{R/\sqrt{2}}^{R} x\pi(R^2-x^2)dx = \dots = \frac{\pi R^4}{16}; \quad (2)$$
$$\frac{\pi(2-\sqrt{2})}{3}R^3\cdot\bar{x} = 2\cdot\frac{\pi R^4}{16} \Leftrightarrow (2-\sqrt{2})R^3\bar{x} = \frac{3R^4}{8} \Leftrightarrow$$
$$\Leftrightarrow \bar{x} = \frac{3}{8(2-\sqrt{2})}R = \frac{3(2+\sqrt{2})}{8(2-\sqrt{2})(2+\sqrt{2})}R = \underline{\frac{3}{16}(2+\sqrt{2})R}.$$

Övning 7.43 (Sid. 137)

Lösning



Jag arbetar med triangeln ABC i figuren.

Hypotenusans ekvation är AB: $y = \frac{R}{h}x,\ 0<x<h$. När triangeln roterar ett varv kring x-axeln alstras en kon med volymen $V = \frac{1}{3}\pi R^2 h$. Om $\bar{x}$ är tyngdpunkten (masscentret), så får ja

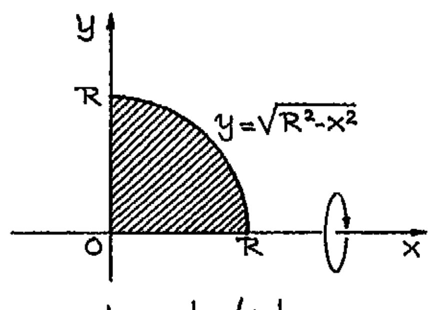
$$\frac{\pi R^3 h}{3}\cdot\bar{x} = \int_0^h x\,dV = \int_0^h x\cdot\pi\left(\frac{Rx}{h}\right)^2 dx = \frac{\pi R^2}{h^2}\int_0^h x^3 dx =$$
$$= \frac{\pi R^2}{h^2}\left[\frac{1}{4}x^4\right]_0^h = \frac{\pi R^2 h}{4} \Leftrightarrow \bar{x} = \frac{3}{4}h.$$

Resultat: Tyngdpunkten ligger på symmetriaxeln och $\frac{1}{4}h$ l.e. från basytan.

Övning 7.44 (Sid. 138)

Lösning

Jag arbetar med kvartscirkeln i figuren:



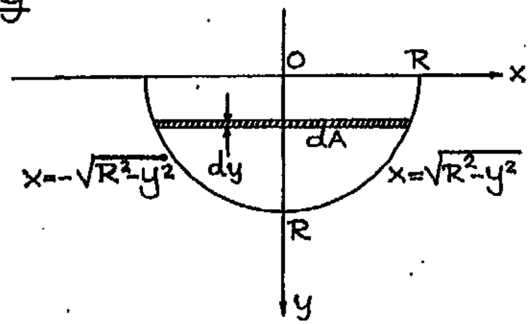
Vid rotation av kvartsdisken ovan alstras ett halvklot med volymen $V = \frac{2}{3}\pi R^3$. Halvklotets volym hamnar på x-axeln, av symmetriskäl.

$$\frac{2\pi R^3}{3}\cdot\bar{x} = \int_0^R x\,dV = \int_0^R x\cdot\pi(R^2-x^2)\,dx = \pi\int_0^R (R^2x-x^3)\,dx =$$
$$= \pi\left[\frac{R^2x^2}{2}-\frac{x^4}{4}\right]_0^R = \frac{\pi R^4}{4} \Leftrightarrow \bar{x} = \frac{3}{8}R = 0{,}375R.$$

Resultat: Halvklotets tyngdpunkt ligger på symmetriaxeln $\frac{3R}{8}$ l.e. från den plana ytan. (Sådana data finns tabellerade i handböcker).

Övning 7.45 (Sid. 138)

Lösning



$$dA = (\sqrt{R^2-y^2}-(-\sqrt{R^2-y^2}))\,dy = 2\sqrt{R^2-y^2}\,dy \Rightarrow dF =$$
$$= p\,dA = \rho g y\cdot 2\sqrt{R^2-y^2}\,dy = -\rho g(-2y\sqrt{R^2-y^2})\,dy \Rightarrow$$
$$\Rightarrow F = -\rho g\int_0^R \sqrt{R^2-y^2}\,(-2y)\,dy \left[\begin{array}{l|l} t = R^2-y^2 & R \to 0 \\ dt = -2y\,dy & 0 \to R^2 \end{array}\right] =$$
$$= -\rho g\int_{R^2}^0 \sqrt{t}\,dt = \rho g\int_0^{R^2} t^{1/2}\,dt = \rho g\left[\frac{2}{3}t^{3/2}\right]_0^{R^2} = \frac{2}{3}\rho g R^3.$$

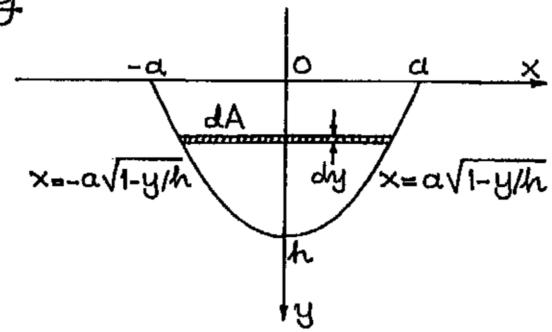
På grund av symmetrin faller tyngdpunkten på y-axeln; momentbalansen ger

$$\frac{2}{3}\rho g R^3\,\bar{y} = \int_0^R y\cdot dF = \int_0^R y\cdot\rho g y\cdot 2\sqrt{R^2-y^2}\,dy =$$
$$= 2\rho g\int_0^R y^2\sqrt{R^2-y^2}\,dy \left[\begin{array}{l|l} y = R\sin\theta & y = R \Rightarrow \theta = \frac{\pi}{2} \\ dy = R\cos\theta\,d\theta & y = 0 \Rightarrow \theta = 0 \end{array}\right] =$$
$$= 2\rho g\int_0^{\pi/2} R^4\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{2}\rho g R^4\int_0^{\pi/2}\sin^2 2\theta\,d\theta =$$
$$= \frac{1}{4}\rho g R^4\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta = \frac{1}{4}\rho g R^4\cdot\frac{\pi}{2} = \frac{1}{8}\pi\rho g R^4 \Leftrightarrow$$
$$\Leftrightarrow \bar{y} = \frac{\pi\rho g R^4}{8}\Big/\frac{2\rho g R^3}{3} = \frac{3\pi}{16}R = 0{,}589R.$$

Resultat: Den totala tryckraften på den ena plana ytan är $\frac{2}{3}\rho g R^2$; angreppspunkten för denna ligger på symmetriaxeln och på ett avstånd $\frac{3\pi}{16}R$ från den fria ytan.

Övning 7.46 (Sid. 138)

Lösning



$$dA = (a\sqrt{1-y/h}-(-a\sqrt{1-y/h}))\,dy = 2a\sqrt{1-y/h}\,dy \Rightarrow$$

$$\Rightarrow dF = 2\rho g a y\sqrt{1-y/h}\,dy \Rightarrow F = 2a\rho g\int_0^h y\sqrt{1-y/h}\,dy =$$

$$= \left[\begin{array}{l|l} y=h(1-u^2) & y=h \Rightarrow u=0 \\ dy=-2hu\,du & y=0 \Rightarrow u=1 \end{array}\right] = 2a\rho g\int_1^0 h^2(1-u^2)u(-2u)\,du$$

$$= 4a\rho g h^2\int_0^1 (u^2-u^4)\,du = 4a\rho g h^2\left[\frac{u^3}{3}-\frac{u^5}{5}\right]_0^1 = \frac{8agh^2}{15};$$

Pga symmetrin faller tryckcentret på y-axeln.

$$\frac{8}{15}agh^2\cdot\bar{y} = 2ag\int_0^h y^2\sqrt{1-y/h}\,dy\left[\begin{array}{l} u=\sqrt{1-y/h} \\ y=h(1-u^2) \end{array}\ \text{osv.}\right] =$$

$$= 2ag\int_1^0 h^3(1-u^2)^2u(-2u)\,du = 4agh^3\int_0^1(u^2-2u^4+u^6)\,du =$$

$$= 4agh^3\left[\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right]_0^1 = 4agh^3\left(\frac{1}{3}+\frac{1}{7}-\frac{2}{5}\right) = \frac{32}{105}agh^3$$

$$\Leftrightarrow \bar{y} = \frac{32agh^3}{105}\Big/\frac{8agh^2}{15} = \frac{4h}{7} = 0{,}571h.$$

Resultat: Den totala tryckkraften på luckan är $\frac{8}{15}agh^2$; dess angreppspunkt faller på symmetriaxeln och på avståndet $\frac{4h}{7}$ från den raka kanten.

Integraler och summor

Övning 7.47 (Sid. 138)

Lösning

(Se sid. 341 i grundboken)

$f(x) = 1/\sqrt{x}$, $1 \le x \le 400$, är strängt avtagande.

$$\int_1^{400} x^{-1/2}dx + \frac{1}{\sqrt{400}} \le \sum_{k=1}^{400}\frac{1}{\sqrt{k}} \le \int_1^{400} x^{-1/2}dx + \frac{1}{\sqrt{1}} \Leftrightarrow$$

$$\Leftrightarrow 2(20-1)+0{,}05 \le \sum_{k=1}^{400}\frac{1}{\sqrt{k}} \le 2(20-1)+1 \Leftrightarrow$$

$$\Leftrightarrow 38{,}05 \le \sum_{k=1}^{400}\frac{1}{\sqrt{k}} \le 39 \Rightarrow 35 \le \sum_{k=1}^{400}\frac{1}{\sqrt{k}} \le 40.$$

Övning 7.48 (Sid. 138)

Lösning

$f(x) = \frac{1}{\sqrt{x}(x+1)}$, $x \ge 1$, är positiv och avtagande så Sats 1, sid. 341, ger

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}(k+1)} \le \frac{1}{\sqrt{1}(1+1)} + \int_1^n \frac{dx}{\sqrt{x}(x+1)}\left[\begin{array}{l|l} x=t^2 & x=n \Rightarrow t=n^2 \\ dx=2t\,dt & x=1 \Rightarrow t=1 \end{array}\right]$$

$$= \frac{1}{2} + \int_1^{n^2}\frac{2}{t^2+1}dt = \frac{1}{2} + \left[2\arctan t\right]_1^{n^2} = 2\arctan(n^2) + \frac{1}{2} -$$

$$- 2\arctan 1 = \frac{1}{2} - \frac{\pi}{2} + 2\arctan(n^2) \xrightarrow[n\to\infty]{} \frac{1}{2} + \frac{\pi}{2} = \frac{\pi+1}{2}.$$

Övning 7.49 (Sid. 138)

Lösning

$f(x) = \ln x/x^2$, $x > 1$, är positiv och avtagande, så Sats 1 (sid. 341) ger

$$\sum_{k=1}^{n}\frac{\ln k}{k^2}\le\int_1^n\frac{\ln x}{x^2}dx+\frac{\ln 1}{1^2}=\left[-\frac{\ln x}{x}\right]_1^n+\int_1^n\frac{1}{x^2}dx=-\frac{\ln n}{n}+$$
$$+\left[-\frac{1}{x}\right]_1^n=1-\frac{\ln n+1}{n}\xrightarrow[n\to\infty]{}1\le\frac{2+3\ln 2}{4}\approx 1{,}019.$$

Blandade problem

Övning 7.50 (Sid. 139)

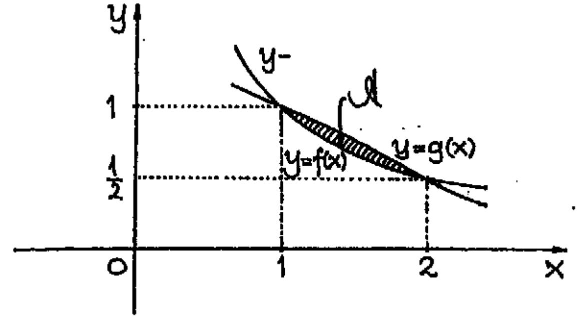
Lösning

Jag sätter $f(x)=\frac{1}{x}$ och $g(x)=\frac{3}{2+x^2}$, $x>0$.

För att bestämma integrationsgränserna så sätter jag y-koordinaterna lika!

$f(x)=\frac{1}{x}=\frac{3}{2+x^2}=g(x)\Leftrightarrow x^2+2=3x\Leftrightarrow \underline{x=1}\vee\underline{x=2}.$

x	0,5	1,2	1,5	1,7	1,9
f(x)	2	0,83	0,67	0,59	0,53
g(x)	1,33	0,87	0,71	0,61	0,53



$$A=\int_1^2(g(x)-f(x))\,dx=\int_1^2\left(\frac{3}{2+x^2}-\frac{1}{x}\right)dx=\left[\frac{3}{\sqrt2}\arctan\frac{x}{\sqrt2}-\right.$$
$$\left.-\ln x\right]_1^2=\frac{3}{\sqrt2}\left(\arctan\sqrt2-\arctan\frac{1}{\sqrt2}\right)-\ln 2\approx 0{,}028\text{ ae.}$$

Övning 7.51 (Sid. 139)

Lösning

a) $r=\sin^3\frac{\theta}{3}\Rightarrow\frac{dr}{d\theta}=3\sin^2\frac{\theta}{3}\cdot\cos\frac{\theta}{3}\cdot\frac{1}{3}=\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}\Rightarrow$

$\Rightarrow r^2+\left(\frac{dr}{d\theta}\right)^2=\sin^4\frac{\theta}{3}\cdot\cos^2\frac{\theta}{2}+\sin^6\frac{\theta}{3}=\sin^4\frac{\theta}{3}\left(\cos^2\frac{\theta}{2}+\right.$

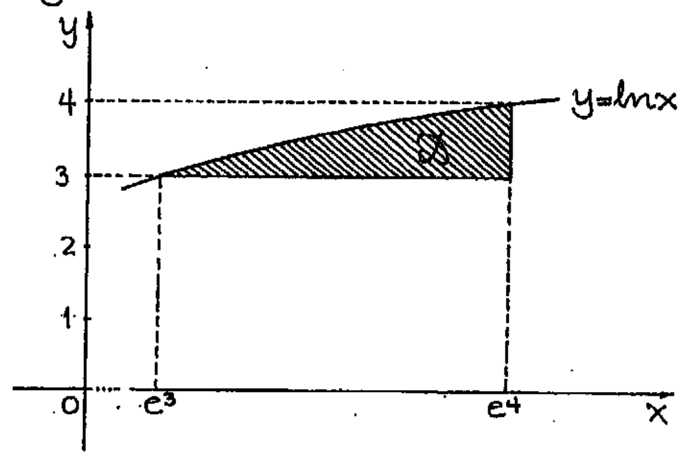
$\left.+\sin^2\frac{\theta}{3}\right)=\sin^4\frac{\theta}{3}\Rightarrow ds=\left(\left(\frac{dr}{d\theta}\right)^2+r^2\right)^{1/2}d\theta=\sin^2\frac{\theta}{3}d\theta=$

$=\frac{1}{2}\left(1-\cos\frac{2\theta}{3}\right)d\theta\Rightarrow s=\int_0^{3\pi}\frac{1}{2}\left(1-\cos\frac{2\theta}{3}\right)d\theta=\frac{3\pi}{2}\approx 4{,}712\text{ le.}$

b) $r=\sin^3\frac{\theta}{3}$, $0\le\theta\le 3\pi$, finns uppritad i facit...

Övning 7.52 (Sid. 139)

Lösning



deras produkt z^2-2z+3. Divisionen ger

$$\begin{array}{rl} z^4 + 4 & \\ \hline z^6-2z^5+3z^4+4z^2-8z+12 & \big\lfloor\, \underline{z^2-2z+3} \\ (-)\ \underline{z^6-2z^5+3z^4 } & \\ 4z^2-8z+12 & \\ (-)\ \underline{4z^2-8z+12} & \\ 0 & \end{array}$$

c) $z^4+4=0 \Leftrightarrow z^2=\pm 2i=\pm(1+i)^2 \Leftrightarrow \begin{cases} z=\pm(1+i) \\ z=\pm i(1+i) \end{cases}$

<u>Resultat</u>: a) $1-i\sqrt{2}$, b) z^2-2z+3; c) $1+i$, $1-i$, $-1+i$, $-1-i$.

$\mu(\mathcal{D}) = \int_{e^3}^{e^4} (\ln x - 3)\,dx = [x\ln x - 4x]_{e^3}^{e^4} = 4e^4 - 4e^4 -$
$-3e^3 + 4e^3 = e^3 \approx 20{,}10$ ae.

Övning 7.53 (Sid. 139)

Lösning

a) $V = \pi\int_0^1 e^{2x}dx = \frac{\pi}{2}(e^2-1)$; $(dV = \pi y^2 dx)$.

b) $\rho_0 V x_T = \int_0^1 x\,dm = \int_0^1 x\cdot\rho_0\,dV = \int_0^1 x\rho_0\pi(e^x)^2dx =$
$= \pi\rho_0\int_0^1 xe^{2x}dx = \pi\rho_0\left[\frac{1}{2}xe^{2x}\right]_0^1 - \pi\rho_0\cdot\frac{1}{2}\int_0^1 e^{2x}dx =$
$= \frac{1}{2}\pi\rho_0 e^2 - \frac{1}{4}\pi\rho_0(e^2-1) = \frac{1}{4}\pi\rho_0(2e^2-e^2+1) = \frac{1}{4}\pi\rho_0(e^2+1)$
$\Leftrightarrow \frac{1}{2}\pi\rho_0(e^2-1)\cdot x_T = \frac{1}{4}\pi\rho_0(e^2+1) \Leftrightarrow \underline{x_T = \frac{1}{2}\cdot\frac{e^2+1}{e^2-1}} \approx 0{,}657$.

Resultat: a) Volymen av K är $\frac{\pi}{2}(e^2-1) \approx 10{,}0$ ve.
b) K:s tyngdpunkt ligger på rotationsaxeln (och) 0,66 l.e. från den mindre plana ytan.

Övning 7.54 (Sid. 139)

Lösning

$F = kx$ (Hooke's lag); k fjäderkonstanten.

$250 = k\cdot 0{,}05 \Leftrightarrow \underline{k = 5000} \Rightarrow W = \int_0^{0,1} kx\,dx = \frac{1}{2}k\cdot 0{,}1^2 =$
$= \frac{1}{2}\cdot 5000\cdot 0{,}01 = \underline{25\text{ Nm}}$.

Övning 7.55 (Sid. 140)

Lösning

$W = \int_R^\infty \frac{mgR^2}{x^2}dx = mgR^2 \lim_{X\to\infty}\left[-\frac{1}{x}\right]_R^X = mgR$.

Energiprincipen ger (W_k = kinetisk energi)

$W_k = W \Rightarrow \frac{1}{2}mv^2 = mgR \Leftrightarrow v^2 = 2gR \Rightarrow v = \sqrt{2gR}$.

Övning 7.56 (Sid. 140)

Lösning

Volymen

$y = \sin x \Rightarrow dV = \pi y^2dx = \pi\sin^2x\,dx = \frac{\pi}{2}(1-\cos 2x)dx$
$\Rightarrow V = \frac{\pi}{2}\int_0^\pi (1-\cos 2x)dx = \frac{\pi}{2}\cdot\pi = \frac{\pi^2}{2} \approx 4{,}93$ ve.

Arean

$d\sigma = 2\pi y\,ds = 2\pi y\sqrt{y'^2+1}\,dx = 2\pi\sin x\sqrt{\cos^2x+1}\,dx$

$\Rightarrow S = 2\pi\int_0^\pi \sqrt{\cos^2x+1}\cdot\sin x\,dx \left[\begin{array}{l|l} t = \cos x & \pi \to -1 \\ dt = -\sin x\,dx & 0 \to 1 \end{array}\right] =$
$= 2\pi\int_1^{-1}\sqrt{t^2+1}(-dt) = 2\pi\int_{-1}^1\sqrt{t^2+1}\,dt = 2\pi\int_0^1 2\sqrt{t^2+1}\,dt =$
$= 2\pi\left[t\sqrt{t^2+1} + \ln(t+\sqrt{t^2+1})\right]_0^1 = 2\pi(\sqrt{2}+\ln(1+\sqrt{2})) = 14{,}42$.

Svar: Volymen är $\frac{\pi^2}{2}$ och arean $2\pi(\sqrt{2}+\ln(1+\sqrt{2}))$.

Övning 7.57 (Sid. 140)

Lösning

Vi arbetar i SI-enheter och då blir $1\,dm^3 =$

$= (1\,dm)^3 = (10^{-1}\,m)^3 = 10^{-3}\,m^3$ samt $1\,dm^2 = 10^{-2}\,m^2$.

$P_0 = 2000\,N/dm^2 = 2000 \cdot 10^{-2}\,N/m^2 = 20\,N/m^2$;

$V_0 = 10\,dm^3 = 10^{-2}\,m^3$ och $V_1 = 5\,dm^3 = 0{,}005\,m^3$

$pV^{1,4} = k = P_0 V_0 = 2 \cdot 10^{2,2} \Leftrightarrow p = 2 \cdot 10^{2,2} \cdot V^{-1,4} \Rightarrow$

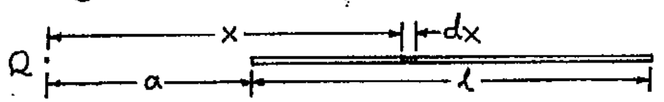
$\Rightarrow W = -\int_{V_0}^{V_1} p\,dV = -2 \cdot 10^{2,2} \int_{0,01}^{0,05} V^{-1,4}\,dV = \left[\frac{5 \cdot 10^{2,2}}{V^{0,4}}\right]_{0,01}^{0,005} =$

$= 5 \cdot 10^{2,2}(5^{-0,4} \cdot 10^{1,2} - 10^{0,8}) = 1{,}6 \cdot 10^3$.

Svar: Det erforderliga arbetet är 1,6 kJ.

Övning 7.58 (Sid. 140)

Lösning



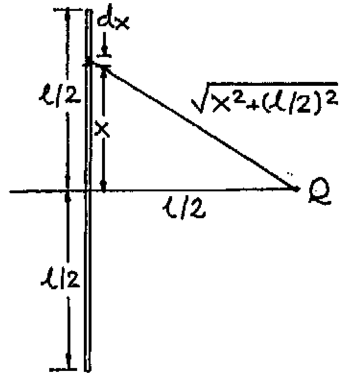
Laddningstätheten i staven är q så elementet ovan bär laddningen $dQ = q\,dx$.

$dF = -k \cdot \frac{Q \cdot q}{x^2}\,dx \Rightarrow F = -qQ \cdot k \int_a^{a+l} \frac{dx}{-x^2} = \frac{klQq}{a(a+l)}$

Det här tas normalt upp i elläran.

Övning 7.59 (Sid. 140)

Lösning



$dF = -kQ \cdot \frac{q\,dx}{r^2} = -kQq \frac{dx}{x^2+(l/2)^2} \Rightarrow F = -kQq \int_{-l/2}^{l/2} \frac{dx}{x^2+l^2/4} =$

$= -\frac{2kQq}{l}\left[\arctan\frac{2x}{l}\right]_{-l/2}^{l/2} = -\frac{4kQq}{l}\arctan 1 = -\frac{kqQ\pi}{l}$.

Minustecknet anger att kraften är attraktiv.

Övning 7.60 (Sid. 140)

Lösning

$\left.\begin{cases} x = \cos 3t + 3\cos t \Rightarrow \frac{dx}{dt} = -3\sin 3t - 3\sin t \\ y = \sin 3t + 3\sin t \Rightarrow \frac{dy}{dt} = 3\cos 3t + 3\cos t \end{cases}\right\} \Rightarrow \dot{x}^2 + \dot{y}^2 =$

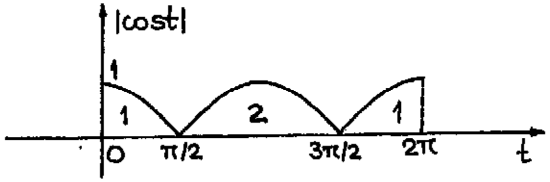
$= 9 + 9 + 18\sin 3t \sin t + 18\cos 3t \cos t = 18(1 + \cos 3t \cos t +$

$+ \sin 3t \sin t) = 18(1 + \cos(3t - t)) = 18(1 + \cos 2t) = 36\cos^2 t$

$\Rightarrow ds = \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = 6|\cos t|\,dt \Rightarrow s = \int_0^{2\pi} 6|\cos t|\,dt =$

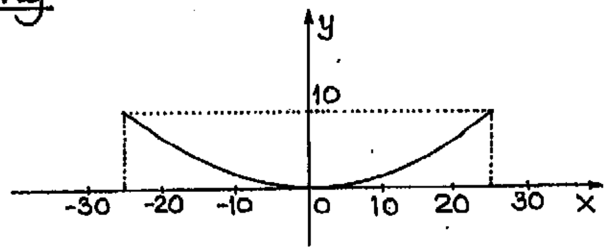
$= (\text{Se fig.}) = 6\left(\int_0^{\pi/2} + \int_{\pi/2}^{3\pi/2} + \int_{3\pi/2}^{2\pi}\right)|\cos t|\,dt = 6\int_0^{\pi/2}\cos t\,dt +$

$+ 6\int_{\pi/2}^{3\pi/2}(-\cos t)\,dt + 6\int_{3\pi/2}^{2\pi}\cos t\,dt = 6\cdot 1 + 6\cdot 2 + 6\cdot 1 = 24\text{ le.}$



Övning 7.61 (Sid. 141)

Lösning



$y = kx^2 \Rightarrow 10 = k\cdot 25^2 \Leftrightarrow k = 0{,}016;$

$y' = 2kx \Rightarrow ds = \sqrt{(2kx)^2+1}\,dx \Rightarrow s = \int_{-25}^{25}\sqrt{(2kx)^2+1}\,dx =$

$= 2\int_0^{25}\sqrt{(2kx)^2+1}\,dx \left[\begin{array}{l|l} x = t/2k & x = 25 \Rightarrow t = 50k \\ dx = dt/2k & x = 0 \Rightarrow t = 0 \end{array}\right] =$

$= \frac{1}{k}\int_0^{50k}\sqrt{t^2+1}\,dt = \frac{1}{2k}\left[t\sqrt{t^2+1} + \ln(t+\sqrt{t^2+1})\right]_0^{50k} =$

$= 25\sqrt{(50k)^2+1} + \frac{1}{2k}\ln(50k + \sqrt{(50k)^2+1}) = 25\sqrt{1{,}64} +$

$+ \frac{125}{4}\ln(0{,}8 + \sqrt{1{,}64}) \approx 55$ meter.

Anm. Det som ges som facit är ett skämt.

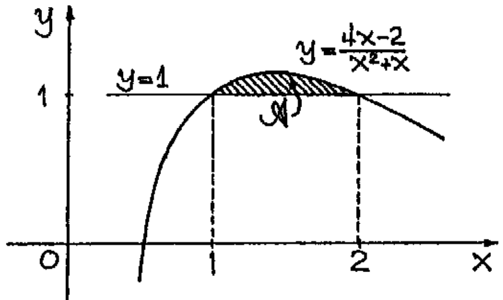
Övning 7.62 (Sid. 141)

Lösning

$f(x) = 1 \Rightarrow \frac{4x-2}{x^2+x} = 1 \Leftrightarrow 4x - 2 = x^2 + x \Leftrightarrow x^2 - 3x + 2 = 0 \Leftrightarrow$

$\Leftrightarrow \underline{x=1} \vee \underline{x=2}$ (integrationsgränserna).

x	0,8	1,2	1,4	1,6	1,8
$f(x)$	0,83	1,07	1,07	1,06	1,03



(i) $\mathcal{A} = \int_1^2\left(\frac{4x-2}{x^2+x} - 1\right)dx = \int_1^2\left(\frac{6}{x+1} - \frac{2}{x} - 1\right)dx = \left[6\ln(x+1) - 2\ln x - x\right]_1^2 =$

$= 6\ln 3 - 2\ln 2 - 2 - 6\ln 2 + 1 = 6\ln\frac{3}{2} - 2\ln 2 - 1 \approx 0{,}046\,m^2.$

(ii) $J = \int_K r^2\,dm = \int_K r^2\cdot\sigma\cdot dA = \int_1^2 x^2\cdot\frac{\overline{m}}{\mathcal{A}}\cdot\left(\frac{4x-2}{x^2+x} - 1\right)dx =$

$= \frac{\overline{m}}{\mathcal{A}}\cdot\int_1^2\left(4x - 6 + \frac{6}{x+1} - x^2\right)dx = \frac{\overline{m}}{\mathcal{A}}\left[2x^2 - 6x + 6\ln(x+1) - \frac{x^3}{3}\right]_1^2 =$

$= \frac{\overline{m}}{\mathcal{A}}\left(8 - 12 + 6\ln 3 - \frac{8}{3} - 2 + 6 - 6\ln 2 + \frac{1}{3}\right) = \left(6\ln\frac{3}{2} - \frac{7}{3}\right)\frac{\overline{m}}{\mathcal{A}}.$

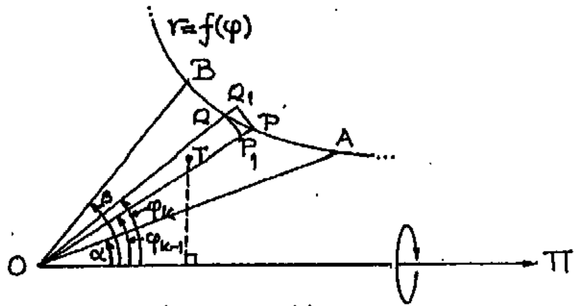
Svar: Arean är $4{,}6\,dm^2$; tröghetsmomentet är $(2{,}16\overline{m})\,kgm^2$, där $\overline{m}$ är ytstyckets massa.

Anm. De exakta värdena ges i facit.

Övning 7.63 (Sid. 141)

Lösning

Jag kommer att lösa problemet med hjälp av polära koordinater, så jag härleder motsvarande formler för rotationsvolymer. Polära systemet definieras av polen O och polaraxeln Π.



Sektorn OAB roterar ett varv kring axeln Π.
Kurvbågen $\widehat{AB}$ har ekvationen $r = f(\varphi)$, $\alpha \leq \varphi \leq \beta$.
Denna antas vara kontinuerlig.
Jag tar och delar intervallet $[\alpha, \beta]$ genom punkterna $\alpha = \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}, \varphi_n = \beta$. Antag att

$$M_k = \max\{f(\varphi) : \varphi_{k-1} \leq \varphi \leq \varphi_k\} \quad (\text{Jfr } OP)$$

och

$$m_k = \min\{f(\varphi) : \varphi_{k-1} \leq \varphi \leq \varphi_k\} \quad (\text{Jfr } OQ).$$

Sektorn OPQ som bestäms av $[\varphi_{k-1}, \varphi_k]$ ligger helt inom cirkelsektorn OPQ_1 vars radie är M_k och omfattar cirkelsektorn OP_1Q vars radie är m_k. Den större cirkelsektorn alstrar vid rotationen en kropp vars volym är

$$\frac{2}{3}\pi M_k (\cos\varphi_{k-1} - \cos\varphi_k) = \frac{2}{3} M_k \cdot \sin\hat{\varphi} \cdot \Delta\varphi_k,$$

med $\hat{\varphi} = \arg OT =$ argumentet för tyngdpunkten T; detta enligt differentialkalkylens medelvärdessats; $\Delta\varphi_k = \varphi_k - \varphi_{k-1}$. OPQ alstrar således en volym, vars storlek ligger mellan

$$\frac{2}{3}\pi M_k^3 \sin\hat{\varphi} \cdot \Delta\varphi_k \quad \text{och} \quad \frac{2}{3}\pi m_k^3 \sin\hat{\varphi} \cdot \Delta\varphi_k.$$

Vid summation från $k=1$ till $k=n$ erhålles att ytstycket OAB alstrar en rotationsfigur, vars volym ligger mellan

$$\sum_{k=1}^{n} \frac{2\pi}{3} M_k^3 \sin\hat{\varphi}\, \Delta\varphi_k \quad \text{och} \quad \sum_{k=1}^{n} \frac{2\pi}{3} m_k^3 \sin\hat{\varphi}\, \Delta\varphi_k.$$

Emedan bägge dessa summor har samma gränsvärde, nämligen $\frac{2\pi}{3}\int_\alpha^\beta r^3 \sin\varphi\, d\varphi$, då intervallindelningen görs tätare, s.a. $\max \Delta\varphi_k \to 0$ kan vi formulera följande:

Sats: Då ytan mellan kurvan $r=f(\varphi), \alpha\le\varphi\le\beta$, som ligger helt på ena sidan om polaraxeln och radierna med argumenten α och β, roterar ett varv kring polaraxeln, alstras en kropp, vars volym är

$$V=\frac{2\pi}{3}\int_\alpha^\beta r^3\sin\varphi\,d\varphi. \qquad (1)$$

Anm. Rotationen kan även ske kring en annan linje (genom polen) än polaraxeln. Om denna linje bildar vinkeln γ med polaraxeln blir formeln

$$V=\frac{2\pi}{3}\int_\alpha^\beta r^3\sin(\varphi-\gamma)\,d\varphi.$$

V är alltid positiv, så även

$$V=\frac{2\pi}{3}\int_\alpha^\beta r^3\sin(\gamma-\varphi)\,d\varphi$$

kan användas.

Om man anpassar systemet så att polen O förläggs till origo i ett cartesiskt xy-system och polaraxeln längs den positiva x-axeln så har vi för $\gamma=\pi/2$ (y-axeln)

$$V=\frac{2\pi}{3}\int_\alpha^\beta r^3\cos\varphi\,d\varphi. \qquad (2)$$

Så långt med "förberedelserna"!

Låt oss införa polära koordinater (r,φ); vi har $x=r\cos\varphi$ och $y=r\sin\varphi$. (*)

$(x^2+y^2)^2=4(x^2-y^2) \Rightarrow (*) \Rightarrow r^4=4r^2\cos2\varphi \Leftrightarrow r=f(\varphi)=$

$=2\sqrt{\cos2\varphi}\ ;\ 0<2\varphi\le\frac{\pi}{2} \Leftrightarrow \underline{0\le\varphi\le\pi/4}.$

(i) $dV=\frac{2\pi}{3}r^3\sin\varphi\,d\varphi=\frac{2\pi}{3}\cdot8(2\cos^2\varphi-1)^{3/2}\sin\varphi\,d\varphi,\ 0\le\varphi\le\frac{\pi}{4};$

$$V=\frac{16\pi}{3}\int_0^{\pi/4}(2\cos^2\varphi-1)^{3/2}\sin\varphi\,d\varphi\left[\begin{array}{l|l} u=\sqrt{2}\cos\varphi & \frac{\pi}{4}\to1 \\ du=-\sqrt{2}\sin\varphi\,d\varphi & 0\to\sqrt{2}\end{array}\right]$$

$$=\frac{16\pi}{3}\int_{\sqrt{2}}^1(u^2-1)^{3/2}\left(-\frac{du}{\sqrt{2}}\right)=\frac{16\pi}{3\sqrt{2}}\int_1^{\sqrt{2}}(u^2-1)^{3/2}du=\ldots=$$

$$=\frac{2\pi}{3\sqrt{2}}\left[2u(u^2-1)^{3/2}-3u(u^2-1)^{1/2}+3\ln\left(u+\sqrt{u^2-1}\right)\right]_1^{\sqrt{2}}=$$

$$=\frac{2\pi}{3\sqrt{2}}\left(2\sqrt{2}-3\sqrt{2}+3\ln(1+\sqrt{2})\right)=\underline{\frac{2\pi}{3\sqrt{2}}\left(3\ln(1+\sqrt{2})-\sqrt{2}\right)};$$

(ii) $x\,dm=x\cdot1\cdot dV=r\cos\varphi\cdot\frac{2\pi}{3}r^3\cdot\sin\varphi\,d\varphi=\frac{\pi}{3}r^4\sin2\varphi\,d\varphi$

$=\frac{\pi}{3}\cdot4r^2\cos2\varphi\sin2\varphi\,d\varphi=\frac{16\pi}{3}\cos^22\varphi\cdot\sin2\varphi\,d\varphi;$

$$V\cdot x_T=\int_0^{\pi/4}\frac{16\pi}{3}\cos^22\varphi\sin2\varphi\,d\varphi\left[\begin{array}{l} u=\cos2\varphi \\ \sin2\varphi\,d\varphi=-\frac{du}{2}\end{array}\right]=$$

$$=\frac{16\pi}{3}\int_1^0u^2\left(-\frac{du}{2}\right)=\frac{8\pi}{3}\int_0^1u^2du=\frac{8\pi}{3}\left[\frac{u^3}{3}\right]_0^1=\frac{8\pi}{9}\Leftrightarrow$$

$$\Leftrightarrow x_T=\frac{8\pi}{9}\cdot\frac{3\sqrt{2}}{2\pi}\cdot\frac{1}{3\ln(1+\sqrt{2})-\sqrt{2}}=\frac{4\sqrt{2}/3}{3\ln(1+\sqrt{2})-\sqrt{2}}\approx1{,}533.$$

$\bar{r}_T=(x_T,y_T,z_T)=(1{,}5331,0,0)$ och inte ...

Övning 7.65 (Sid. 142)

Lösning

$d\Phi = v\cdot dA = v\cdot 2\pi r dr = 2\pi k(R^2-r^2)r dr,\ 0\le r\le R;$

$\Phi = 2\pi k\int_0^R (R^2r-r^3)dr = 2\pi k\left[\frac{1}{2}R^2r^2-\frac{1}{4}r^4\right]_0^R = \frac{\pi k R^4}{2}.$

Anm. Φ står för "flöde".

Övning 7.66 (Sid. 142)

Lösning

Antag att det började snöa t_0 timmar före kl. 12.00, som sätts $t=0$ på tidsaxeln. Om snödjupet ökar med en takt s m/h, så är $x=s(t+t_0)$; modellen är $\frac{dy}{dt}=\frac{k}{x}$, så att vi får

$\frac{dy}{dt}=\frac{a}{t+t_0},\ t>t_0;\ y(0)=0.$

Ekvationen integreras och vi får

$$y=a\cdot\ln(t+t_0)+b,\ b\in\mathbb{R}.$$

$y(0)=0 \Rightarrow a\cdot\ln t_0+b=0 \Rightarrow y=a\cdot\ln(t+t_0)-a\ln t_0 =$

$= a\ln(1+\frac{t}{t_0}) \Rightarrow \left.\begin{cases} y(1)=y_1 \Rightarrow a\ln(1+\frac{1}{t_0})=y_1 \\ y(2)=\frac{1}{2}y_1 \Rightarrow a\ln(1+\frac{2}{t_0})=\frac{3}{2}y_1\end{cases}\right\} \Rightarrow$

$\Rightarrow a\ln(1+\frac{1}{t_0})=\frac{3}{2}a\ln(1+\frac{2}{t_0}) \Leftrightarrow 2\ln(1+\frac{1}{t_0})=3\ln(1+\frac{2}{t_0})$

$\Leftrightarrow \ln(1+\frac{1}{t_0})^2=\ln(1+\frac{2}{t_0})^3 \Leftrightarrow (1+\frac{1}{t_0})^2=(1+\frac{2}{t_0})^3 \Leftrightarrow$

$\Leftrightarrow t_0(t_0+1)^2=(t_0+2)^3 \Leftrightarrow t_0^3+2t_0^2+t_0=t_0^3+3t_0^2+3t_0+1 \Leftrightarrow$

$\Leftrightarrow t_0^2+t_0-1=0 \Leftrightarrow t_0=-\frac{1}{2}+\frac{\sqrt{5}}{2}\approx 0{,}618$ (timmar)

Svar: Det började snöa kl. 11.23.

Övning 7.67 (Sid. 142)

Lösning

Klotets volym är $V=V_0=\frac{4\pi}{3}R^3$ (hela klotet).

Det stympade klotet (utan hålet) fås genom kapning av 2 identiska kalotter av höjden $h=\frac{2R-6}{2}=R-3$. Dessa sammanlagda volym är

$V=V_1=2\int_3^R \pi(R^2-x^2)dx=2\pi\left[R^2x-\frac{x^3}{3}\right]_3^R=2\pi\left(\frac{2R_3}{3}-\right.$

$\left.-3R^2+9\right)=\frac{4\pi}{3}R^3-6\pi(R^2-3)=V_0-6\pi(R^2-3);$

Den urborrade cylinder har radien $\sqrt{R^2-9}$; dess volym är $V_2=\pi(R^2-9)\cdot 6=6\pi R^2-54\pi;$

Det som återstår av klotet har volymen

$V_0-V_1-V_2=\frac{4\pi R^3}{3}-\frac{4\pi R^3}{3}+6\pi(R^2-3)-6\pi(R^2-9)=36\pi.$

Resultat: Den ringformade kroppens volym är 36π cm^2 (oberoende av klotets radie).

Övning 7.68 (Sid. 142)

Lösning

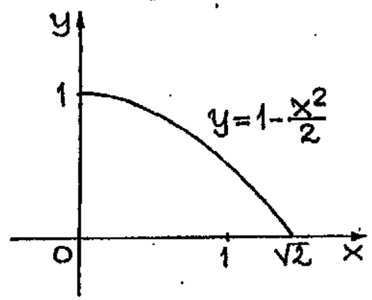
$r=\theta^2 \Rightarrow ds=\sqrt{r'^2+r^2}\,d\theta=\theta\sqrt{\theta^2+4}\,d\theta \Rightarrow s=\int_0^2\sqrt{\theta^2+4}\,\theta\,d\theta=$

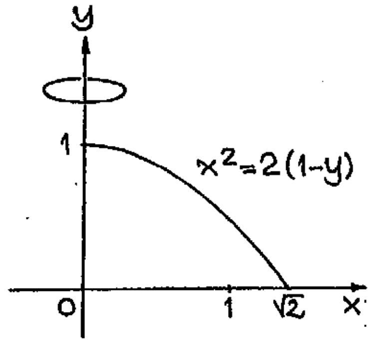
$=\left[\begin{array}{l|l}\theta^2+4=u^2 & 2\to\sqrt{8}\\ \theta d\theta=u\,du & 0\to 2\end{array}\right]=\int_2^{2\sqrt{2}}u\cdot u\,du=\int_2^{\sqrt{8}}u^2du=\frac{8\sqrt{8}}{3}-$

$-\frac{8}{3}=\frac{8}{3}(2\sqrt{2}-1)\approx 4{,}876$ le.

Övning 7.69 (Sid. 142)

Lösning





a) $y=1-\frac{1}{2}x^2 \Rightarrow y'^2+1=x^2+1 \Rightarrow ds=\sqrt{x^2+1}\,dx,\ 0\le x\le\sqrt{2};$

$S=\int_0^{\sqrt{2}}\sqrt{x^2+1}\,dx=\frac{1}{2}\left[x\sqrt{x^2+1}+\ln(x+\sqrt{x^2+1})\right]_0^{\sqrt{2}}=\frac{1}{2}(\sqrt{6}+$

$+\ln(\sqrt{2}+\sqrt{3}))\approx 1{,}798$ le.

b) $V=\int_0^1\pi x^2dy=\pi\int_0^1 2(1-y)dy=\pi\left[-(1-y)^2\right]_0^1=\pi$ ve.

Anm. $V=\int_0^{\sqrt{2}}2\pi xy\,dx$ kan också användas.

Övning 7.70 (Sid. 143)

Lösning

$\tau(x)=kx$, k konstant ; $\tau(1)=1 \Rightarrow k=1 \Rightarrow \tau(x)=x.$

$y=x^2 \Rightarrow y'=2x \Rightarrow ds=\sqrt{y'^2+1}\,dx=\sqrt{(2x)^2+1}\,dx$;

$dm=\tau ds=x\sqrt{(2x)^2+1}\,dx \Rightarrow m=\int_1^4\sqrt{(2x)^2+1}\cdot x\,dx=$

$=\frac{1}{4}\int_1^4(2x)\sqrt{(2x)^2+1}\,d(2x)\left[\begin{array}{l|l}u=2x & x=4\Rightarrow u=8\\ du=d(2x) & x=1\Rightarrow u=2\end{array}\right]=$

$=\frac{1}{4}\int_2^8 u\sqrt{u^2+1}\,du=\left[\frac{1}{12}(u^2+1)^{3/2}\right]_2^8=\frac{65^{3/2}-5^{3/2}}{12}\approx 42{,}74.$

Svar: Trådens massa är 42,74 kg.

Övning 7.71 (Sid. 143)

Lösning

a) $y=x^{1/2} \Rightarrow y'=\frac{1}{2\sqrt{x}} \Rightarrow y'^2+1=\frac{1}{4}\frac{4x+1}{x} \Rightarrow ds=\frac{\sqrt{4x+1}}{2\sqrt{x}}dx$

$\Rightarrow d\sigma=2\pi y\,ds=\pi\sqrt{4x+1}\,dx \Rightarrow S=\pi\int_0^1\sqrt{4x+1}\,dx=$

$=\left[\frac{\pi}{6}(4x+1)^{3/2}\right]_0^1=\frac{\pi}{6}(5\sqrt{5}-1)\approx 5{,}33$ ae.

b) $x_T=\frac{1}{m}\int x\,dm=\frac{1}{\rho S}\int x\rho\,d\sigma=\frac{1}{S}\int_0^1 x\cdot\pi\sqrt{4x+1}\,dx=$

$=\left[\begin{array}{c}t=\sqrt{4x+1}\Leftrightarrow x=(t^2-1)/4\\ dx=t\,dt/2\\ x=0\Rightarrow t=1;\ x=1\Rightarrow t=\sqrt{5}\end{array}\right]=\frac{\pi}{8S}\int_1^{\sqrt{5}}(t^2-1)t\cdot t\,dt=$

$=\frac{\pi}{8S}\left[\frac{t^5}{5}-\frac{t^3}{3}\right]_1^{\sqrt{5}}=\frac{\pi}{8S}\left(5\sqrt{5}-\frac{5\sqrt{5}}{3}-\frac{1}{5}+\frac{1}{3}\right)=\frac{313+15\sqrt{5}}{630}\approx 0{,}55.$

8. Differentialekvationer

Linjära ekvationer av första ordningen

Övning 8.1 (Sid. 156)

Lösning

$y' = \frac{dy}{dx} = x^2 - e^{-x} \Leftrightarrow dy = (x^2 - e^{-x})dx \Rightarrow \int_1^y d\eta = \int_0^x (\xi^2 - e^{-\xi})d\xi$

$\Leftrightarrow [\eta]_1^y = [\frac{1}{3}\xi^3 + e^{-\xi}]_0^x \Leftrightarrow y - 1 = \frac{1}{3}x^3 + e^{-x} - 1 \Leftrightarrow \underline{y = \frac{1}{3}x^3 + e^{-x}}$.

Övning 8.2 (Sid. 156)

Lösning

a) $\underline{xy' = \ln x,\ y(1) = 2.}$

$x\frac{dy}{dx} = \ln x \Leftrightarrow dy = \frac{\ln x}{x}dx \Rightarrow \int_2^y d\eta = \int_1^x \frac{\ln \xi}{\xi}d\xi \Leftrightarrow$

$\Leftrightarrow [\eta]_2^y = [\frac{1}{2}\ln^2 \xi]_1^x \Leftrightarrow y - 2 = \frac{1}{2}\ln^2 x \Leftrightarrow \underline{y = 2 + \frac{1}{2}\ln^2 x}$.

b) $\underline{y'' = 4e^{2x} + x^2,\ y(0) = 2,\ y'(0) = 1.}$

$y' = 2e^{2x} + \frac{1}{3}x^3 + C_1$; $y'(0) = 1 \Rightarrow 2 + C_1 = 1 \Leftrightarrow C_1 = -1$;

$y' = 2e^{2x} + \frac{1}{3}x^3 - 1 \Leftrightarrow y = e^{2x} + \frac{1}{12}x^4 - x + C \Rightarrow y(0) = 1 + C$;

$y(0) = 2 \Rightarrow 1 + C_2 = 2 \Leftrightarrow C_2 = 1$.

<u>Resultat</u>: $y = e^{2x} + \frac{1}{12}x^4 - x + 1$.

Övning 8.3 (Sid. 156)

Lösning

Se facit (s. 178).

Övning 8.4 (Sid. 156)

Lösning

$y' - 2y = 3 \Rightarrow g(x) = -2 \Rightarrow G(x) = -2x \Rightarrow \mu(x) = e^{-2x}$ I.F.

$y'e^{-2x} - 2e^{-2x}y = 3e^{-2x} \Leftrightarrow (ye^{-2x})' = 3e^{-2x} \Leftrightarrow ye^{-2x} =$

$= -\frac{3}{2}e^{-2x} + C \Leftrightarrow \underline{y = Ce^{2x} - \frac{3}{2}}$.

Övning 8.5 (Sid. 156)

Lösning

a) $y' + 2y = 0 \Rightarrow g(x) = 2 \Rightarrow G(x) = 2x \Rightarrow \mu(x) = e^{2x}$ I.F.

$y'e^{2x} + 2e^{2x}y = 0 \Leftrightarrow (ye^{2x})' = 0 \Leftrightarrow ye^{2x} = C \Leftrightarrow \underline{y = Ce^{-2x}}$.

b) $y' - 3y = 0 \Rightarrow g(x) = -3 \Rightarrow G(x) = -3x \Rightarrow \mu(x) = e^{-3x}$ I.F.

$y'e^{-3x} - 3e^{-3x}y = (ye^{-3x})' = 0 \Leftrightarrow ye^{-3x} = C \Leftrightarrow \underline{y = Ce^{3x}}$.

c) $y' - ky = 0 \Rightarrow g(x) = -k \Rightarrow G(x) = -kx \Rightarrow \mu(x) = e^{-kx}$ I.F.

$y'e^{-kx} - ke^{-kx}y = (ye^{-kx})' = 0 \Leftrightarrow ye^{-kx} = C \Leftrightarrow \underline{y = Ce^{kx}}$

d) $y' + xy = 0 \Rightarrow g(x) = x \Rightarrow G(x) = \frac{1}{2}x^2 \Rightarrow \mu(x) = e^{x^2/2}$ I.F.

$y'e^{x^2/2}+yxe^{x^2/2}=(ye^{x^2/2})'=0 \Leftrightarrow ye^{x^2/2}=C \Rightarrow \underline{y=Ce^{-x^2/2}}$

Övning 8.6 (Sid. 156)

Lösning

a) $y'+2y=1 \Rightarrow g(x)=2 \Rightarrow G(x)=2x \Rightarrow \mu(x)=e^{2x}$ I.F.

$y'e^{2x}+2e^{2x}y=e^{2x} \Leftrightarrow (ye^{2x})'=e^{2x} \Leftrightarrow ye^{2x}=\frac{1}{2}e^{2x}+C$

$\Leftrightarrow \underline{y=\frac{1}{2}+Ce^{-2x},\ C\in\mathbb{R}}.$

b) $y'-3y=1 \Rightarrow g(x)=-3 \Rightarrow G(x)=-3x \Rightarrow \mu(x)=e^{-3x}$ I.F.

$y'e^{-3x}-3e^{-3x}y=e^{-3x} \Leftrightarrow (ye^{-3x})'=e^{-3x} \Leftrightarrow ye^{-3x}=-\frac{1}{3}e^{-3x}+C$

$\Leftrightarrow \underline{y=-\frac{1}{3}+Ce^{3x},\ C\in\mathbb{R}}.$

c) $y'+(\cos x)y=4\cos x \Rightarrow g(x)=\cos x \Rightarrow G(x)=\sin x \Rightarrow$

$\Rightarrow \mu(x)=e^{\sin x}$ I.F. ; $(ye^{\sin x})'=4\cos x\, e^{\sin x} \Leftrightarrow$

$\Leftrightarrow e^{\sin x}\cdot y=4e^{\sin x}+C \Leftrightarrow \underline{y=4+Ce^{-\sin x},\ C\in\mathbb{R}}.$

d) $y'+xy=x \Rightarrow g(x)=x \Rightarrow G(x)=\frac{1}{2}x^2 \Rightarrow \mu(x)=e^{x^2/2}$ I.F.

$(ye^{x^2/2})'=xe^{x^2/2}=(e^{x^2/2})' \Leftrightarrow ye^{x^2/2}=e^{x^2/2}+C \Leftrightarrow$

$\Leftrightarrow \underline{y=1+Ce^{-x^2/2},\ C\in\mathbb{R}}.$

Övning 8.7 (Sid. 156)

Fullständig lösning på s. 178-179.

Övning 8.8 (Sid. 156)

Lösning

a) $y'+2xy=0 \Rightarrow g(x)=2x \Rightarrow G(x)=x^2 \Rightarrow \mu(x)=e^{x^2}$ I.F.

$(ye^{x^2})'=0 \Leftrightarrow ye^{x^2}=C \Leftrightarrow \underline{y=Ce^{-x^2},\ C\in\mathbb{R}}.$

b) $xy'+10y=\ln x \Rightarrow y'+\frac{10}{x}y=\frac{\ln x}{x} \Rightarrow g(x)=\frac{10}{x} \Rightarrow G(x)=$

$=10\ln x=\ln x^{10} \Rightarrow \mu(x)=e^{G(x)}=x^{10}$ I.F.

$y'\cdot x^{10}+10x^9y=x^9\ln x \Leftrightarrow (y\cdot x^{10})'=x^9\ln x \Leftrightarrow$

$\Leftrightarrow y\cdot x^{10}=\int x^9\ln x\,dx=\frac{1}{10}x^{10}\ln x-\frac{1}{10}\int x^9dx=\frac{x^{10}\ln x}{10}-$

$-\frac{x^{10}}{100}+C \Leftrightarrow \underline{y=\frac{\ln x}{10}-\frac{1}{100}+Cx^{-10},\ C\in\mathbb{R}}.$

c) $y'+y\cdot\cot x=\tan^2 x \Leftrightarrow y'+y\cdot\frac{\cos x}{\sin x}=\frac{1}{\cos^2 x}-1 \Leftrightarrow$

$\Leftrightarrow \sin x\cdot y'+\cos x\cdot y=\frac{\sin x}{\cos^2 x}-\sin x \Leftrightarrow (y\cdot\sin x)'=$

$=\left(\frac{1}{\cos x}+\cos x\right)' \Leftrightarrow y\sin x=\frac{1}{\cos x}+\cos x+C \Leftrightarrow y=$

$=\frac{1}{\sin x\cos x}+\frac{\cos x}{\sin x}+\frac{C}{\sin x} \Leftrightarrow y=\frac{2}{\sin 2x}+\cot x+\frac{C}{\sin x} \Leftrightarrow$

$\Leftrightarrow \underline{y=2\cdot\operatorname{cosec}2x+\cot x+C\cdot\operatorname{cosec}x}$ (Se sid. 111).

Övning 8.9 (Sid. 157)

Lösning

a) $y'+x^2y=x^2 \Rightarrow g(x)=x^2 \Rightarrow G(x)=\frac{x^3}{3} \Rightarrow \mu(x)=e^{x^3/3}$ I.F.

$y' \cdot e^{x^3/3} + x^2 e^{x^3/3} y = x^2 e^{x^3/3} \Leftrightarrow (y e^{x^3/3})' = (e^{x^3/3})' \Leftrightarrow$

$\Leftrightarrow y e^{x^3/3} = e^{x^3/3} + C \Leftrightarrow y = 1 + Ce^{-x^3/3} \Rightarrow y(0) = 1 + C;$

$y(0) = 2 \Rightarrow 1 + C = 2 \Leftrightarrow C = 1 \Rightarrow \underline{y = 1 + e^{-x^3/3}}.$

b) $(1-x^2)y' + xy = x \Leftrightarrow y' + \frac{x}{1-x^2} y = \frac{x}{1-x^2} \Rightarrow g(x) = \frac{x}{1-x^2} \Rightarrow$

$\Rightarrow G(x) = -\frac{1}{2}\ln(1-x^2) = \ln(1-x^2)^{-1/2} \Rightarrow \mu(x) = (1-x^2)^{-1/2}$ I.F.

$((1-x^2)^{-1/2} y)' = x(1-x^2)^{-3/2} = (+(1-x^2)^{-1/2})' \Leftrightarrow y \cdot (1-x^2)^{-1/2} =$

$= +(1-x^2)^{-1/2} + C \Leftrightarrow y = +1 + C\sqrt{1-x^2} \Rightarrow y(0) = C + 1;$ (*)

$y(0) = 3 \Rightarrow C + 1 = 3 \Leftrightarrow C = 2 \Rightarrow \underline{y = 2\sqrt{1-x^2} - 1,\ |x| < 1.}$

c) $(1+x^2)y' - 2xy = (1+x^2)\arctan x \Leftrightarrow y' - \frac{2x}{1+x^2} y = \arctan x$

$\Rightarrow g(x) = -\frac{2x}{1+x^2} \Rightarrow G(x) = -\ln(1+x^2) = \ln(1+x^2)^{-1} \Rightarrow \mu(x) =$

$= (1+x^2)^{-1}$ I.F. ; $\left(\frac{y}{x^2+1}\right)' = \frac{\arctan x}{x^2+1} = \left(\left(\frac{\arctan x}{\sqrt{2}}\right)^2\right)'$

$\Leftrightarrow \frac{y}{x^2+1} = \frac{1}{2}(\arctan x)^2 + C \Leftrightarrow y = \left(\frac{(\arctan x)^2}{2} + C\right)(x^2+1);$

$y(1) = 2 \Rightarrow \left(\frac{1}{2}(\arctan 1)^2 + C\right) \cdot 2 = 2 \Leftrightarrow C = -\frac{\pi^2}{32} + 1 \Rightarrow$

$\Rightarrow \underline{y = \frac{1}{2}(x^2+1)(\arctan x)^2 + \left(1 - \frac{\pi^2}{32}\right)(x^2+1).}$

d) $(x+1)(x+2)y' - y = 1 \Leftrightarrow y' - \frac{1}{(x+1)(x+2)} y = \frac{1}{(x+1)(x+2)} \Rightarrow$

$\Rightarrow g(x) = -\frac{1}{(x+1)(x+2)} = \frac{1}{x+2} - \frac{1}{x+1} \Rightarrow G(x) = \ln\frac{x+2}{x+1} \Rightarrow$

$\Rightarrow \mu(x) = \frac{x+2}{x+1} \Rightarrow \left(y \cdot \frac{x+2}{x+1}\right)' = \frac{1}{(x+1)^2} \Rightarrow y \cdot \frac{x+2}{x+1} = -\frac{1}{x+1} + C$

$\Leftrightarrow y = C \cdot \frac{x+1}{x+2} - \frac{1}{x+2} \Rightarrow y(0) = \frac{C}{2} - \frac{1}{2};\ y(0) = 2 \Rightarrow C = 5;$

Resultat: $y = \frac{5x+4}{x+2},\ x > -1.$

Övning 8.10 (Sid. 157)

Lösning

$\sqrt{1+x^2}\, y' + y = \sqrt{1+x^2} \Leftrightarrow y' + \frac{1}{\sqrt{1+x^2}} y = 1 \Rightarrow g(x) = \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow G(x) = \ln(x + \sqrt{x^2+1}) \Rightarrow \mu(x) = x + \sqrt{x^2+1}$ I.F. $\Rightarrow$

$\Rightarrow ((x+\sqrt{x^2+1})y)' = x + \sqrt{x^2+1} \Leftrightarrow (x+\sqrt{x^2+1})y = \frac{x^2}{2} +$

$+ \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ln(x+\sqrt{x^2+1}) + C \Rightarrow y(0) = C = 7$ (villkor);

Resultat: $y = \frac{1}{2} \cdot \frac{x^2 + x\sqrt{x^2+1} + \ln(x+\sqrt{x^2+1}) + 7}{x + \sqrt{x^2+1}}.$

Övning 8.11 (Sid. 157)

Lösning

$y = y(t)$ = antalet bakterier vid en tidpunkt t.

a) $\frac{dy}{dt} = 0{,}1 \cdot y$, $y(0) = 1000$; $y(t) = 1000e^{0{,}1t}$ (kurs E).

$y(6) = 1000e^{0{,}6} \approx 1800$ bakterier.

b) $y(\tau) = 2000 \Rightarrow e^{0{,}1\tau} = 2 \Leftrightarrow 0{,}1\tau = \ln 2 \Leftrightarrow \tau = 10\ln 2 \approx 6{,}9h.$

Övning 8.12 (Sid. 157)

Lösning

a) $y = y(t)$ = mängden av ämnet vid tiden t.

$\frac{dy}{dt} = -0{,}2y \wedge y(0)=3 \Rightarrow y(t)=3e^{-0{,}2t} \Rightarrow y(10)=\underline{3e^{-2}}.$

b) $y(t_{1/2}) = \frac{3}{2} \Rightarrow 3e^{-0{,}2t_{1/2}} = \frac{3}{2} \Leftrightarrow e^{0{,}2t_{1/2}} = 2 \Leftrightarrow 0{,}2t_{1/2} = \ln 2 \Leftrightarrow$

$\Leftrightarrow \underline{t_{1/2} = 5\ln 2}.$

<u>Resultat</u>: a) Efter 10s finns 0,40g kvar.

b) Ämnets halveringstid är 3,45s.

Övning 8.13 (Sid. 157)

Lösning

$m\frac{dv}{dt} = -k\cdot v \Leftrightarrow \frac{dv}{dt} + \frac{k}{m}v = 0 \Rightarrow g(t) = \frac{k}{m} \Rightarrow G(t) = \frac{kt}{m} \Rightarrow$

$\Rightarrow \mu(t) = e^{kt/m}$ I.F. $\Rightarrow \frac{d}{dt}(ve^{kt/m}) = 0 \Leftrightarrow v.e^{kt/m} = C$

$\Rightarrow v = Ce^{-kt/m}$; $v(0)=v_0 \Rightarrow C = v_0 \Rightarrow \underline{v(t) = v_0 e^{-kt/m}}$

Övning 8.14 (Sid. 157)

Lösning

$\underline{u = -RC\frac{du}{dt},\ u(0) = E.}$

$\frac{du}{dt} + \frac{1}{RC}u = 0 \Rightarrow g(t) = \frac{1}{RC} \Rightarrow G(t) = \frac{t}{RC} \Rightarrow \mu(t) = e^{t/RC}$ I.F.

$\Rightarrow \frac{d}{dt}(u\cdot e^{t/RC}) = 0 \Leftrightarrow u\cdot e^{t/RC} = \varsigma \Leftrightarrow u(t) = \varsigma e^{-t/RC}$;

$u(0) = E \Rightarrow \varsigma = E \Rightarrow \underline{u(t) = E\cdot e^{-t/RC}};$

$u(\tau) = \frac{1}{2}E \Rightarrow \frac{1}{2}E = E\cdot e^{-\tau/RC} \Leftrightarrow e^{\tau/RC} = 2 \Leftrightarrow \underline{t_{1/2} = \tau = RC\ln 2}.$

Övning 8.15 (Sid. 158)

Lösning

$E = RC\frac{du}{dt} + u \Leftrightarrow \frac{du}{dt} + \frac{1}{RC}u = \frac{E}{RC} \Rightarrow g(t) = \frac{1}{RC} \Rightarrow G(t) = \frac{t}{RC}$

$\Rightarrow \mu(t) = e^{t/RC}$ I.F. $\Rightarrow \frac{d}{dt}(u\cdot e^{t/RC}) = \frac{E}{RC}e^{t/RC} = \frac{d}{dt}Ee^{t/RC}$

$\Leftrightarrow ue^{t/RC} = E\cdot e^{t/RC} + \varsigma \Leftrightarrow u(t) = E + \varsigma e^{-t/RC}$, $\varsigma \in \mathbb{R}$;

$u(0) = 0 \Rightarrow E + \varsigma = 0 \Leftrightarrow \varsigma = -E \Rightarrow \underline{u(t) = E(1 - e^{-t/RC})}.$

Övning 8.16 (Sid. 158)

Lösning

$E = R\cdot i + L\frac{di}{dt} \Leftrightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \Rightarrow g(t) = \frac{R}{L} \Rightarrow G(t) = \frac{Rt}{L} \Rightarrow$

$\Rightarrow \mu(t) = e^{Rt/L}$ I.F $\Rightarrow \frac{d}{dt}(i\cdot e^{Rt/L}) = \frac{E}{L}e^{Rt/L} = \frac{d}{dt}(\frac{E}{R}e^{Rt/L})$

$\Leftrightarrow i\cdot e^{Rt/L} = \frac{E}{R}e^{Rt/L} + \varsigma \Leftrightarrow i(t) = \frac{E}{R} + \varsigma e^{-Rt/L}$, $\varsigma \in \mathbb{R}$.

$i(0) = 0 \Rightarrow \frac{E}{R} + \varsigma = 0 \Leftrightarrow \varsigma = -\frac{E}{R} \Rightarrow \underline{i(t) = \frac{E}{R}(1 - e^{-Rt/L})}.$

Övning 8.17 (Sid. 158)

Lösning

Tangentens ekvation är $y = f'(a)(x-a) + f(a)$;

$y(\frac{1}{a}) = 0 \Rightarrow f'(a)\cdot(\frac{1}{a} - a) + f(a) = 0 \Rightarrow f'(x) + \frac{x}{1-x^2}f(x) = 0 \Rightarrow$

$\Rightarrow g(x) = \frac{x}{1-x^2} \Rightarrow G(x) = \ln(1-x^2)^{-1/2} \Rightarrow \mu(x) = \frac{1}{\sqrt{1-x^2}}$ I.F. $\Rightarrow$

$\Rightarrow \frac{d}{dx}\left(y\cdot\frac{1}{\sqrt{1-x^2}}\right)=0 \Leftrightarrow \frac{y}{\sqrt{1-x^2}}=C \Leftrightarrow \underline{y=C\sqrt{1-x^2},\ C\in\mathbb{R}}.$

Övning 8.18 (Sid. 158)

Lösning

$y=y(t)$ = koncentrationen av föroreningen vid tiden t.

$y(0)=0{,}05\cdot 3000=150$; $\underline{y(\tau)=0{,}01\cdot 3000=30}$.

$\frac{dy}{dt}=$(smuts in − smuts ut) $=0-50\frac{y}{3000}=-\frac{1}{60}y \Leftrightarrow$

$\Leftrightarrow \frac{dy}{dt}+\frac{1}{60}y=0 \Rightarrow g(t)=\frac{1}{60} \Rightarrow G(t)=\frac{t}{60} \Rightarrow \mu(t)=e^{t/60}$ I.F.

$\Rightarrow \frac{d}{dt}(y\cdot e^{t/60})=0 \Leftrightarrow y\cdot e^{t/60}=C \Leftrightarrow y(t)=Ce^{-t/60}$;

$y(0)=150 \Rightarrow C=150 \Rightarrow y(t)=150e^{-t/60},\ t\geq 0.$

$\underline{y(\tau)=30} \Rightarrow 150e^{-\tau/60}=30 \Leftrightarrow e^{\tau/60}=5 \Leftrightarrow \tau=60\ln 5$ (h).

Svar: Föroreningens koncentration har gått ner till 1% efter 1 timme och 37 minuter.

Övning 8.19 (Sid. 158)

Lösning

Jag sätter $x=x(t)$ = antalet mjuka julklappar och $y=y(t)$ = antalet hårda julklappar; $x+y=M$.

$\frac{dx}{dt}=Ky-2Kx=K(M-x)-2Kx=K(M-3x)=-3Kx+KM$;

$\frac{dx}{dt}+3Kx=KM \Rightarrow g(t)=3K \Rightarrow G(t)=3Kt \Rightarrow \mu(t)=e^{3Kt}$ I.F.

$\Rightarrow \frac{d}{dt}(x\cdot e^{3Kt})=KMe^{3Kt} \Leftrightarrow x\cdot e^{3Kt}=\frac{M}{3}e^{3Kt}+C,\ C\in\mathbb{R}, \Leftrightarrow$

$\Leftrightarrow x(t)=\frac{1}{3}M+Ce^{-3Kt} \Rightarrow x(0)=\frac{M}{3}+C$; $x(0)=\frac{1}{2}M \Rightarrow C=\frac{M}{6}$;

Svar: $x(t)=\frac{M}{6}(2-e^{-3Kt})$, f.ö. se ovan.

Övning 8.20 (Sid. 158)

Lösning

$\frac{dv}{dt}=-kv,\ v(x=0)=v_0,\ v(x=b)=0.$

$\frac{dv}{dx}\frac{dx}{dt}=\frac{dv}{dx}\cdot v=-k\cdot v \Leftrightarrow \frac{dv}{dx}=-k \Leftrightarrow v(x)=-kx+C,\ C\in\mathbb{R}.$

$v(x=0)=v_0 \Rightarrow C=v_0 \Rightarrow v(x)=v_0-kx$; $v(b)=0 \Rightarrow \underline{b=\frac{v_0}{k}}.$

Separabla variabler

Övning 8.21 (Sid. 158)

Lösning

a) $2yy'=3x^2 \Leftrightarrow 2y\frac{dy}{dx}=3x^2 \Leftrightarrow 2y\,dy=3x^2dx \Leftrightarrow$

$\Leftrightarrow \int 2y\,dy=\int 3x^2dx \Leftrightarrow y^2=x^3+C$; $y(1)=2 \Rightarrow C=3$;

$\therefore y=\sqrt{x^3+3}$. (Jag tar plustecken, ty $y(1)=2>0$).

b) $3y^2y'=2x \Leftrightarrow 3y^2\frac{dy}{dx}=2x \Leftrightarrow 3y^2dy=2x\,dx \Leftrightarrow \int 3y^2dy=$

$=\int 2x\,dx \Leftrightarrow y^3=x^2+C$; $y(1)=2 \Rightarrow C=7 \Rightarrow \underline{y=(x^2+7)^{1/3}}.$

c) $yy' = 3x^2 \Leftrightarrow 2y\frac{dy}{dx} = 6x^2 \Leftrightarrow 2y\,dy = 6x^2dx \Leftrightarrow \int 2y\,dy =$
$= \int 6x^2dx \Leftrightarrow y^2 = 2x^3 + C$; $y(1) = 2 \Rightarrow C = 2 \Rightarrow \underline{y = \sqrt{2x^3+2}}.$

d) $y^2y' = 2x \Leftrightarrow 3y^2\frac{dy}{dx} = 6x \Leftrightarrow 3y^2dy = 6x\,dx \Rightarrow \int 3y^2dy =$
$= \int 6x\,dx \Leftrightarrow y^3 = 3x^2 + C$; $y(1) = 2 \Rightarrow C = 5 \Rightarrow \underline{y = (3x^2+5)^{1/3}}.$

Övning 8.22 (Sid. 159)

Lösning

a) $4xy^3y' = 1 \Leftrightarrow 4y^3\frac{dy}{dx} = \frac{1}{x} \Leftrightarrow 4y^3dy = \frac{1}{x}dx \Leftrightarrow \int 4y^3dy =$
$= \int \frac{1}{x}dx \Leftrightarrow y^4 = \ln x + C$; $y(1) = 1 \Rightarrow C = 1 \Rightarrow \underline{y = (\ln x + 1)^{1/4}}.$

b) $xy' + y^2 = 1 \Leftrightarrow x\frac{dy}{dx} = 1 - y^2 \Rightarrow y = \pm 1$ lösningar.

$y \neq \pm 1 \Rightarrow \frac{2}{1-y^2}dy = \frac{2}{x}dx \Leftrightarrow \left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy = \frac{2}{x}dx \Leftrightarrow$
$\Leftrightarrow \int\left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy = \int \frac{2}{x}dx \Leftrightarrow \ln\left|\frac{1+y}{1-y}\right| = \ln Cx^2 \Leftrightarrow \frac{1+y}{1-y} =$
$= Cx^2 \Leftrightarrow y + 1 = (1-y)Cx^2 = Cx^2 - yCx^2 \Leftrightarrow y\cdot(1 + Cx^2) = Cx^2 - 1$
$\Leftrightarrow y = \frac{Cx^2-1}{Cx^2+1}$, $C \neq 0$;

Resultat: a) $y = (\ln x + 1)^{1/2}$, $x > e^{-1}$;

b) $y = \frac{Cx^2-1}{Cx^2+1}$, $C \neq 0$; $y = \pm 1$ (singulära).

Övning 8.23 (Sid. 159)

Lösning

a) $yy' = -x \Leftrightarrow 2y\frac{dy}{dx} = -2x \Leftrightarrow 2y\,dy = -2x\,dx \Leftrightarrow \int 2y\,dy =$
$= -\int 2x\,dx \Leftrightarrow y^2 = -x^2 + C^2 \Leftrightarrow \underline{x^2 + y^2 = C^2}.$

b) $y' = e^{x+y} \Leftrightarrow \frac{dy}{dx} = e^x e^y \Leftrightarrow e^{-y}dy = e^x dx \Leftrightarrow \int e^{-y}dy =$
$= \int e^x dx \Leftrightarrow -e^{-y} = e^x + C \Leftrightarrow \underline{e^{-y} = -e^x - C}.$

c) $y' = y^2 ; \Rightarrow y = y(x) = 0$ lösning som dock inte uppfyller begynnelsevillkoret $y(1) = 1$.

Jag väljer $y > 0$, ty $y(1) > 0$, och får
$\frac{dy}{dx} = y^2 \Leftrightarrow \frac{1}{y^2}dy = dx \Leftrightarrow \int\frac{dy}{y^2} = \int dx \Leftrightarrow -\frac{1}{y} = x + C$;
$y(1) = 1 \Rightarrow -1 = 1 + C \Leftrightarrow C = -2 \Rightarrow -\frac{1}{y} = x - 2 \Leftrightarrow \underline{y = \frac{1}{2-x}, x < 2}.$

d) $y' = y^2 \wedge y(1) = 0 \Rightarrow y(x) \equiv 0$ (Se c) ovan).

e) $x^2y\frac{dy}{dx} = 1 + x^2 \Leftrightarrow y\,dy = \left(\frac{1}{x^2} + 1\right)dx \Leftrightarrow 2y\,dy = 2\left(\frac{1}{x^2} + 1\right)dx$
$\Leftrightarrow \int 2y\,dy = 2\int\left(\frac{1}{x^2} + 1\right)dx \Leftrightarrow y^2 = -\frac{2}{x} + 2x + C$, $C \in \mathbb{R}$;
$y(2) = 2 \Rightarrow 4 = -1 + 4 + C \Leftrightarrow C = 1 \Rightarrow y^2 = 2\frac{x^2-1}{x} + 1 \Rightarrow$
$\Rightarrow \underline{y = \left(2\frac{x^2-1}{x} + 1\right)^{1/2}, x > 0}.$

Övning 8.24 (Sid. 159)

Lösning

a) $\underline{y' = (y^2-1)x,\ y(0) = 0}$

$y = \pm 1$ är lösningar som dock inte uppfyller villkoret $y(0)=0$. Dessa s.k. stationära lösningar delar xy-planet i tre områden: $y<-1, -1<y<1$ och $y>1$; jag väljer $-1<y<1$, ty $-1<y(0)<1$.

$\frac{dy}{dx}=(y^2-1)x \Leftrightarrow \frac{2}{1-y^2}dy=-2xdx \Leftrightarrow (\frac{1}{1+y}+\frac{1}{1-y})dy=-2xdx$

$\Leftrightarrow \int(\frac{1}{1+y}+\frac{1}{1-y})dy=-\int 2xdx \Leftrightarrow \ln\frac{1+y}{1-y}=-x^2+C,\ C\in\mathbb{R}.$

$y(0)=0 \Rightarrow \ln 1=0+C \Leftrightarrow C=0 \Rightarrow \ln\frac{1+y}{1-y}=-x^2 \Leftrightarrow$

$\Leftrightarrow \frac{1}{2}\ln\frac{1+y}{1-y}=-\frac{x^2}{2} \Leftrightarrow \operatorname{artanh} y=-\frac{x^2}{2} \Leftrightarrow y=\tanh(-\frac{x^2}{2})$

$\Leftrightarrow y=-\tanh\frac{x^2}{2}=-\frac{e^{x^2/2}-e^{-x^2/2}}{e^{x^2/2}+e^{x^2/2}}=\frac{1-e^{x^2}}{1+e^{x^2}}$

b) $xy'=y^2-2y=y(y-2) \Rightarrow y=0$ och $y=2$ lösningar. Dessa (stationära) lösningar uppfyller inte begynnelsevillkoret $y(1)=1$; de delar dock planet i tre områden $y<0$, $0<y<2$ och $y>2$. Jag väljer "bandet" $0<y<2$, ty $0<y(1)<2$.

$x\frac{dy}{dx}=y(y-2) \Leftrightarrow \frac{2}{y(2-y)}dy=-\frac{2}{x}dx \Leftrightarrow (\frac{1}{y}+\frac{1}{2-y})dy=$

$=-\frac{2}{x}dx \Leftrightarrow \int(\frac{1}{y}-\frac{1}{2-y})dy=-\int\frac{2}{x}dx \Leftrightarrow \ln\frac{y}{2-y}=\ln\frac{C}{x^2} \Leftrightarrow$

$\Leftrightarrow \frac{y}{2-y}=\frac{C}{x^2}$; $y(1)=1 \Rightarrow 1=C \Rightarrow \frac{2-y}{y}=x^2 \Leftrightarrow \frac{2}{y}-1=x^2$

$\Leftrightarrow \frac{2}{y}=x^2+1 \Leftrightarrow \frac{y}{2}=\frac{1}{x^2+1} \Leftrightarrow y=\frac{2}{x^2+1}.$

(Separabla ekvationer är ofta svårlösta.)

Övning 8.26 (Sid. 159)

Lösning

a) $y'+y=3$; linjär; $g(x)=1$, $h(x)=3$.

b) $y'y=3$; icke-linjär; $g(y)=y$, $h(x)=3$.

c) $y'+x^2y=3$; linjär; $g(x)=x^2$, $h(x)=3$.

d) $y'=y^2+3 \Leftrightarrow \frac{1}{y^2+3}y'=1$; icke-linjär, separabel.

e) $xyy'=x^2+y^2$; icke-linjär, icke-separabel.

f) $xyy'=e^y\sin x \Leftrightarrow ye^{-y}y'=\frac{\sin x}{x}$; separabel.

Övning 8.27 (Sid. 159)

Lösning

$2\frac{dv}{dt}=-5v^2 \Leftrightarrow -\frac{dv}{v^2}=\frac{5}{2}dt \Leftrightarrow \int(-\frac{1}{v^2})dv=\frac{5}{2}\int dt \Leftrightarrow \frac{1}{v}=\frac{5t}{2}+C.$

$v(0)=3 \Rightarrow \frac{1}{3}=C \Rightarrow \frac{1}{v}=\frac{5t}{2}+\frac{1}{3}=\frac{15t+2}{6} \Leftrightarrow v=\frac{6}{15t+2}.$

Övning 8.28 (Sid. 159)

Lösning

a) $m\frac{dv}{dt}=-kv^\alpha$, $\alpha\neq 1$, $k>0$; $v(0)=v_0$.

$\frac{dv}{dt}=-\frac{k}{m}v^\alpha \Leftrightarrow v^{-\alpha}dv=-\frac{k}{m}dt \Leftrightarrow \int v^{-\alpha}dv=-\frac{k}{m}\int dt \Leftrightarrow$

$\Leftrightarrow \frac{v^{-\alpha+1}}{-\alpha+1}=-\frac{kt}{m}+C$; $v(0)=v_0 \Rightarrow C=v_0^{-\alpha+1}/1-\alpha$;

forts.

$$v^{-(\alpha-1)}/(1-\alpha) = v_0^{-(\alpha-1)}/(1-\alpha) - \frac{kt}{m} \Leftrightarrow v^{-(\alpha-1)} = \frac{(\alpha-1)kt}{m} + v_0^{1-\alpha}$$

$$\Leftrightarrow v^{1-\alpha} = \frac{(\alpha-1)kt + mv_0^{1-\alpha}}{m} \Leftrightarrow v = \left(\frac{(\alpha-1)kt + mv_0^{1-\alpha}}{m}\right)^{1/(1-\alpha)}$$

b) Givna: $\alpha = 1{,}5$, $m = 1$, $v_0 = 10$, $v(4) = 5$.

$$5 = \left(\frac{0{,}5 \cdot 4k + 10^{-1/2}}{1}\right)^{-2} \Leftrightarrow 2k + \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{5}} \Leftrightarrow 2k = \frac{\sqrt{2}}{\sqrt{10}} - \frac{1}{\sqrt{10}} =$$

$$= \frac{\sqrt{2}-1}{\sqrt{10}} \Leftrightarrow k = \frac{\sqrt{2}-1}{2\sqrt{10}} \approx 0{,}065.$$

Övning 8.28 (Sid. 159)

Lösning

a) Newtons andra lag ger $m\frac{dv}{dt} = mg - kv^2$; $k > 0$.

b) $\frac{dv}{dt} = 1 - v^2$, $v(0) = 3$; $v > 1$ är ett lämpligt val.

$$\frac{dv}{v^2-1} = -dt \Leftrightarrow \left(\frac{1}{v-1} - \frac{1}{v+1}\right)dv = -2dt \Leftrightarrow \int\left(\frac{1}{v-1} - \frac{1}{v+1}\right)dv =$$

$$= -2\int dt \Leftrightarrow \ln\frac{v-1}{v+1} = -2t + C \Leftrightarrow \frac{v-1}{v+1} = Ae^{-2t} \Leftrightarrow 1 - \frac{2}{v+1} =$$

$$= Ae^{-2t} \Leftrightarrow \frac{2}{v+1} = 1 - Ae^{-2t} \Leftrightarrow v+1 = \frac{2}{1-Ae^{-2t}} \Leftrightarrow v = \frac{2}{1-Ae^{-2t}}$$

$$-1 = \frac{2-1+Ae^{-2t}}{1-Ae^{-2t}} = \frac{1+Ae^{-2t}}{1-Ae^{-2t}} = \frac{e^{2t}+A}{e^{2t}-A}; \quad v(0) = 3 \Rightarrow A = 1/2.$$

Resultat: a) Se ovan; b) $v(t) = \frac{2e^{2t}+1}{2e^{2t}-1}$; $\lim_{t\to\infty} v(t) = 1$.

Övning 8.29 (Sid. 160)

Lösning

$\frac{dy}{dt} = -k\sqrt{y}$; minustecken, ty djupet minskar.

$$\frac{dy}{\sqrt{y}} = -kdt \Rightarrow \int_7^0 y^{-1/2}dy = -\int_0^4 kdt \Leftrightarrow [2\sqrt{y}]_7^0 = -k[t]_0^4 \Leftrightarrow$$

$$\Leftrightarrow -2\sqrt{7} = -4k \Leftrightarrow k = \frac{\sqrt{7}}{2}.$$

Övning 8.30 (Sid. 160)

Lösning:

Fullständigt löst på sidan 160.

Övning 8.31 (Sid. 160)

Lösning

$c = c(t)$ = koncentrationen av butandien vid en godtycklig tidpunkt $t \geq 0$.

$$\frac{dc}{dt} = -kc^2 \Leftrightarrow -\frac{1}{c^2}dc = kdt \Rightarrow \int_{c_0}^{c}\left(-\frac{1}{\gamma^2}\right)d\gamma = \int_0^t kd\tau \Leftrightarrow \left[\frac{1}{\gamma}\right]_{c_0}^{c} =$$

$$= kt \Leftrightarrow \frac{1}{c} - \frac{1}{c_0} = kt \Leftrightarrow \frac{1}{c} = \frac{1}{c_0} + kt = \frac{1+kc_0t}{c_0} \Leftrightarrow c(t) = \frac{c_0}{1+kc_0t}$$

Övning 8.3 (Sid. 160)

Lösning

$\frac{dy}{dt} = ry(K-y)$, $y(0) = 10^4$, $y(1) = 2 \cdot 4^{10}$.

$y(0) < y(1) < y(\infty) \Rightarrow \frac{dy}{dt} > 0 \Rightarrow 0 < y < K = y(\infty) = 10^5$.

$$\frac{K}{y(K-y)}dy = rKdt \Leftrightarrow \left(\frac{1}{y} + \frac{1}{K-y}\right)dy = Krdt \Leftrightarrow Kr\int dt =$$

$$= \int\left(\frac{1}{y} - \frac{1}{K-y}\right)dy = Kr{\cdot}t \Leftrightarrow \ln\frac{y}{K-y} = Krt + C, \text{ } c \text{ konstant.}$$

$y(0)=10^4 \Rightarrow \ln\frac{10^4}{10^5-10^4}=0+C \Leftrightarrow C=\ln\frac{1}{9}=-2\ln 3 \Rightarrow$

$\Rightarrow \ln\frac{y}{K-y}=Krt-2\ln 3$; (*)

$y(1)=2\cdot10^4 \Rightarrow (*) \Rightarrow \ln\frac{2\cdot10^4}{8\cdot10^4}=\ln\frac{1}{4}=Kr+\ln\frac{1}{9} \Leftrightarrow Kr=\ln\frac{9}{4}$

$\Leftrightarrow r=\frac{1}{K}\ln\left(\frac{3}{2}\right)^2=\frac{2}{K}\ln\frac{3}{2}=2\cdot10^{-5}\cdot\ln\frac{3}{2}.$

Svar: $K=10^5$, $r=2(\ln 1{,}5)\cdot10^{-5}\approx 8{,}1\cdot10^{-6}$.

Integralekvationer

Övning 8.34 (Sid. 160)

Lösning

$f(x)=x+\int_0^x \frac{2t}{1+t^2}f(t)\,dt \Rightarrow f'(x)=1+\frac{2x}{x^2+1}f(x) \wedge f(0)=0$;

$f'(x)-\frac{2x}{x^2+1}f(x)=1 \Rightarrow g(x)=-\frac{2x}{x^2+1} \Rightarrow G(x)=\ln(x^2+1)^{-1}$

$\Rightarrow \mu(x)=e^{G(x)}=\frac{1}{x^2+1}$ I.F $\Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x^2+1}\right)=\frac{1}{x^2+1} \Leftrightarrow$

$\Leftrightarrow \frac{f(x)}{x^2+1}=\arctan x+C$; $f(0)=0 \Rightarrow C=0$;

Resultat: $f(x)=(x^2+1)\arctan x$.

Övning 8.35 (Sid. 160)

Lösning

(i) $xf(x)=x+\int_1^x \frac{t}{t+1}f(t)\,dt \Rightarrow \lim_{x\to1^+} xf(x)=\lim_{x\to1^+}\left(x+\int_1^x \frac{tf(t)}{t+1}dt\right)$

$\Leftrightarrow 1\cdot f(1)=1+0 \Leftrightarrow f(1)=1$ (begynnelsevillkor);

(ii) $xf(x)=x+\int_1^x \frac{t\cdot f(t)}{t+1}dt \Rightarrow \frac{d}{dx}(xf(x))=\frac{d}{dx}\left(x+\int_1^x \frac{tf(t)}{t+1}dt\right)$

$\Leftrightarrow f(x)+xf'(x)=1+\frac{x}{x+1}f(x) \Leftrightarrow xf'(x)+\left(1-\frac{x}{x+1}\right)f(x)=1 \Leftrightarrow$

$\Leftrightarrow xf'(x)+\frac{1}{x+1}f(x)=1 \Leftrightarrow f'(x)+\frac{1}{x(x+1)}f(x)=\frac{1}{x} \Rightarrow g(x)=$

$=\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1} \Rightarrow G(x)=\ln\frac{x}{x+1} \Rightarrow \mu(x)=\frac{x}{x+1}$ I.F. $\Rightarrow$

$\Rightarrow \frac{d}{dx}\left(\frac{x}{x+1}f(x)\right)=\frac{1}{x+1} \Leftrightarrow \frac{x}{x+1}f(x)=\ln(x+1)+C$; (*)

$f(1)=1 \Rightarrow (*) \Rightarrow \frac{1}{2}f(1)=\ln 2+C \Leftrightarrow C=\frac{1}{2}-\ln 2 \Rightarrow (*) \Rightarrow$

$\Rightarrow \frac{x}{x+1}f(x)=\ln(x+1)+\frac{1}{2}-\ln 2=\ln\frac{x+1}{2}+\frac{1}{2}$;

Resultat: $f(x)=\frac{x+1}{x}\left(\ln\frac{x+1}{2}+\frac{1}{2}\right)$, $x>1$.

Övning 8.36 (Sid. 161)

Lösning

$y(x)=1+\int_0^x y(t)\,dt \Rightarrow y'(x)=y(x) \wedge y(0)=1 \Rightarrow y=e^x$.

Övning 8.37 (Sid. 161)

Lösning

$y(x)=2-\int_0^x e^{y(t)-t}dt \Rightarrow y'(x)=e^{y(x)-x} \wedge y(0)=2$;

$\frac{dy}{dx}=e^{y-x}=e^y\cdot e^{-x} \Leftrightarrow -e^{-y}dy=-e^{-x}dx \Leftrightarrow \int(-e^{-y})dy=$

$=\int(-e^{-x})dx \Leftrightarrow e^{-y}=e^{-x}+C$; $y(0)=2 \Rightarrow C=e^{-2}-1 \Rightarrow$

$\Rightarrow e^{-y}=e^{-x}+e^{-2}-1 \Leftrightarrow y=-\ln(e^{-x}+e^{-2}-1)$, $x<\ln\frac{e^2}{e^2-1}$

Linjära ekvationer av andra ordningen

a) ## Homogena ekvationer

Övning 8.38 (Sid. 161)

Lösning

a) $y''-3y'+2y=0 \Leftrightarrow r^2-3r+2=0 \Leftrightarrow r=1 \vee r=2 \Rightarrow y=e^x \vee$
$\vee\ y=e^{2x} \Rightarrow \underline{y=C_1e^x+C_2e^{2x}}.$

b) $y''-4y'+4y=0 \Leftrightarrow r^2-4r+4=(r-2)^2=0 \Leftrightarrow r=r_1=r_2=2 \Rightarrow$
$\Rightarrow y=e^{2x} \vee y=xe^{2x} \Rightarrow y=C_1e^{2x}+C_2xe^{2x}=\underline{(C_1+C_2x)e^{2x}}.$

c) $y''-6y'+10y=0 \Leftrightarrow r^2-6r+10=0 \Leftrightarrow r=3+i \vee r=3-i \Rightarrow$
$\Rightarrow y=e^{3x}\cos x \vee y=e^{3x}\sin x \Rightarrow \underline{y=e^{3x}(C_1\cos x+C_2\sin x)}.$

Övning 8.39 (Sid. 161)

Lösning

a) $y''-y'-2y=0 \Leftrightarrow r^2-r-2=(r+1)(r-2)=0 \Leftrightarrow r=-1 \vee r=2$
$\Leftrightarrow y=e^{-x} \vee y=e^{2x} \Rightarrow \underline{y=C_1e^{-x}+C_2e^{2x}}.$

b) $y''-10y'+61y=0 \Leftrightarrow r^2-10r+61=0 \Leftrightarrow r=5-6i \vee r=5+6i$
$\Leftrightarrow y=e^{5x}\cos 6x \vee y=e^{5x}\sin 6x \Rightarrow y=C_1e^{5x}\cos 6x+$
$+C_2e^{5x}\sin 6x \Leftrightarrow \underline{y=e^{5x}(C_1\cos 6x+C_2\sin 6x)}.$

c) $y''-2y'+5y=0 \Leftrightarrow r^2-2r+5=0 \Leftrightarrow r=1+2i \vee r=1-2i \Leftrightarrow$
$\Leftrightarrow y=e^x\cos 2x \vee y=e^x\sin 2x \Rightarrow y=C_1e^x\cos 2x+C_2e^x\sin 2x$
$\Leftrightarrow \underline{y=e^x(C_1\cos 2x+C_2\sin 2x)}.$

d) $y''+6y'+9y=0 \Leftrightarrow r^2+6r+9=(r+3)^2=0 \Leftrightarrow r=r_1=r_2=-3 \Rightarrow$
$\Rightarrow y=e^{-3x} \vee y=xe^{-3x} \Rightarrow y=C_1e^{-3x}+C_2xe^{-3x}=\underline{(C_1+C_2x)e^{-3x}}.$

Övning 8.40 (Sid. 161)

Lösning

a) $y''-4y=0 \Leftrightarrow r^2-4=r^2-2^2=(r+2)(r-2)=0 \Leftrightarrow r=2 \vee r=-2$
$\Rightarrow y=e^{2x} \vee y=e^{-2x} \Rightarrow y=C_1e^{2x}+C_2e^{-2x} \Rightarrow y'=2(C_1e^{2x}-C_2e^{-2x}).$

$$\begin{cases} y(0)=0 \Rightarrow C_1+C_2=0 \\ y'(0)=1 \Rightarrow C_1-C_2=\frac{1}{2} \end{cases} \Leftrightarrow C_1=\frac{1}{4}=-C_2 \Rightarrow \underline{y=\frac{1}{2}\sinh 2x}.$$

b) $y''+6y'+9y=0 \Leftrightarrow r^2+6r+9=(r+3)^2=0 \Leftrightarrow r=r_1=r_2=-3 \Rightarrow$
$\Rightarrow y=(C_1+C_2x)e^{-3x} \Rightarrow y'=(C_2-3C_1-3C_2x)e^{-3x};$

$$\begin{cases} y(0)=-1 \Rightarrow C_1=-1 \\ y'(0)=1 \Rightarrow -3C_1+C_2=1 \end{cases} \Leftrightarrow \begin{cases} C_1=-1 \\ C_2=-2 \end{cases} \Rightarrow \underline{y=-(1+2x)e^{-3x}}.$$

Övning 8.41 (Sid. 161)

Lösning

Se nästföljande sida.

a) $y''+4y=0 \Leftrightarrow r^2+4=0 \Leftrightarrow r=\pm 2i \Rightarrow y=C_1\cos 2x+C_2\sin 2x$;

$y(0)=0 \Rightarrow C_1=0 \Rightarrow y=C_2\sin 2x \Rightarrow y(\frac{\pi}{2})=C_2\sin\pi=0$.

Resultat: $y=A\cdot\sin 2x,\ A\in\mathbb{R}\setminus\{0\}$.

b) $y''+4y=0 \Leftrightarrow y=C_1\cos 2x+C_2\sin 2x$ (enl. ovan) $\Rightarrow$

$\Rightarrow y'=-2C_1\sin 2x+2C_2\cos 2x \Rightarrow y'(0)=-2C_2$; (*)

$y'(0)=0 \Rightarrow C_2=0 \Rightarrow y'=-2C_1\sin 2x \Rightarrow y'(\frac{\pi}{2})=0,\ \forall C_1$.

Resultat: $y=A\cos 2x,\ A\in\mathbb{R}\setminus\{0\}$.

Övning 8.42 (Sid. 161)

Lösning

Jag skiljer mellan 3 olika fall: $\lambda<0, \lambda=0$ & $\lambda>0$.

(i) $\lambda<0 \Rightarrow \lambda=-\mu^2 \Rightarrow y''-\mu^2y=0 \Leftrightarrow y=C_1e^{\mu x}+C_2e^{-\mu x}$;

$y(0)=0 \Rightarrow C_1+C_2=0 \Leftrightarrow C_2=-C_1 \Rightarrow y=2C_1\sinh\mu x$;

$y(\ell)=0 \Rightarrow 2C_1\sinh\mu\lambda=0 \Leftrightarrow C_1=0 \Rightarrow C_2=0 \Rightarrow y\equiv 0$.

(ii) $\lambda=0 \Rightarrow y''=0 \Leftrightarrow y=C_1x+C_2$;

$y(0)=0 \Rightarrow C_2=0 \Rightarrow y=C_1x$; $y(\ell)=0 \Rightarrow C_1\cancel{\ell}=0 \Rightarrow y\equiv 0$. $C_1=0$.

(iii) $\lambda>0 \Rightarrow \lambda=\nu^2 \Rightarrow y''+\nu^2y=0 \Leftrightarrow y=C_1\cos\nu x+C_2\sin\nu x$;

$y(0)=0 \Rightarrow C_1=0 \Rightarrow y=C_2\sin\nu x \Rightarrow y(\ell)=C_2\sin\nu\ell$;

$y(\ell)=0 \wedge y\not\equiv 0 \Rightarrow \sin\nu\ell=0 \Leftrightarrow \nu\ell=n\pi \Leftrightarrow \nu=\frac{n\pi}{\ell}$,

Resultat: $y_n=A_n\cdot\sin\frac{n\pi x}{\ell},\ n=1,2,3,\ldots$

Anm. Konstanten A_n beror av n.

Övning 8.43 (Sid. 161)

Lösning

(i) $\lambda>0 \Rightarrow \lambda=\mu^2 \Rightarrow y=C_1e^{\mu x}+C_2e^{-\mu x} \Rightarrow y'=\mu(C_1e^{\mu x}-C_2e^{-\mu x})$;

$y'(0)=0 \Rightarrow C_1=C_2 \Rightarrow y'=\mu C_1(e^{\mu x}-e^{-\mu x})=2\mu C_1\sinh\mu x$;

$y'(\ell)=0 \Rightarrow 2\mu C_1\cdot\sinh\mu\ell=0 \Rightarrow C_1=0 \Rightarrow C_2=0 \Rightarrow y\not\equiv 0$.

(ii) $\lambda=0 \Rightarrow y''=0 \Rightarrow y=C_1x+C_2 \Rightarrow y'=C_1 \Rightarrow y'(\ell)=C_1 \Rightarrow y\equiv 0$.

(iii) $\lambda<0 \Rightarrow \lambda=-\nu^2 \Rightarrow y'=\nu(-C_1\sin\nu x+C_2\cos\nu x)$; (*)

$y'(0)=0 \overset{(*)}{\Rightarrow} C_2=0 \Rightarrow y'=-C_1\mu\sin\nu x \Rightarrow y'(\ell)=-C_1\mu\sin\nu\ell$;

$y'(\ell)=0 \Rightarrow \sin\nu\ell=0 \Leftrightarrow \nu\ell=n\pi \Leftrightarrow \nu=\frac{n\pi}{\ell},\ n=1,2,\ldots$.

Resultat: $y=A_n\cos\frac{n\pi x}{\ell},\ n=1,2,3,\ldots$

Övning 8.44 (Sid. 162)

Lösning

$3°=0{,}052$ rad; $g=9{,}8,\ L=0{,}2$; $\frac{g}{L}=49$;

$\frac{d^2\alpha}{dt^2}+49\alpha=0,\ \alpha(0)=0{,}052,\ \alpha'(0)=0$.

$r^2+49=0 \Leftrightarrow r=\pm 7i \Rightarrow \alpha(t)=C_1\cos 7t+C_2\sin 7t$;

$\alpha'(t) = 7(-C_1\sin 7t + C_2\cos 7t)$; $\alpha'(0) = 0 \Rightarrow C_2 = 0 \Rightarrow$

$\Rightarrow \alpha(t) = C_1\cos 7t \Rightarrow \alpha(0) = C_1 = 0{,}052 \Rightarrow \alpha(t) = 0{,}052\cos 7t$

$\Rightarrow \alpha(1) = 0{,}052\cos 7 = 0{,}039 \approx 2{,}25°$

Övning 8.45 (Sid. 162)

Lösning

$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \Leftrightarrow r^2 + \frac{c}{m}r + \frac{k}{m} = 0 \Leftrightarrow r = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$;

$r = r_1 = r_2 = -\frac{c}{2m} \Rightarrow \Delta = \frac{c^2}{4m^2} - \frac{k}{m} = 0 \Leftrightarrow c^2 = 4mk$;

$y = (A + Bt)e^{-ct/2m} \Rightarrow y' = (B - \frac{Ac}{2m} - \frac{Bct}{2m})e^{-ct/2m}$;

Derivatan har ett nollställe i det fall $B \neq 0$,

så motsvarande lösning har ett extremvärde.

Övning 8.46 (Sid. 162)

Lösning

$0{,}2\frac{d^2I}{dt^2} + R\frac{dI}{dt} + 1{,}25\cdot 10^6 I = 0 \Leftrightarrow \frac{d^2I}{dt^2} + 5R\frac{dI}{dt} + 6{,}25\cdot 10^6 I = 0$

$\Leftrightarrow r^2 + 5Rr + 1{,}25\cdot 10^6 I = 0 \Leftrightarrow r = -\frac{5R}{2} \pm \sqrt{6{,}25R^2 - 6{,}25\cdot 10^6}$;

Kritisk dämpning $\Rightarrow 6{,}25R^2 - 6{,}25\cdot 10^6 = 0 \Rightarrow \underline{R = 1\,k\Omega}$.

b) Partikulärlösning

Övning 8.47 (Sid. 162) (Löst på sidan 185).

Övning 8.48 (Sid. 162)

Lösning

$y'' + 4y' = x + 1 \Rightarrow (y' + 4y)' = x + 1 \Leftrightarrow y' + 4y = \frac{1}{2}(x+1)^2 + C_1 \Leftrightarrow$

$\Leftrightarrow (ye^{4x})' = e^{4x}(\frac{1}{2}(x+1)^2 + C) \Leftrightarrow ye^{4x} = \int e^{4x}(\frac{(x+1)^2}{2} + C_1)dx =$

$= \frac{1}{4}e^{4x}(\frac{(x+1)^2}{2} + C_1) - \frac{1}{4}\int e^{4x}(x+1)dx = \frac{1}{4}e^{4x}(\frac{(x+1)^2}{2} + C_1) -$

$- \frac{1}{16}e^{4x}(x+1) + \frac{1}{16}\int e^{4x}dx = \frac{1}{4}e^{4x}(\frac{(x+1)^2}{2} + C_1) - \frac{1}{16}e^{4x}(x+1) +$

$+ \frac{1}{64}e^{4x} + C_2 = e^{4x}(\frac{1}{8}(x^2+2x+1) - \frac{1}{16}x - \frac{1}{16} + \frac{1}{64} + \frac{C_1}{4}) + C_2 =$

$= e^{4x}(\frac{1}{8}x^2 + \frac{3}{16}x + A) + B \Leftrightarrow \underline{y = \frac{1}{8}x^2 + \frac{3}{16}x + A + Be^{-4x}}$

Övning 8.49 (Sid. 163)

Lösning

a) $y'' - 3y' + 2y = 0 \Leftrightarrow r^2 - 3r + 2 = 0 \Leftrightarrow r = 1 \vee r = 2 \Rightarrow y = e^x \vee$

$\vee\ y = e^{2x} \Rightarrow y_h = C_1e^x + C_2e^{2x}$.

$y_p = a \Rightarrow VL = y_p'' - 3y_p' + 2y_p = 0 + 0 + 2a = 6 = HL \Rightarrow a = 3$

Resultat: $y = C_1e^x + C_2e^{2x} + 3$.

b) $y_p = ax^2 + bx + c \Rightarrow y_p' = 2ax + b \Rightarrow y_p'' = 2a$;

$VL = y_p'' - 3y_p' + 2y_p = 2a - 3(2ax + b) + 2(ax^2 + bx + c) =$

$= 2ax^2 + (-6a + 2b)x + 2a - 3b + 2c \equiv x^2 = HL \Leftrightarrow 2a = 1 \wedge$

$\wedge\ 3a - b = 0 = 2a - 3b + 2c \Leftrightarrow a = \frac{1}{2} \wedge b = \frac{3}{2} \wedge c = \frac{7}{4}$;

<u>Resultat</u>: $y = C_1e^{x} + C_2e^{2x} + \frac{1}{4}(2x^2+6x+7)$.

<u>Anm.</u> y_h är densamma som under a).

c) $y'' + 3y' + 2y = 0 \Leftrightarrow r^2+3r+2=0 \Leftrightarrow r=-1 \vee r=-2 \Rightarrow$

$\Rightarrow y = e^{-x} \vee y = e^{-2x} \Leftrightarrow \underline{y_h = C_1e^{-x} + C_2e^{-2x}}$;

$y_p = ax^3+bx^2+cx+d \Rightarrow y_p' = 3ax^2+2bx+c \Rightarrow y_p'' = 6ax+2b$;

$VL = y_p''+3y_p'+2y_p = 2ax^3+(9a+2b)x^2+(6a+6b+2c)x +$

$+2b+3c+2d$; $HL = x^3+x+1$;

$$VL = HL \Rightarrow \begin{cases} 2a=1 \\ 9a+2b=0 \\ 6a+6b+2c=1 \\ 2b+3c+2d=1 \end{cases} \Leftrightarrow \begin{cases} a=1/2 \\ b=-9/4 \\ c=23/4 \\ d=-47/8 \end{cases} \Rightarrow \underline{y_p = \frac{x^3}{8} - \frac{9x^2}{4} + \frac{23x}{4} - \frac{47}{8}}.$$

<u>Resultat</u>: $y = C_1e^{-x}+C_2e^{-2x}+\frac{1}{8}(4x^3-18x^2+46x-47)$.

d) $y''+2y'=0 \Leftrightarrow r^2+2r = r(r+2)=0 \Leftrightarrow r=0 \vee r=-2 \Rightarrow$

$\Rightarrow y=1 \vee y=e^{-2t} \Rightarrow \underline{y_h = C_1+C_2e^{-2t}}$.

$y_p = x(ax^2+bx+c) = ax^3+bx^2+cx \Rightarrow y_p' = 3ax^2+2bx+c$

$\Rightarrow y_p'' = 6ax+2b \Rightarrow VL = y_p''+2y_p' = 6ax+2b+2(3ax^2+$

$+2bx+c) = 6ax^2+(6a+2b)x+2b+2c = x^2+1 = HL \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 6a=1 \\ 3a+2b=0 \\ 2(b+c)=1 \end{cases} \Leftrightarrow \begin{cases} a=1/6 \\ b=-3a/2 \\ c=1/2-b \end{cases} \Leftrightarrow \begin{cases} a=1/6 \\ b=-1/4 \\ c=3/4 \end{cases} \Rightarrow \underline{y_p = \frac{x^3}{6}-\frac{x^2}{4}+\frac{3}{4}x}.$$

<u>Resultat</u>: $y = A+Be^{-2x}+x^3/6-x^2/4+3x/4$.

<u>Övning 8.50 (Sid. 163)</u>

<u>Lösning</u>

$y_p = (ax+b)e^{-3x} \Rightarrow y_p' = ae^{-3x}-3y_p \Rightarrow y_p'' = -3ae^{-3x}-$

$-3y_p' = -3ae^{-3x}-3(ae^{-3x}-3y_p) = 9y_p-6ae^{-3x}$;

$VL = y_p''+3y_p'+2y_p = 9y_p-6ae^{-3x}+3(ae^{-3x}-3y_p)+2y_p =$

$= 2y_p-3ae^{-3x} = (2ax-3a+2b)e^{-3x} = (x+1)e^{-3x} = HL$

$\Leftrightarrow 2a=1 \wedge -3a+2b=1 \Leftrightarrow a=\frac{1}{2} \wedge b=\frac{5}{4} \Rightarrow \underline{y_p = \frac{2x+5}{4}e^{-3x}}$.

y_h hämtas från föregående övning.

<u>Resultat</u>: $y = C_1e^{-x}+C_2e^{-2x}+(\frac{x}{2}+\frac{5}{4})e^{-3x}$.

<u>Övning 8.51 (Sid. 163)</u>

<u>Lösning</u>

$y_p = ae^{5x} \Rightarrow y_p' = 5y_p \Rightarrow y_p'' = 5^2y_p \Rightarrow VL = y_p''-3y_p'+$

$+2y_p = 25y_p-15y_p+2y_p = 12y_p = e^{5x} = HL \Leftrightarrow y_p = \frac{1}{12}e^{5x}$.

<u>Resultat</u>: $y = C_1e^{x}+C_2e^{2x}+\frac{1}{12}e^{5x}$.

<u>Anm.</u> y_h hämtas från Ö. 8.49 a).

b) $y=e^{2x}$ ingår i y_h, så en lämplig ansats är

$y_p = axe^{2x}$ (Glöm förskjutningsregeln).

$y_p' = ae^{2x}+2y_p \Rightarrow y_p'' = 2ae^{2x}+2y_p' = 4y_p+4ae^{2x}$;

$VL = y_p'' - 3y_p' + 2y_p = 4y_p + 4ae^{2x} - 3(ae^{2x} + 2y_p) + 2y_p =$

$= ae^{2x} = e^{2x} = HL \Leftrightarrow a = 1 \Rightarrow \underline{y_p = xe^{2x}}.$

Resultat: $y = C_1e^x + (C_2 + x)e^{2x}$. (För y_h se Ö. 8.49).

c) $\underline{y = ze^{-x}} \Rightarrow y' = (z' - z)e^{-x} \Rightarrow y'' = (z'' - 2z' + z)e^{-x}$;

$VL = y'' + 6y' + 9y = (z'' - 2z' + z + 6z' - 6z + 9z)e^{-x} = 4e^{-x} = HL$

$\Leftrightarrow z'' + 4z' + 4z = 4 \Leftrightarrow z = (C_1 + C_2x)e^{-2x} + 1$ (enkelt!?)

$\underline{y = (C_1 + C_2x)e^{-3x} + e^{-x}} \Rightarrow y' = C_2e^{-3x} - 3(C_1 + C_2x)e^{-3x} - e^{-x}$;

$$\begin{cases} y(0) = 2 \Rightarrow C_1 + 1 = 2 \\ y'(0) = -2 \Rightarrow C_2 - 3C_1 - 1 = -2 \end{cases} \Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = 2 \end{cases} \Rightarrow \underline{y = (2x+1)e^{-3x} + e^{-x}}.$$

d) $\underline{y = ze^{-x}} \Rightarrow VL = y'' + 2y' + y = (z'' - 2z' + z + 2z' - 2z + z)e^{-x} =$

$= z''e^{-x} = xe^{-x} = HL \Leftrightarrow z'' = x \Leftrightarrow z' = \frac{x^2}{2} + C_1 \Leftrightarrow z = \frac{x^3}{6} +$

$+ C_1x + C_2 \Leftrightarrow \underline{y = (\frac{x^3}{6} + C_1x + C_2)e^{-x}}$; (*)

$y(0) = 1 \overset{(*)}{\Rightarrow} C_2 = 1 \overset{(*)}{\Rightarrow} y' = -y + (\frac{x^2}{2} + C_1)e^{-x} \Rightarrow y'(0) = -C_2 +$

$+ C_1 = 0 \Leftrightarrow C_1 = C_2 = 1 \Rightarrow \underline{y = (1 + x + \frac{1}{6}x^3)e^{-x}}.$

Övning 8.52 (Sid. 163)

Lösning

$y = ze^{-x} \Rightarrow y' = (z' - z)e^{-x} \Rightarrow y'' = (z'' - 2z' + z)e^{-x}$;

$VL = y'' + 2y' + y = (z'' - 2z' + z + 2z' - 2z + z)e^{-x} = z''e^{-x} =$

$= e^x + e^{-x} = HL \Leftrightarrow z'' = e^{2x} + 1 \Leftrightarrow z' = \frac{1}{2}e^{2x} + x + C_1 \Leftrightarrow$

$\Leftrightarrow z = \frac{1}{4}e^{2x} + \frac{1}{2}x^2 + C_1x + C_2 \Leftrightarrow \underline{y = \frac{1}{4}e^x + (\frac{1}{2}x^2 + C_1x + C_2)e^{-x}}.$

Övning 8.53 (Sid. 163)

Lösning (Studera fall F (sid. 393)).

(i) $y'' - 3y' - 4y = 0 \Leftrightarrow r^2 - 3r - 4 = 0 \Leftrightarrow r = -1 \vee r = 4 \Rightarrow y = e^{-x} \vee$

$\vee\ y = e^{4x} \Rightarrow \underline{y_h = C_1e^{-x} + C_2e^{4x}}.$

(ii) $y'' - 3y' - 4y = 5e^{-x}$; $y = e^{-x}$ ingår i y_h.

$y_p = axe^{-x} \Rightarrow y_p' = a(1-x)e^{-x} \Rightarrow y_p'' = a(x-2)e^{-x}$;

$VL = y_p'' - 3y_p' - 4y_p = a(x - 2 - 3 + 3x - 4x)e^{-x} = -5ae^{-x} =$

$= 5e^{-x} = HL \Leftrightarrow a = -1 \Rightarrow \underline{y_{p_1} = -xe^{-x}}.$

(iii) $y'' - 3y' - 4y = 4x$; lämplig ansats $y = ax + b$;

$VL = 0 - 3a - 4ax - 4b = 4x = HL \Leftrightarrow -4a = 4 \wedge 3a + 4b = 0 \Leftrightarrow$

$\Leftrightarrow a = -1 \wedge b = \frac{3}{4} \Rightarrow \underline{y_{p_2} = -x + \frac{3}{4}}.$

(iv) $y = (C_1 - x)e^{-x} + C_2e^{4x} - x + \frac{3}{4} \Rightarrow y(0) = C_1 + C_2 + 3/4$;

$y' = (x - C_1 - 1)e^{-x} + 4C_2e^{4x} - 1 \Rightarrow y'(0) = -C_1 - 1 + 4C_2 - 1$;

$$\begin{cases} y(0) = 1 \Rightarrow C_1 + C_2 = \frac{1}{4} \\ y'(0) = -1 \Rightarrow -C_1 + 4C_2 = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 = 0 \\ C_2 = \frac{1}{4} \end{cases} \Rightarrow y = -xe^{-x} + \frac{e^{4x}}{4} - x + \frac{3}{4}.$$

Resultat. $y = \frac{1}{4}(3 + e^{4x}) - x(1 + e^{-x}).$

Övning 8.54 (Sid. 163)

Lösning

$y = ze^{-x} \Rightarrow y' = (z'-z)e^{-x} \Rightarrow y'' = (z''-2z'+z)e^{-x}$;

$VL = y''+2y'+y = z''e^{-x} = xe^{-x}+1 = HL \Leftrightarrow z'' = x+e^{x} \Leftrightarrow$

$\Leftrightarrow z' = \frac{1}{2}x^2+e^x+C_1 \Leftrightarrow z = \frac{1}{6}x^3+e^x+C_1x+C_2$;

$y(0) = z(0)e^{-0} = \underline{z(0)=0}$; $y'(0) = (z'(0)-z(0))e^{-0} = \underline{z'(0)=0}$.

$$\left.\begin{array}{l} z(0)=0 \Rightarrow C_2=-1 \\ z'(0)=0 \Rightarrow C_1=-1 \end{array}\right\} \Rightarrow z = \frac{1}{6}x^3-x-1+e^x \Rightarrow \underline{y = 1+\left(\frac{x^3}{6}-x-1\right)e^{-x}}.$$

Övning 8.56 (Sid. 163)

Lösning

a) $y''-2y'-y=0 \Leftrightarrow r^2-2r-1=0 \Leftrightarrow r = 1-\sqrt{2} \lor r = 1+\sqrt{2} \Rightarrow$

$\Rightarrow y = e^{(1-\sqrt{2})x} \lor y = e^{(1+\sqrt{2})x} \Rightarrow \underline{y_h = e^x(C_1e^{-\sqrt{2}x}+C_2e^{\sqrt{2}x})}$.

$y_p = A\cos 3x+B\sin 3x \Rightarrow y_p' = 3(-A\sin 3x+B\cos 3x)$;

$VL = y_p''-2y_p'-y_p = -9y_p-y_p-2y_p' = -10y_p-2y_p' = \dots =$

$= -(10A+6B)\cos 3x+(6A-10B)\sin 3x = \sin 3x = HL \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 5A+3B=0 \\ 6A-10B=1 \end{cases} \Leftrightarrow \begin{cases} A=-\frac{3}{68} \\ B=\frac{5}{68} \end{cases} \Rightarrow y_p = \frac{1}{68}(5\sin 3x-3\cos 3x).$$

Resultat: $y = e^x(C_1e^{-\sqrt{2}x}+C_2e^{\sqrt{2}x})+\frac{1}{68}(5\sin 3x-3\cos 3x)$.

b) $y''+4y=0 \Leftrightarrow r^2+4=0 \Leftrightarrow r=\pm 2i \Rightarrow \underline{y_h = C_1\cos 2x+C_2\sin 2x}$.

$y_p = A\cos x+B\sin x \Rightarrow y_p''+4y_p = -y_p+4y_p = 3y_p =$

$= 2\sin x-\cos x = HL \Leftrightarrow \underline{y_p = \frac{1}{3}(2\sin x-\cos x)}$.

Resultat: $y = C_1\cos 2x+C_2\sin 2x+\frac{2}{3}\sin x-\frac{1}{3}\cos x$.

c) $\underline{y_h = C_1e^x+C_2e^{2x}}$ (Se 8.51 a)).

$y_p = A\cos 2x+B\sin 2x \Rightarrow y_p' = 2(-A\sin 2x+B\cos 2x) \Rightarrow$

$\Rightarrow VL = y_p''-3y_p'+2y_p = -4y_p-3y_p'+2y_p = -2y_p-3y_p' =$

$= -2A\cos 2x-2B\sin 2x+6A\sin 2x-6B\cos 2x =$

$= -2(A+3B)\cos 2x+2(3A-B)\sin 2x = \cos 2x+\sin 2x =$

$$= HL \Leftrightarrow \begin{cases} A+3B=-1/2 \\ 3A-B=1/2 \end{cases} \Leftrightarrow \begin{cases} 10A=1 \\ B=3A-\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} A=1/10 \\ B=-1/5 \end{cases};$$

Resultat: $y = C_1e^x+C_2e^{2x}+\frac{1}{10}\cos 2x-\frac{1}{5}\sin 2x$.

d) $y''-2y'+5y=0 \Leftrightarrow r^2-2r+5=0 \Leftrightarrow r=1+2i \lor r=1-2i \Rightarrow$

$\Rightarrow y_h = e^x(C_1\cos 2x+C_2\sin 2x)$.

$y_{p_1} = \frac{1}{5}$ är en partikulärlösning till $y''-2y'+5y=1$.

$y_p = A\cos x+B\sin x \Rightarrow y_p' = -A\sin x+B\cos x \Rightarrow y_p'' = -y_p$;

$VL = y_p''-2y_p'+5y_p = -y_p-2y_p'+5y_p = 4y_p-2y_p' =$

$= \underbrace{(4A-2B)}_{0}\cos x+\underbrace{(2A+4B)}_{1}\sin x = 0\cdot\cos x+1\cdot\sin x = HL$

deras produkt z^2-2z+3. Divisionen ger

$$\begin{array}{rl}
z^4 + \qquad 4 & \\
\hline
z^6-2z^5+3z^4+4z^2-8z+12 & \big|\, \underline{z^2-2z+3} \\
(-)\ \underline{z^6-2z^5+3z^4} & \\
4z^2-8z+12 & \\
(-)\ \underline{4z^2-8z+12} & \\
0 &
\end{array}$$

c) $z^4+4=0 \Leftrightarrow z^2=\pm 2i=\pm(1+i)^2 \Leftrightarrow \begin{cases} z=\pm(1+i) \\ z=\pm i(1+i) \end{cases}$

<u>Resultat</u>: a) $1-i\sqrt{2}$, b) z^2-2z+3; c) $1+i, 1-i,$

$-1+i, -1-i$.

$\Leftrightarrow \begin{cases} 2A = B \\ A+2B = 1/2 \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{10} \\ B = \frac{1}{5} \end{cases} \Rightarrow y_{P_2} = \frac{1}{10}\cos x + \frac{1}{5}\sin x$;

<u>Resultat</u>: $y = e^x(C_1\cos x + C_2\sin x) + \frac{1}{10}\cos x + \frac{1}{5}\sin x + \frac{1}{5}$.

e) $y'' + 4y = 0 \Leftrightarrow y_h = C_1\cos 2x + C_2\sin 2x$. (Se 8.47).

$y_{P_1} = 1/4$ är en partikulärlösning till $y'' + 4y = 1$.

Betrakta ekvationen $y'' + 4y = \cos 2x$. $\cos 2x$ är en jämn funktion, så vi ansätter $y_p = Ax \cdot \sin 2x$.

<u>Anm.</u> f jämn $\Rightarrow f'$ udda $\Rightarrow f''$ jämn ...

$y_p' = A\sin 2x + 2Ax\cos 2x \Rightarrow y_p'' = 4A\cos 2x - 4y_p \Leftrightarrow$

$\Leftrightarrow VL = y_p'' + 4y_p = 4A\cos 2x = \cos 2x = HL \Leftrightarrow A = 1/4$;

<u>Resultat</u>: $y = C_1\cos 2x + (C_2 + \frac{x}{4})\sin 2x + \frac{1}{4}$.

<u>Övning 8.57 (Sid. 164)</u>

<u>Lösning</u>

$\underline{y = ze^{3x}} \Rightarrow y' = (z' + 3z)e^{3x} \Rightarrow y'' = (z'' + 6z' + 9z)e^{3x}$;

$VL = y'' - 3y' + 2y = (z'' + 6z' + 9z - 3z' - 9z + 2z)e^{3x} =$

$= (z'' + 3z' + 2z)e^{3x} = e^{3x}\cos x = HL \Leftrightarrow \underline{z'' + 3z' + 2z = \cos x.}$

$z'' + 3z' + 2z = 0 \Leftrightarrow z_h = C_1e^{-x} + C_2e^{-2x}$ (Se Ö. 8.50)

$z_p = A\cos x + B\sin x \Rightarrow z_p' = -A\sin x + B\cos x \Rightarrow z_p'' = -z_p$;

$VL = z_p'' + 3z_p' + 2z_p = -z_p + 3z_p' + 2z_p = 3z_p' + z_p = -3A\sin x +$

$+ 3B\cos x + A\cos x + B\sin x = (A + 3B)\cos x + (-3A + B)\sin x$;

$VL = \cos x = HL \Rightarrow \begin{cases} A + 3B = 1 \\ B = 3A \end{cases} \Leftrightarrow \begin{cases} 10A = 1 \\ B = 3A \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{10} \\ B = \frac{3}{10} \end{cases}$;

$z = C_1e^{-x} + C_2e^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x$; $(z = ye^{-3x})$;

<u>Resultat</u>: $y = C_1e^{2x} + C_2e^{x} + \frac{1}{10}e^{3x}(\cos x + 3\sin x)$.

<u>Övning 8.58 (Sid. 164)</u>

<u>Lösning</u>

a) $\underline{y = ze^{3x}} \Rightarrow y' = (z' + 3z)e^{3x} \Rightarrow y'' = (z'' + 6z' + 9z)e^{3x}$;

$VL = y'' - 6y' + 9y = (z'' + 6z' + 9z - 6z' - 18z + 10z)e^{3x} = (z'' + z)e^{3x}$;

$VL = e^{3x}\cos x = HL \Leftrightarrow \underline{z'' + z = \cos x.}$

$z'' + z = 0 \Leftrightarrow z_h = C_1\cos x + C_2\sin x$.

$\cos x$ är jämn och dessutom ingår i z_h så jag ansätter $z_p = Ax \cdot \sin x$ (jämn); $z_p'' = 2A\cos x - z_p$;

$VL = z_p'' + z_p = 2A\cos x = \cos x = HL \Leftrightarrow A = \frac{1}{2} \Rightarrow z_p = \frac{1}{2}x\sin x$;

$z = C_1\cos x + (C_2 + \frac{x}{2})\sin x$;

<u>Resultat</u>: $y = e^{3x}(C_1\cos x + (C_2 + \frac{1}{2}x)\sin x)$.

b) $y = ze^{-2x} \Rightarrow y' = (z' - 2z)e^{-2x} \Rightarrow y'' = (z'' - 4z' + 4z)e^{-2x}$;

$VL = y'' + 6y' + 8y = (z'' - 4z' + 4z + 6z' - 12z + 8z)e^{-2x} = (z'' + 2z')e^{-2x}$;

$VL = e^{-2x}\sin x = HL \Leftrightarrow \underline{z'' + 2z' = \sin x}$.

$z'' + 2z' = 0 \Leftrightarrow r^2 + 2r = 0 \Leftrightarrow r = 0 \vee r = -2 \Rightarrow z_h = C_1 + C_2 e^{-2x}$;

$z_p = A\cos x + B\sin x \Rightarrow z_p' = -A\sin x + B\cos x \Rightarrow z_p'' = -z_p$;

$VL = z_p'' + 2z_p' = 2z_p' - z_p = -A\cos x - B\sin x - 2A\sin x + 2B\cos x =$

$= (-A + 2B)\cos x - (2A + B)\sin x = \sin x = HL \Leftrightarrow A = 2B \wedge$

$\wedge\ 2A + B = -1 \Leftrightarrow A = -\frac{2}{5} \wedge B = -\frac{1}{5} \Rightarrow z_p = -\frac{2}{5}\cos x - \frac{1}{5}\sin x$;

$z = C_1 + C_2 e^{-2x} - \frac{1}{5}(2\cos x + \sin x)$;

Resultat: $y = C_1 e^{-2x} + C_2 e^{-4x} - \frac{1}{5}e^{-2x}(2\cos x + \sin x)$.

Övning 8.59 (Sid. 164)

Lösning

$y'' + \frac{1}{4}y = 0 \Leftrightarrow r^2 + \frac{1}{4} = 0 \Leftrightarrow r = \pm\frac{i}{2} \Rightarrow y_h = A\cos\frac{x}{2} + B\sin\frac{x}{2}$.

$y_p = A\sin x$ (udda ansats, ty HL udda).

$VL = y_p'' + \frac{1}{4}y_p = -y_p + \frac{1}{4}y_p = -\frac{3}{4}y_p = -\sin x = HL \Leftrightarrow y_p = \frac{4}{3}\sin x$.

$y = C_1\cos\frac{x}{2} + C_2\sin\frac{x}{2} + \frac{4}{3}\sin x$; $y(0) = 0 \Rightarrow C_1 = 0$;

$y = C_2\sin\frac{x}{2} + \frac{4}{3}\sin x$, $y(\pi) = 0 \Rightarrow C_2 = 0$.

Resultat: $y = \frac{4}{3}\sin x$.

y_h har "försvunnit" i detta fall.

Övning 8.60 (Sid. 164)

Lösning

$y'' + 9y = 0 \Leftrightarrow r^2 + 9 = 0 \Leftrightarrow r = \pm 3i \Rightarrow y_h = C_1\cos 3x + C_2\sin 3x$.

$\sin 3x$ ingår i y_h, så $\alpha = 3$.

Anm. $y = \sin\alpha x \Rightarrow y'' = -\alpha^2\sin\alpha x$;

$VL = y'' + 9y = -\alpha^2\sin\alpha x + 9\sin\alpha x = 0 = HL \Rightarrow \underline{\alpha = 3}$.

$y'' + 9y = \sin 3x$, $y_p = Ax\cos 3x$ (en udda ansats).

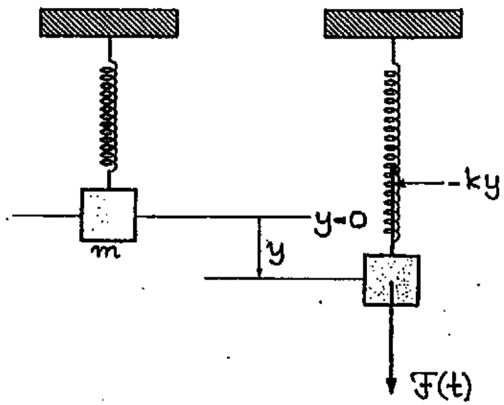
$y'' = -6A\sin 3x - 9y_p \Leftrightarrow VL = y_p'' + 9y_p = -6A\sin 3x =$

$= \sin 3x = HL \Leftrightarrow A = -\frac{1}{6} \Rightarrow y_p = -\frac{1}{6}x\cdot\cos 3x$.

Resultat: $y = (C_1 - \frac{1}{6}x)\cos 3x + C_2\sin 3x$, för $\alpha = 3$.

Övning 8.61 (Sid. 164)

Lösning



forts.

Newtons andra lag ger

$m\frac{du}{dt} = -ky + F(t) \Rightarrow 2\frac{d^2y}{dt^2} = -32y + 12\sin\omega t \Leftrightarrow$

$\Leftrightarrow \underline{y'' + 16y = 6\sin\omega t,\ y(0) = y'(0) = 0.}$

$y'' + 16y = 0 \Leftrightarrow r^2 + 16 = 0 \Leftrightarrow r = \pm 4i \Rightarrow y_h = C_1\cos 4t + C_2\sin 4t.$

(i) $\underline{\omega = 4}$. $\sin 4t$ ingår i y_h, så jag ansätter

$y_p = At\cdot\cos 4t \Rightarrow y_p'' = -8A\sin 4t - 16y_p \Rightarrow VL = y_p'' + 16y_p =$

$= -8A\sin 4t = 6\sin 4t = HL \Leftrightarrow A = -\frac{3}{4} \Rightarrow y_p = -\frac{3}{4}t\cos 4t;$

$y = (C_1 - \frac{3}{4}t)\cos 4t + C_2\sin 4t$; $y(0) = 0 \Rightarrow C_1 = 0$;

$y = C_2\sin 4t - \frac{3}{4}t\cos 4t \Rightarrow y' = 4C_2\cos 4t - \frac{3}{4}\cos 4t +$

$+ 3t\sin 4t$; $y'(0) = 0 \Rightarrow 4C_2 - \frac{3}{4} = 0 \Leftrightarrow C_2 = \frac{3}{16}$;

$\underline{y = \frac{3}{16}\sin 4t - \frac{3}{4}t\cos 4t.}$

(ii) $\underline{\omega \neq 4}$: $y_p = B\sin\omega t \Rightarrow y_p'' = -\omega^2 y_p \Rightarrow VL = y_p'' +$

$+ 16y_p = (16 - \omega^2)y_p = 12\sin\omega t = HL \Leftrightarrow y_p = \frac{12}{16-\omega^2}\sin\omega t.$

$y = C_1\cos 4t + C_2\sin 4t + \frac{12}{16-\omega^2}\sin\omega t;$

$y(0) = 0 \Rightarrow C_1 = 0 \Rightarrow y = C_2\sin 4t + \frac{12}{16-\omega^2}\sin\omega t \Rightarrow$

$\Rightarrow y' = 4C_2\cos 4t + \frac{12\omega}{16-\omega^2}\cos\omega t \Rightarrow y'(0) = 4C_2 + \frac{12\omega}{16-\omega^2}$;

$y'(0) = 0 \Rightarrow 4C_2 + \frac{12\omega}{16-\omega^2} = 0 \Leftrightarrow C_2 = \frac{3\omega}{\omega^2-16}$;

$y = \frac{3}{\omega^2-16}(\omega\sin 4t - 4\sin\omega t).$

Övning 8.62 (Sid. 164)

Lösning

$y^{(4)} - 3y'' - 4y = 0 \Leftrightarrow r^4 - 3r^2 - 4 = (r^2-4)(r^2+1) = 0 \Leftrightarrow r = \pm 2 \lor$

$\lor\ r = \pm i \Rightarrow y_h = C_1e^{2x} + C_2e^{-2x} + C_3\cos x + C_4\sin x.$

$y_p = Axe^{-2x} \Rightarrow D^n y_p \overset{I}{=} \sum_{j=1}^{n} A(D^{n-k}e^{-2x})D^k x = A(D^n e^{-2x})x +$

$+ A(D^{n-1}e^{-2x})n \Rightarrow D^2 y_p = A(4x-4)e^{2x} \land D^4 y_p = A\cdot 16xe^{-2x} +$

$+ 4A(-8e^{-2x}) = A(16x - 32)e^{-2x}$;

Anm. I $\overset{I}{=}$ tillämpar jag Leibniz' regel.

$VL = y^{(4)} - 3y'' - 4y = A(16x - 32 - 12x + 12 - 4x)e^{-2x} = -20Ae^{-2x}$;

$VL = e^{-2x} = HL \Rightarrow A = -\frac{1}{20} \Rightarrow y_p = -\frac{x}{20}e^{-2x}.$

Resultat: $y = C_1e^{2x} + (C_2 - \frac{x}{20})e^{-2x} + C_3\cos x + C_4\sin x.$

Övning 8.63 (Sid. 164)

Lösning

a) $y''' + 6y'' + 11y' + 6y = 0 \Leftrightarrow r^3 + 6r^2 + 11r + 6 = (r+1)(r+2)(r+3) =$

$= 0 \Leftrightarrow r = -1 \lor r = -2 \lor r = -3 \Rightarrow y = C_1e^{-x} + C_2e^{-2x} + C_3e^{-3x}.$

b) $y^{(4)} - 2y''' + 2y' - y = 0 \Leftrightarrow r^4 - 2r^3 + 2r - 1 = (r-1)(r^3 - r^2 - r + 1) =$

$= (r-1)(r^2(r-1) - (r-1)) = (r-1)^2(r^2-1) = (r-1)^3(r+1) = 0 \Leftrightarrow$

$\Leftrightarrow r = r_1 = r_2 = r_3 = 1 \lor r = -1 \Leftrightarrow y = e^x \lor y = xe^x \lor y = x^2e^x$

$\vee\ y = e^{-x} \Rightarrow y_h = C_1 e^x + C_2 x e^x + C_3 x^2 e^x + C_4 e^{-x}$;

$\underline{y = ze^x} \Rightarrow y' = (z'+z)e^x \Rightarrow y'' = (z''+2z'+z)e^x \Rightarrow y''' = (z''' + 3z'' + 3z' + z)e^x \Rightarrow y^{(4)} = (z^{(4)} + 4z''' + 6z'' + 4z' + z)e^x$;

$VL = y^{(4)} - 2y''' + 2y' - y = (z^{(4)} + 4z''' + 6z'' + 4z' + z - 2z''' - 6z'' - 6z' - 2z + 2z' + 2z - z)e^x = (z^{(4)} + 2z''')e^x = e^x = HL \Leftrightarrow$

$\Leftrightarrow \underline{z^{(4)} + 2z''' = 1.}$ (z_p sökes.)

$z_p = Ax^3 \Rightarrow z_p''' = 6A \Rightarrow VL = z_p^{(4)} + 2z_p''' = 12A = 1 = HL \Leftrightarrow$

$\Leftrightarrow A = \frac{1}{12} \Rightarrow z_p = \frac{1}{12}x^3 \Leftrightarrow y_p = \frac{1}{12}x^3 e^x$.

Resultat: $y = (C_1 + C_2 x + C_3 x^2 + \frac{1}{12}x^3)e^x + C_4 e^{-x}$.

c) $y''' + 9y' = 0 \Leftrightarrow r^3 + 9r = r(r^2+9) = 0 \Leftrightarrow r = 0 \vee r = \pm 3i \Rightarrow$

$\Rightarrow y = 1 \vee y = \cos 3x \vee y = \sin 3x \Rightarrow y_h = C_1 + C_2 \cos 3x + C_3 \sin 3x$.

$y_p = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx \Rightarrow y_p' = 3Ax^2 + 2Bx + C$

$\Rightarrow y_p'' = 6Ax + 2B \Rightarrow y_p''' = 6A$;

$VL = y_p''' + 9y_p' = 6A + 27Ax^2 + 18Bx + 9C = x^2 + 5 = HL \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 27A = 1 \\ 18B = 0 \\ 6A + 9C = 5 \end{cases} \Leftrightarrow \begin{cases} A = 1/27 \\ B = 0 \\ C = 16/81 \end{cases} \Rightarrow y_p = \frac{1}{81}(3x^3 + 16x);$$

Resultat: $y = C_1 + C_2 \cos 3x + C_3 \sin 3x + \frac{1}{81}(3x^3 + 16x)$.

d) $y''' - (a+2)y'' + (2a+1)y' - ay = 0 \Leftrightarrow r^3 - (a+2)r + (2a+1)r - a =$

$= (r-a)(r\ 1) = 0 \Leftrightarrow r = a \vee r = r_1 = r_2 = 1$;

(i) $\underline{a = +1} \Rightarrow r = r_1 = r_2 = r_3 = 1 \Rightarrow y_h = (C_1 + C_2 x + C_3 x^2)e^x$

$y_p = -1$ med blotta ögat, så $y = (C_1 + C_2 x + C_3 x^2)e^x - 1$.

(ii) $\underline{a \neq 1} \Rightarrow y = (C_1 + C_2 x)e^x + C_3 e^{ax} - 1/a$.

(iii) $\underline{a = 0} \Rightarrow y_h = C_1 + (C_2 + C_3 x)e^x$;

$y_p = Ax \Rightarrow VL = y_p''' - 2y_p'' + y_p' = A = 1 = HL \Rightarrow y_p = x$;

$\therefore\ y = C_1 + (C_2 + C_3 x)e^x + x$.

$$\text{Resultat:} \begin{cases} a = 1 \Rightarrow y = (C_1 + C_2 x + C_3 x^2)e^x - 1 \\ a \neq 1 \Rightarrow y = (\hat{C}_1 + \hat{C}_2 x)e^x + \hat{C}_3 e^{ax} - 1/a. \\ a = 0 \Rightarrow y = \check{C}_1 + (\check{C}_2 + \check{C}_3 x)e^x + x \end{cases}$$

Övning 8.64 (Sid. 164)

Lösning

$\lambda = \mu^4 \Rightarrow y^{(4)} - \lambda y = y^{(4)} - \mu^4 y = 0 \Leftrightarrow r^4 - \mu^4 = 0 \Leftrightarrow$

$\Leftrightarrow r = \pm\mu \vee r = \pm\mu i \Rightarrow y = C_1 e^{\mu x} + C_2 e^{-\mu x} + C_3 \cos\mu x + C_4 \sin\mu x$;

$\Rightarrow y'' = \mu^2(C_1 e^{\mu x} + C_2 e^{-\mu x} - C_3 \cos\mu x - C_4 \sin\mu x)$;

$$\left.\begin{aligned} y(0) = 0 &\Rightarrow C_1 + C_2 + C_3 = 0 \\ y''(0) = 0 &\Rightarrow C_1 + C_2 - C_3 = 0 \end{aligned}\right\} \Leftrightarrow \begin{cases} C_1 + C_2 = 0 \\ C_3 = 0 \end{cases} \Leftrightarrow \begin{cases} C_3 = 0 \\ C_2 = -C_1 \end{cases} \Rightarrow$$

$\Rightarrow y = 2C_1 \sinh\mu x + C_4 \sin\mu x$; forts.

$y'' = \mu^2(2C_1 \sinh \mu x - C_4 \sin \mu x)$

$$\left.\begin{array}{l} y(\pi)=0 \Rightarrow 2C_1 \sinh \mu\pi + C_4 \sin \mu\pi = 0 \\ y''(\pi) = 0 \Rightarrow 2C_1 \sinh \mu\pi - C_4 \sin \mu\pi = 0 \end{array}\right\} \Rightarrow \begin{cases} C_1 = 0 \\ \sin \mu\pi = 0 \end{cases}$$

$\sin \mu\pi = 0 \Leftrightarrow \mu\pi = n\pi \Rightarrow \mu = n = 1,2,3,\dots$

<u>Resultat:</u> $y = y_n = C_n \cdot \sin(nx), \quad n = 1,2,3,\dots$

Ekvationer av speciella typer

Övning 8.65 (Sid. 165)

<u>Lösning</u>

$xyy' = x^2+y^2 \Leftrightarrow y' = \frac{x^2+y^2}{xy} = \frac{y}{x} + \frac{x}{y}$ (homogen).

$z = \frac{y}{x} \Leftrightarrow y = xz \Rightarrow y' = z + xz' = z + \frac{1}{z} \Leftrightarrow \frac{dz}{dx}\cdot x = \frac{1}{z} \Leftrightarrow$

$\Leftrightarrow 2z\,dz = \frac{2}{x}dx \Leftrightarrow z^2 = \ln x^2 + C \Leftrightarrow y^2 = x^2(\ln x^2 + C)$.

Övning 8.66 (Sid. 165)

<u>Lösning</u>

$xy' = y(1+\ln y - \ln x) = y(1+\ln\frac{y}{x}) \Leftrightarrow y' = \frac{y}{x} + \frac{y}{x}\ln\frac{y}{x}$;

$y = xz \Rightarrow y' = z + xz' = z + z\ln z \Leftrightarrow x\frac{dz}{dx} = z\ln z \Leftrightarrow$

$\Leftrightarrow \frac{1}{\ln z}\frac{dz}{z} = \frac{dx}{x} \Leftrightarrow \int\frac{dz}{z\ln z} = \int\frac{dx}{x} \Leftrightarrow \ln\ln z = \ln Cx \Leftrightarrow$

$\Leftrightarrow \ln z = Cx \Leftrightarrow z = e^{Cx} \Leftrightarrow y = xe^{Cx} \Rightarrow y(1) = e^C$;

$y(1) = e \Rightarrow e^C = e \Leftrightarrow C = 1 \Rightarrow \underline{y = xe^x}$.

Övning 8.67 (Sid. 165)

<u>Lösning</u>

$(x-y)y' - y = 0$; $\underline{y=0}$ är en lösning, den triviala.

$(1-\frac{y}{x})y' = \frac{y}{x}$; $y = xz \Rightarrow y' = xz' + z \Rightarrow (1-z)(z+xz') = z \Leftrightarrow$

$\Leftrightarrow z + xz' = \frac{z}{1-z} \Leftrightarrow xz' = \frac{z}{1-z} - z = \frac{z^2}{1-z} \Leftrightarrow \frac{1-z}{z^2}\frac{dz}{dx} = \frac{1}{x} \Leftrightarrow$

$\Leftrightarrow \int(-\frac{1}{z}+\frac{1}{z^2})dz = \int\frac{1}{x}dx \Leftrightarrow -\ln z - \frac{1}{z} = \ln Cx \Leftrightarrow \frac{1}{z} + \ln Cxz = 0$

$\Leftrightarrow \frac{x}{y} + \ln Cy = 0 \Leftrightarrow \underline{x + y\ln Cy = 0}$.

Övning 8.68 (Sid. 165)

<u>Lösning</u>

$y' + y^2 = \frac{1}{x^2}$; $z = xy \Rightarrow y = \frac{z}{x} \Rightarrow y' = \frac{xz'-z}{x^2}$;

$\frac{xz'-z}{x^2} + \frac{z^2}{x^2} = \frac{1}{x^2} \Leftrightarrow xz' - z + z^2 = 1 \Leftrightarrow xz' = -(z^2 - z \;\; 1)$;

(i) $z^2 - z - 1 = 0 \Leftrightarrow z = \frac{y}{x} = \frac{1\pm\sqrt5}{2} \Leftrightarrow y = \frac{1\pm\sqrt5}{2}x$ lösningar.

(ii) <u>Anm.</u> $\alpha \neq \beta \Rightarrow \frac{1}{(u-\alpha)(u-\beta)} = \frac{1}{\alpha-\beta}(\frac{1}{u-\alpha} - \frac{1}{u-\beta})$.

$x\frac{dz}{dx} = -(z - \frac{1+\sqrt5}{2})(z - \frac{1-\sqrt5}{2}) \Leftrightarrow \frac{dz}{(z-(1+\sqrt5)/2)(z-(1-\sqrt5)/2)} = -\frac{dx}{x}$

$\Leftrightarrow \frac{1}{\sqrt5}(\frac{1}{z-(1+\sqrt5)/2} - \frac{1}{z-(1-\sqrt5)/2})dz = -\frac{dx}{x} \Leftrightarrow \frac{1}{\sqrt5}\ln\frac{z-(1+\sqrt5)/2}{z-(1-\sqrt5)/2} =$

$= \ln Cx^{-1} \Leftrightarrow \ln\frac{z-(1+\sqrt5)/2}{z-(1-\sqrt5)/2} = \sqrt5 \ln Cx^{-1} = \ln Ax^{-\sqrt5} \Leftrightarrow$

$$\Leftrightarrow \frac{2z-1-\sqrt{5}}{2z-1+\sqrt{5}} = Ax^{-\sqrt{5}} \Leftrightarrow \frac{2z-1+\sqrt{5}-2\sqrt{5}}{2z-1+\sqrt{5}} = 1-\frac{2\sqrt{5}}{2z-1+\sqrt{5}} =$$

$$= Ax^{-\sqrt{5}} \Leftrightarrow \frac{2\sqrt{5}}{2z-1+\sqrt{5}} = 1-Ax^{-\sqrt{5}} = \frac{x^{\sqrt{5}}-A}{x^{\sqrt{5}}} \Leftrightarrow 2z-1+\sqrt{5} =$$

$$= \frac{2\sqrt{5}x^{\sqrt{5}}}{x^{\sqrt{5}}-A} \Leftrightarrow 2z = 1-\sqrt{5}+\frac{2\sqrt{5}x^{\sqrt{5}}}{x^{\sqrt{5}}-A} \Leftrightarrow \underline{y = \frac{1-\sqrt{5}}{2}x + \frac{\sqrt{5}x^{\sqrt{5}+1}}{x^{\sqrt{5}}-A}}.$$

Övning 8.69 (Sid. 165)

Lösning

$y'+g(x)y = h(x)y^{\alpha}$; $G(x) = \int g(x)dx$

$z = ye^{G(x)} \Rightarrow z' = y'e^{G}+gye^{G} = h(x)y^{\alpha}e^{G} \Leftrightarrow z' = \frac{dz}{dx} =$

$= h(x)e^{G(x)}z^{\alpha}e^{-\alpha \cdot G(x)} \Leftrightarrow \underline{\frac{dz}{z^{\alpha}} = h(x)e^{(1-\alpha)G(x)}dx}.$ (*)

$\underline{y'-xy = x^3y^3} \Rightarrow g(x) = -x \Rightarrow G(x) = -\frac{x^2}{2} \overset{(*)}{\Rightarrow} \frac{dz}{z^3} = x^3e^{x^2}dx$

$\Leftrightarrow -\frac{2}{z^3}dz = -2x^3e^{x^2}dx \Leftrightarrow \int(-\frac{2}{z^3})dz = -\int x^2e^{x^2}\cdot 2xdx \Leftrightarrow$

$\Leftrightarrow \frac{1}{z^2} = -(x^2-1)e^{x^2}+C \Leftrightarrow \frac{1}{y^2}e^{x^2} = (1-x^2)e^{x^2}+C \Leftrightarrow$

$\Leftrightarrow y^{-2} = 1-x^2+Ce^{-x^2} \Leftrightarrow y^2 = 1/(Ce^{-x^2}+1-x^2)$ el. $y=0$.

Övning 8.70 (Sid. 165)

Lösning

$x = e^t \Leftrightarrow t = \ln x \Rightarrow y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x} \Leftrightarrow xy' = \frac{dy}{dt}.$ (1)

$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{1}{x}\frac{dy}{dt}) = -\frac{1}{x^2}\frac{dy}{dt}+\frac{1}{x}\frac{d}{dx}\frac{dy}{dt} =$

$= -\frac{1}{x^2}\frac{dy}{dt} - \frac{d^2y}{dt^2}\frac{dt}{dx}\cdot\frac{1}{x} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt} \Leftrightarrow x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$ (2)

$x^2y''-2xy'+2y = 2x^2 \Rightarrow (1)+(2) \Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt}+2y = 2e^{2t}$

$\Leftrightarrow D^2y-3Dy+2y = 2e^{2t} \Leftrightarrow (D^2-3D+2)y = 2e^{2t}$; $D = \frac{d}{dt}$;

$r^2-3r+2 = 0 \Leftrightarrow r=1 \lor r=2 \Leftrightarrow y_h = C_1e^t+C_2e^{2t}$;

$y_p = Ate^{2t} \Rightarrow y_p' = A(2t+1)e^{2t} \Rightarrow y_p'' = A(4t+4)e^{2t}$;

$VL = D^2y_p-3Dy_p+2y_p = A(4t+4-6t-3+2t) = Ae^{2t} = 2e^{2t} =$

$= HL \Leftrightarrow A=2 \Rightarrow y_p = 2te^{2t}$

$\therefore\ y = C_1e^t+(C_2+2t)e^{2x} \Leftrightarrow \underline{y = C_1x+(C_2+2\ln x)x^2}.$

Övning 8.71 (Sid. 165)

Lösning

a) $x^2y''+3xy'+2y = x^3 \Leftrightarrow D^2y-Dy+3Dy+2y = e^{3t} \Leftrightarrow$

$\Leftrightarrow \underline{(D^2+2D+2)y = e^{3t}}$;

$r^2+2r+2 = 0 \Leftrightarrow r = -1\pm i \Rightarrow y_h = e^{-t}(C_1\cos t+C_2\sin t)$; (*)

$y_p = Ae^{3t} \Rightarrow y_p' = 3y_p \Rightarrow y_p'' = 9y_p$;

$VL = D^2y_p+2Dy_p+2y_p = 9y_p+6y_p+2y_p = 17y_p = e^{3t} =$

$= HL \Leftrightarrow y_p = \frac{1}{17}e^{3t} \Rightarrow y = e^{-t}(C_1\cos t+C_2\sin t)+\frac{1}{17}e^{3t}$;

Resultat: $y = \frac{1}{x}(C_1\cos(\ln x)+C_2\sin(\ln x))+\frac{1}{17}x^3$.

b) Jag har redan visat (Ö. 8.70) att

$$\frac{d}{dx} = \frac{1}{x}\frac{d}{dt} \text{ och } \frac{d^2}{dx^2} = \frac{1}{x^2}(\frac{d^2}{dt^2}-\frac{d}{dt});$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right)=\frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)-$$
$$-\frac{2}{x^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)=\frac{1}{x^2}\left(\frac{d^3y}{dt^3}\frac{dt}{dx}-\frac{d^2y}{dt^2}\frac{dt}{dx}\right)-\frac{2}{x^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)=$$
$$=\frac{1}{x^3}\left(\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}\right)-\frac{2}{x^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\Leftrightarrow x^3y'''=D^3y-3D^2y+2Dy;$$

$$VL=4x^3y'''+3xy'-3y=4D^3y-12D^2y+8Dy+3Dy-3y=$$
$$=4D^3y-12D^2y+11Dy-3y=(4D^3-12D^2+11D-3)y;$$

$VL=HL\Rightarrow \underline{(4D^3-12D^2+11D-3)y=3e^{2t}};\quad D=\frac{d}{dt}.$

$p(r)=4r^3-12r^2+11r-3\ \Rightarrow p(1)=0\Leftrightarrow r-1$ faktor i $p(r)$.

Divisionen ger $p(r)=(r-1)(4r^2-8r+3)$;

$p(r)=0\Rightarrow r=1 \vee 4r^2-8r+3=0\Leftrightarrow r=1\vee r=\frac{1}{2}\vee r=\frac{3}{2}$

$\Rightarrow y_h=C_1e^t+C_2e^{t/2}+C_3e^{3t/2}=C_1x+C_2\sqrt{x}+C_3x\sqrt{x}.$

$y_P=Ae^{2t}\Rightarrow y_P'=2y_P\Rightarrow y_P''=4y_P\Rightarrow y_P'''=8y_P$;

$VL=(4D^3-12D^2+11D-3)y_P=3y_P=3e^{2t}=HL\Leftrightarrow y_P=e^{2t}.$

<u>Resultat</u>: $y=C_1x+(C_2+C_3x)\sqrt{x}+x^2$.

Övning 8.72 (Sid. 165)

<u>Lösning</u>

$\tilde{y}=e^x\Rightarrow \tilde{y}'=\tilde{y}'=e^x\ \Rightarrow x\tilde{y}''-(2x+1)\tilde{y}'+(x+1)\tilde{y}=xe^x-$

$-2xe^x-e^x+xe^x+e^x=0\Rightarrow \tilde{y}=e^x$ är lösning.

$y=ze^x\Rightarrow y'=(z'+z)e^x\Rightarrow y''=(z''+2z'+z)e^x$; fortS.

$$VL=(x(z''+2z'+z)-(2x+1)(z'+z)+(x+1)z)e^x=$$
$$=(xz''+2xz'+xz-2xz'-2xz-z'-z+xz+z)e^x=$$
$$=(xz''-z')e^x;$$

$VL=HL=0\ \Rightarrow xz''-z'=0\Leftrightarrow z''-\frac{1}{x}z'=0\Leftrightarrow \frac{1}{x}z''-\frac{1}{x^2}z'=0$

$\Leftrightarrow \left(\frac{z'}{x}\right)'=0\Leftrightarrow \frac{z'}{x}=2C_1\Leftrightarrow z'=2C_1x\Leftrightarrow z=C_1x^2+C_2$;

<u>Resultat</u>: Den allmänna lösningen är

$$y=(C_1x^2+C_2)e^x,\quad C_1,C_2\in\mathbb{R}.$$

Övning 8.73 (Sid. 166)

<u>Lösning</u>

$\frac{dT}{dt}=\frac{1}{k}(\hat{T}-T)$, $T=T(t)$; $\hat{T}$ = omgivningens temperatur.

$\phi=A\cdot\frac{dx}{dt}$ är det s.k. flödet (enhet m^3/s).

$\frac{dT}{dt}=\frac{dT}{dx}\frac{dx}{dt}=\frac{dT}{dx}\frac{\phi}{A}=\frac{1}{k}(\hat{T}-T)\Leftrightarrow \frac{dT}{dx}=\frac{A}{k\phi}(\hat{T}-T)$;

$\hat{T}=5°$ = markens, dvs. omgivningens, temperatur.

$\frac{dT}{dx}=\lambda(5-T)$, $T(x=0)=85$; $T(\xi)=65$, $\xi=?$ $\quad\left(\lambda=\frac{A}{k\phi}\right)$.

$\frac{dT}{dx}+\lambda T=5\lambda\Rightarrow g(x)=\lambda\Rightarrow G(x)=\lambda x\Rightarrow \mu(x)=e^{\lambda x}\Rightarrow$

$\Rightarrow\frac{d}{dx}(T\cdot e^{\lambda x})=5\lambda e^{\lambda x}\Leftrightarrow Te^{\lambda x}=5e^{\lambda x}+C\Leftrightarrow T(x)=$

$=5+Ce^{-\lambda x}\Rightarrow T(0)=5+C$; $T(0)=85=5+C\Leftrightarrow C=80$

$T(x)=5+80e^{-\lambda x}$, $T(\xi)=65\Leftrightarrow\cdots\Leftrightarrow \xi=\frac{1}{\lambda}\ln\frac{4}{3}$.

Svar: $T(x)=5+80\exp\{-\frac{Ax}{k\phi}\}$; vattnets temperatur är 65° på ett avstånd $\frac{k\phi}{A}\ln\frac{4}{3}$ meter från värmeverket.

Övning 8.74 (Sid. 166)

Lösning

Newtons avsvalningslag $\Rightarrow \frac{dT}{dt}=k(-10-T)$, $T(0)=20$.

$\frac{dT}{dt}+kT=-10k \Rightarrow e^{kt}\frac{dT}{dt}+ke^{kt}T=-10ke^{kt} \Leftrightarrow \frac{d}{dt}Te^{kt}=$

$=-10ke^{kt} \Leftrightarrow T\cdot e^{kt}=-10e^{kT}+C \Leftrightarrow T=-10+Ce^{-kt}$;

$T(0)=20 \Rightarrow -10+C=20 \Leftrightarrow C=30. \Rightarrow \underline{T(t)=30e^{-kt}-10.}$

$T(2)=15 \Rightarrow 30e^{-2k}-10=15 \Leftrightarrow e^{2k}=\frac{6}{5} \Leftrightarrow e^{k}=1{,}2^{1/2}$

$\Rightarrow T(t)=30\cdot(1{,}2)^{-t/2}-10 \Rightarrow T(24)=30\cdot 1{,}2^{-12}-10\approx -6{,}63.$

Svar: Efter ett dygn är temperaturen $-6{,}6°$.

Övning 8.75 (Sid. 166)

Lösning

Newtons andra lag ger

$$m\frac{dv}{dt}=mg-Vg\rho_0-kv,\quad v(0)=0;$$

$\frac{dv}{dt}+\frac{k}{m}v=g-V\rho_0$; jag sätter $\lambda=\frac{k}{m}$ och $\mu=g-V\rho_0$;

$\frac{dv}{dt}+\lambda v=\mu \Leftrightarrow \frac{d}{dt}(ve^{\lambda t})=\mu e^{\lambda t} \Leftrightarrow ve^{\lambda t}=\frac{\mu}{\lambda}e^{\lambda t}+C$

$\Leftrightarrow v=\frac{\mu}{\lambda}+Ce^{-\lambda t} \Rightarrow v(0)=\frac{\mu}{\lambda}+C$; $v(0)=0 \Rightarrow C=-\frac{\mu}{\lambda}$;

$v(t)=\frac{\mu}{\lambda}(1-e^{-\lambda t}) \Rightarrow v_\infty=\lim_{t\to\infty}v(t)=\frac{\mu}{\lambda}=\frac{g(m-V\rho_0)}{k}.$

Övning 8.76 (Sid. 166)

Lösning

$y'=x(4+y^2) \Leftrightarrow \frac{dy}{4+y^2}=x\,dx \Leftrightarrow \int x\,dx=\int\frac{dy}{y^2+4}\left[\begin{matrix}y=2t\\ dy=2dt\end{matrix}\right]=$

$=\left\{\int\frac{2dt}{4t^2+4}\right\}_{t=y/2}=\left\{\frac{1}{2}\int\frac{dt}{t^2+1}\right\}_{t=y/2}=\frac{1}{2}\arctan\frac{y}{2}+C \Leftrightarrow$

$\Leftrightarrow \int 2x\,dx=\underline{x^2=\arctan\frac{y}{2}+C};$

$y(0)=2 \Rightarrow 0=\arctan 1+C \Leftrightarrow C=-\frac{\pi}{4} \Rightarrow \arctan\frac{y}{2}=$

$=x^2+\pi/4 \Leftrightarrow \underline{y=2\tan(x^2+\frac{\pi}{4})};$

$y(0)=2 \Rightarrow -\frac{\pi}{2}<x^2+\frac{\pi}{4}<\frac{\pi}{2} \Leftrightarrow -\frac{3\pi}{4}<x^2<\frac{\pi}{4} \Leftrightarrow x^2<\frac{\pi}{4} \Leftrightarrow$

$\Leftrightarrow \sqrt{x^2}<\sqrt{\frac{\pi}{4}} \Leftrightarrow |x|<\frac{\sqrt{\pi}}{2} \Leftrightarrow -\frac{\sqrt{\pi}}{2}<x<\frac{\sqrt{\pi}}{2}.$

Resultat: $y=2\tan(x^2+\frac{\pi}{4})$, $-\frac{\sqrt{\pi}}{2}<x<\frac{\sqrt{\pi}}{2}$.

Övning 8.77 (Sid. 166)

Lösning

$y'-\frac{1}{x+1}y=\frac{x+3}{x+1} \Rightarrow g(x)=-\frac{1}{x+1} \Rightarrow G(x)=\ln(x+1)^{-1} \Rightarrow \mu(x)=$

$=\frac{1}{x+1} \Rightarrow \frac{d}{dt}\left(\frac{y}{x+1}\right)=\frac{1}{x+1}+\frac{2}{(x+1)^2} \Leftrightarrow \frac{y}{x+1}=\ln(x+1)-\frac{2}{x+1}+C$

$y(2)=0 \Rightarrow C=\frac{2}{3}-\ln 3 \Rightarrow \underline{y=(x+1)(\ln(x+1)+\frac{2}{3}-\ln 3)-2}.$

Övning 8.78 (Sid. 166)

Lösning

(i) $y''+4y=0 \Leftrightarrow y_h=C_1\cos 2x+C_2\sin 2t.$

(ii) $y''+4y=2\sin^2 x=1-\cos 2x$ (Se fall F i boken).

$y''+4y=1 \Rightarrow y_{P_1}=1/4$ (med blotta ögat).

$y''+4y=-\cos 2x$; cos är jämn så vi ansätter jämn, dvs. $y_{P_2}=Ax\sin 2x$; faktorn x kommer från $\cos 2x$, denna finns i y_h.

$y_{P_2}=Ax\cdot\sin 2x \Rightarrow y''_{P_2}=4A\cos 2x-4y_{P_2} \Leftrightarrow y''_{P_2}+4y_{P_2}=$
$=4A\cos 2x=-\cos 2x \Rightarrow A=-\frac{1}{4} \Rightarrow y_{P_2}=-\frac{1}{4}x\sin 2x.$

(iii) $y=C_1\cos 2x+(C_2-\frac{1}{4}x)\sin 2x+\frac{1}{4} \Rightarrow y(0)=C_1+\frac{1}{4}$;

$y'=-2C_1\sin 2x-\frac{1}{4}\sin 2x+2(C_2-\frac{x}{4})\cos 2x \Rightarrow y'(0)=2C_2$;

$y(0)=0=y'(0) \Rightarrow C_1=-\frac{1}{4} \wedge C_2=0$

Resultat: $y=\frac{1}{4}(1-\cos 2x-x\sin 2x).$

Övning 8.79 (Sid. 167)

Lösning

Se nästföljande sida.

$y(t)$ = mängden salt i kärlet vid tiden t.

Smuts in $=8\frac{l}{min}\cdot 3\frac{g}{l}=24\,g/min$

Smuts ut $=8\frac{l}{min}\cdot\frac{y(t)}{400}g/l=\frac{1}{50}y(t)\,g/min.$

Detta ger begynnelsevärdesproblemet:

$$\frac{dy}{dt}=24-\frac{1}{50}y,\quad y(0)=0.$$

$\frac{dy}{dt}+\frac{1}{50}y=24 \Rightarrow g(t)=\frac{1}{50} \Rightarrow G(t)=\frac{t}{50} \Rightarrow \mu(t)=e^{t/50} \Rightarrow$
$\Rightarrow \frac{d}{dt}(ye^{t/50})=24e^{t/50} \Leftrightarrow ye^{t/50}=1200e^{t/50}+C$; (*)

$y(0)=0 \Rightarrow C=-1200 \Rightarrow y(t)=1200(1-e^{-t/50})$. (gram)

$2\,g/l \leftrightarrow 2\cdot 400\,g=800\,g$ ($\leftrightarrow$ utläses "motsvarar").

$1200(1-e^{-t/50})=800 \Leftrightarrow 1-e^{-t/50}=\frac{2}{3} \Leftrightarrow e^{-t/50}=\frac{1}{3} \Leftrightarrow$
$\Leftrightarrow e^{t/50}=3 \Leftrightarrow t/50=\ln 3 \Leftrightarrow t=50\ln 3\approx 54{,}9$ min.

Svar: Det dröjer ca 1 timme.

Övning 8.80 (Sid. 167)

Lösning

$xy'=1-y^2 \Rightarrow y=\pm 1$ lösningar.

$y(1)=\frac{1}{2} \Rightarrow$ jag väljer $-1<y<1$;

$\frac{2}{1-y^2}dy=\frac{2}{x}dx \Leftrightarrow (\frac{1}{1-y}+\frac{1}{1+y})dy=\frac{2}{x}dx \Rightarrow \int\frac{2}{x}dx=$
$=\int(\frac{1}{1-y}+\frac{1}{1+y})dy \Leftrightarrow \ln\frac{1+y}{1-y}=\ln Cx^2 \Leftrightarrow \frac{1+y}{1-y}=Cx^2$; (*)

$y(1)=\frac{1}{2} \Rightarrow C=3 \Rightarrow \frac{1+y}{1-y}=3x^2 \Leftrightarrow 1+y=3x^2(1-y)=3x^2-$

$-3x^2y \Leftrightarrow y+y\cdot 3x^2=3x^2-1 \Leftrightarrow (3x^2+1)y=3x^2-1 \Leftrightarrow$

$\Leftrightarrow \underline{y=\frac{3x^2-1}{3x^2+1}}.$

Anm. Den "stationära" lösningen $y=1$ uppfyller villkoret $y(1)=1$.

Övning 8.81 (Sid. 167)

Lösning

$u''+u=t^2e^{it}$ tas som hjälpekvation; $y=\operatorname{Im} u$.

$u=ve^{it} \Rightarrow u''=(v''+2iv'-v)e^{it}=(v''+2iv')e^{it}-u \Leftrightarrow$

$\Leftrightarrow u''+u=(v''+2iv')e^{it}=t^2e^{it} \Leftrightarrow \underline{v''+2iv'=t^2};$

$v_p=t(at^2+bt+c)=at^3+bt^2+ct \Rightarrow v_p'=3at^2+2bt+c \Rightarrow$

$\Rightarrow v_p''=6at+2b \Rightarrow VL=6at+2b+2i(3at^2+2bt+c)=$

$=6at+2b+6iat^2+4ibt+2ic=6iat^2+(4bi+6a)t+2b+2ci;$

$VL=HL=t^2 \Rightarrow \begin{cases} 6ai=1 \\ 2b=3ai \\ c=bi \end{cases} \Leftrightarrow \begin{cases} a=-i/6 \\ b=1/4 \\ c=bi \end{cases} \Leftrightarrow \begin{cases} a=-i/6 \\ b=1/4 \\ c=i/4 \end{cases} \Rightarrow$

$\Rightarrow v_p=\frac{1}{4}t^2+i(\frac{1}{4}t-\frac{1}{6}t^3) \Rightarrow u_p=v_p(\cos t+i\sin t)=$

$=\frac{1}{4}t^2\cos t+(\frac{1}{6}t^3-\frac{1}{4}t)\sin t+i[\frac{1}{4}t^2\sin t+(\frac{1}{4}t-\frac{1}{6}t^3)\cos t$

$\Rightarrow y_p=\frac{1}{12}(3t^2\sin t+(3t-2t^3)\cos t];$

Svar: $y=(C_1+\frac{t}{4}-\frac{t^3}{6})\cos t+(C_2+\frac{t^2}{4})\sin t.$

Övning 8.82 (Sid. 167)

Lösning

$y''+2y'=0 \Leftrightarrow r^2-2r=r(r+2)=0 \Leftrightarrow r=0 \vee r=-2 \Leftrightarrow y=1$

$\vee\ y=e^{-2t} \Rightarrow y_h=C_1+C_2e^{-2t}.$

$y_p=x(ax+b)=ax^2+bx \Rightarrow y_p'=2ax+b \Rightarrow y_p''=2a;$

$VL=y_p''+2y_p'=2a+2(2ax+b)=4ax+2(a+b)=x-1=HL$

$\Leftrightarrow 4a=1 \wedge a+b=-1/2 \Leftrightarrow a=\frac{1}{4} \wedge b=-\frac{3}{4} \Rightarrow y_p=\frac{x^2-3x}{4}$

$y=C_1+C_2e^{-2x}+\frac{x^2-3x}{4} \Rightarrow y'=-2C_2e^{-2x}+\frac{2x-3}{4};$

$y(0)=0 \Rightarrow C_1+C_2=0;\quad y'(0)=0 \Rightarrow -2C_2-\frac{3}{4}=0;$

$C_2=-\frac{3}{8}=-C_1 \Rightarrow \underline{y=\frac{3}{8}(1-e^{-2t})+\frac{1}{4}(x^2-3x)}.$

Övning 8.84 (Sid. 167)

Lösning

a) $m\frac{dv}{dt}=mg-kv,\quad k>0.$

b) $\frac{dv}{dt}+\frac{k}{m}v=g \Rightarrow \frac{d}{dt}(ve^{kt/m})=ge^{kt/m} \Leftrightarrow v\cdot e^{kt/m}=$

$=\frac{mg}{k}e^{kt/m}+C \Leftrightarrow v=\frac{mg}{k}+Ce^{-kt/m} \Rightarrow v(0)=\frac{mg}{k}+C;$

$v(0)=v_0 \Rightarrow C=v_0-\frac{mg}{k} \Rightarrow v(t)=\frac{mg}{k}+(v_0-\frac{mg}{k})e^{-kt/m}.$

b) För $v_0 = mg/k$ försvinner den tidsberoende delen.

Övning 8.85 (Sid. 167)

Lösning

$f'(x) = x - \int_0^x f(t)\,dt + 2f(x) \Rightarrow f''(x) = 1 - f(x) + 2f'(x) \Leftrightarrow$

$\Leftrightarrow f''(x) - 2f'(x) + f(x) = 1$; $f'(0) = 2f(0) = 0$;

$y = f(x) \Rightarrow \underline{y'' - 2y' + y = 1,\ y(0) = y'(0) = 0.}$

$y'' - 2y' + y = 0 \Leftrightarrow r^2 - 2r + 1 = 0 \Leftrightarrow r = r_1 = r_2 = 1 \Rightarrow \underline{y = (C_1 + C_2 x)e^x}.$

$y_P = 1$ (by inspection).

$y = (C_1 + C_2 x)e^x + 1 \Rightarrow y' = (C_1 + C_2 + C_2 x)e^x$;

$$\left.\begin{array}{l} y(0) = 0 \Rightarrow C_1 + 1 = 0 \Leftrightarrow C_1 = -1 \\ y'(0) = 0 \Rightarrow C_1 + C_2 = 0 \Leftrightarrow C_2 = -C_1 \end{array}\right\} \Leftrightarrow \begin{cases} C_1 = -1 \\ C_2 = 1 \end{cases}$$

Resultat: $y = (x-1)e^x + 1$.

Prövning: $VL = f'(x) = xe^x$; $HL = x - \int_0^x ((t-1)e^t + 1)\,dt +$

$+ 2(x-1)e^x + 2 = x - \left(\left[(t-2)e^t + t \right]_0^x \right) + 2(x-1)e^x + 2 =$

$= x - ((x-2)e^x + x + 2) + 2(x-1)e^x + 2 = x - (x-2)e^x - x - 2 +$

$+ 2xe^x - 2e^x + 2 = x - xe^x + 2e^x - x - 2 + 2xe^x - 2e^x + 2 = xe^x = VL.$

Anm. Först prövar man och sen svarar man...

9. Maclaurins och Taylors formler

Inledning

Övning 9.1 (Sid. 191)

Lösning

$f(x) = \ln(1+x).$

a) $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$, $f^{(4)}(x) = -\frac{6}{(1+x)^4}$;

$f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -6$;

$P_1(x) = f(0) + f'(0)x = x$;

$P_2(x) = P_1(x) + \frac{1}{2}f''(0)x^2 = x - \frac{1}{2}x^2$;

$P_3(x) = P_2(x) + \frac{1}{6}f'''(0)x^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$;

$P_4(x) = P_3(x) + \frac{1}{24}f^{(4)}(0)x^4 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$;

b) $f(0{,}1) = \ln 1{,}1 = 0{,}095310179.$

$P_1(0{,}1) = 0{,}1$, $P_2(0{,}1) = 0{,}095$, $P_3(0{,}1) = 0{,}09533333\ldots$,

$P_4(0{,}1) = 0{,}095308333$; kommentar i facit.

c) $f(0{,}1) = \ln 1{,}1 = 0{,}095310179$, $P_2(0{,}1) = 0{,}095.$

$f(0{,}01) = \ln 1{,}01 = 0{,}00995033$, $P_2(0{,}01) = 0{,}00995.$

$f(0{,}001) = \ln 1{,}001 = 0{,}0009995$; $P_2(0{,}001) = 0{,}0009995.$

Övning 9.2 (Sid. 191)

Lösning

a) $f(x)=e^x,\ n=3.$

$f'(x)=f''(x)=f'''(x)=f(x)=e^x \Rightarrow f'(0)=f''(0)=f'''(0)=f(0)=1.$

$P_3(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3.$

b) $f(x)=\sin x,\ n=3.$

$f'(x)=\cos x,\ f''(x)=-\sin x,\ f'''(x)=-\cos x;$

$f(0)=0,\ f'(0)=1,\ f''(0)=0,\ f'''(0)=-1;$

$P_3(x)=x-\frac{1}{6}x^3.$

c) $f(x)=\sin x,\ n=4.$

$f^{(4)}(x)=\sin x;\ f^{(4)}(0)=0;\quad P_4(x)=P_3(x)=x-\frac{1}{6}x^3.$

d) $f(x)=\sqrt{1+x},\ n=2.$

$f(x)=(1+x)^{1/2} \Rightarrow f'(x)=\frac{1}{2}(1+x)^{-1/2} \Rightarrow f''(x)=-\frac{1}{4}(1+x)^{-3/2};$

$f(0)=1,\ f'(0)=\frac{1}{2},\ f''(0)=-\frac{1}{4};$

$P_2(x)=1+\frac{1}{2}x-\frac{1}{8}x^2.$

Övning 9.3 (Sid. 191)

Lösning

Se nästföljande sida.

a) $f(x)=\sqrt{1+x^2};\ a=1,\ n=2.$

$f(x)=(1+x^2)^{1/2},\ f'(x)=x(1+x^2)^{-1/2};\ f''(x)=(1+x^2)^{-1/2}+x^2(1+x^2)^{-3/2}.$

$f(1)=\sqrt{2},\ f'(1)=\frac{1}{\sqrt{2}},\ f''(1)=\frac{1}{2\sqrt{2}};$

$P_2(x)=\sqrt{2}+\frac{1}{\sqrt{2}}(x-1)+\frac{1}{4\sqrt{2}}(x-1)^2=\sqrt{2}+\frac{\sqrt{2}}{2}(x-1)+\frac{\sqrt{2}}{4}(x-1)^2.$

Övning 9.4 (Sid. 191)

Lösning

$f(x)=e^x;\ a=1,\ n=3.$

$f(x)=f'(x)=f''(x)=f'''(x)=e^x;\ f(1)=f'(1)=f''(1)=f'''(1)=e.$

$P_3(x)=e+e(x-1)+\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3.$

Övning 9.5 (Sid. 191)

Lösning

$f(x)=\sqrt{x},\ a=2,\ n=2.$

$f'(x)=\frac{1}{2}x^{-1/2},\ f''(x)=-\frac{1}{4}x^{-3/2};\ f(2)=\sqrt{2},\ f'(2)=\frac{\sqrt{2}}{4},\ f''(2)=-\frac{\sqrt{2}}{32};$

$P_2(x)=\sqrt{2}+\frac{\sqrt{2}}{4}(x-2)-\frac{\sqrt{2}}{32}(x-2)^2.$

Övning 9.6 (Sid. 191)

Lösning

$f(x)=\tan x;\ a=0,\ n=3.$

$f'(x)=1+\tan^2 x$, $f''(x)=2\tan x+2\tan^3 x$, $f'''(x)=2+$

$+8\tan^2 x+6\tan^4 x$; $f(0)=0, f'(0)=1, f''(0)=0, f'''(0)=2$;

$P_3(x)=x+\frac{1}{3}x^3$.

Övning 9.7 (Sid. 191)

Lösning

$\underline{f(x)=\arctan x,\ a=-1,\ n=3.}$

$f'(x)=\frac{1}{1+x^2}$, $f''(x)=-\frac{2x}{(1+x^2)^2}$, $f'''(x)=-\frac{2}{(1+x^2)^2}+\frac{8x^2}{(1+x^2)^3}$;

$f(-1)=-\frac{\pi}{4}$, $f'(-1)=\frac{1}{2}$, $f''(-1)=\frac{1}{2}$, $f'''(-1)=\frac{3}{4}$;

$P_3(x)=-\frac{\pi}{4}+\frac{1}{2}(x+1)+\frac{1}{4}(x+1)^2+\frac{1}{8}(x+1)^3$.

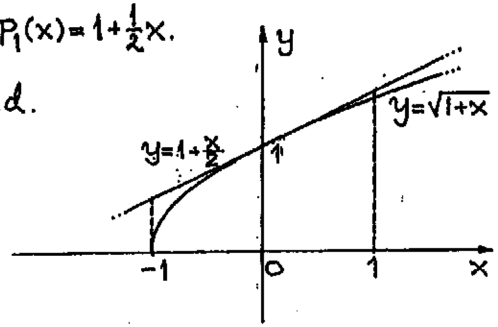
Approximationer

Övning 9.8 (Sid. 192)

Lösning

a) $f(x)=\sqrt{1+x}$, $P_1(x)=1+\frac{1}{2}x$.

b) Se fig. bredvid.



c) $f''(x)=-\frac{1}{4}(x+1)^{-3/2} \Rightarrow f''(\xi)=-\frac{1}{4}(\xi+1)^{-3/2}$;

$f(x)=1+\frac{1}{2}x-\frac{1}{8}(1+\xi)^{-3/2}x^2 \Rightarrow R_2(x)=-\frac{x^2}{8(1+\xi)^{3/2}}$, $0<|\xi|<|x|$.

d) $x\geq 0 \Rightarrow \xi>0 \Rightarrow (1+\xi)^{3/2}\geq 1 \Leftrightarrow \frac{1}{(1+\xi)^{3/2}}\leq 1 \Rightarrow |R_2(x)|=$

$=\left|-\frac{1}{(1+\xi)^{3/2}}\cdot\frac{x^2}{8}\right|=\frac{1}{8(1+\xi)^{3/2}}x^2\leq \underline{\frac{1}{8}x^2}.$

e) $0\leq x\leq 0{,}1 \Rightarrow 0\leq x^2\leq 0{,}01 \Rightarrow |R_2(x)|\leq\frac{1}{8}\cdot 0{,}01=1{,}25\cdot 10^{-3}<5\cdot 10^{-3}$.

Om felet är av ordning 3, så är decimalerna korrekta upp till 2 ; $f(0{,}05)=1{,}025$, $P(0{,}05)=1{,}025$;

$f(0{,}06)=1{,}02956$, $P(0{,}06)=1{,}03$; $f(0{,}07)=1{,}0344$,

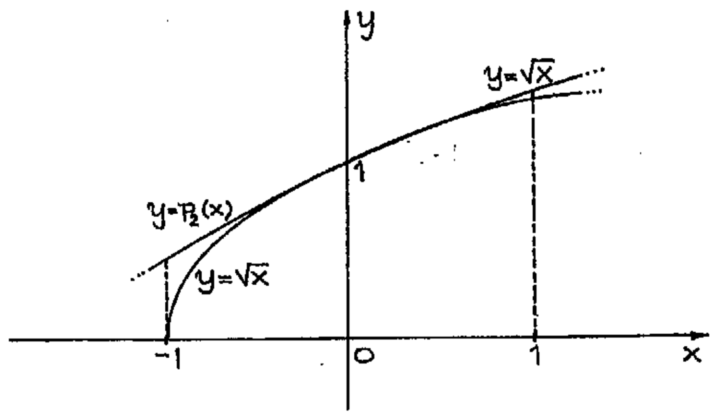
$P(0{,}07)=1{,}035$. Se även facit.

f) $|R_2(a)|<5\cdot 10^{-4} \Rightarrow \frac{a^2}{8}<5\cdot 10^{-4} \Rightarrow a^2<40\cdot 10^{-4} \Rightarrow a<0{,}06$;

$\underline{a\approx 0{,}06.}$

g) $P_2(x)=1+\frac{1}{2}x-\frac{1}{8}x^2$. (Se Ö. 9.3); $P_2(x)=\frac{3}{2}-\frac{1}{8}(x-2)^2$.

h)



För små $|x|$ är $f(x) \approx P_2(x)$.

i) $f'''(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f(x) = P_2(x) + \frac{1}{16(1+\xi)^{5/2}}x^3$, $0<|\xi|<|x|$;

$R_3(x) = \frac{x^2}{16(1+\xi)^{5/2}}$,

j) $x \geq 0 \Rightarrow \xi \geq 0 \Rightarrow 1+\xi \geq 1 \Leftrightarrow (1+\xi)^{5/2} \geq 1 \Leftrightarrow \frac{1}{(1+\xi)^{5/2}} \leq 1 \Rightarrow$

$\Rightarrow |R_3(x)| \leq \frac{1}{16}x^3$, för $x \geq 0$.

k) $0 \leq x \leq 0{,}1 \Rightarrow 0 \leq R_3(0{,}1) \leq \frac{1}{16}\cdot 10^{-3} = 6{,}25\cdot 10^{-5} < 10^{-4}$.

$f(0{,}08) = 1{,}039230485$, $P_2(0{,}08) = 1{,}0392$.

Övning 9.9 (Sid. 193)

Lösning

a) $f(x) = \sin x \Rightarrow f'(x) = \cos x$; $P_1(x) = f(0) + f'(0)x = x$.

$P_2(x) = x + \frac{f''(0)}{2!}x^2 = x^2 = P_1(x)$.

b) $f''(x) = -\sin x$, $f'''(x) = -\cos x$;

$f(x) = P_1(x) + R_2(x) = x - \frac{1}{2}(\sin\xi)x^2$, $0<|\xi|<|x|$.

$f(x) = P_2(x) + R_3(x) = x - \frac{\cos\xi}{6}x^3$, $0<|\xi|<|x|$.

c) $|R_2(x)| = |-\frac{1}{2}\sin\xi\cdot x^2| = \frac{1}{2}|\sin\xi|\cdot x^2 \leq \frac{1}{2}\sin 0{,}1\cdot 0{,}1^2 < 5\cdot 10^{-4}$.

$|R_3(x)| = |-\frac{1}{6}\cos\xi\cdot x^3| \leq \frac{1}{6}\cos\xi\cdot x^3 < \frac{1}{6}\cos 0{,}1\cdot 0{,}1^3 < 1{,}7\cdot 10^{-4}$.

Antalet säkra decimaler är 3.

d) $f(0{,}074) = 0{,}0741$; $P_1(0{,}074) = P_3(0{,}074) = 0{,}074$.

Övning 9.10 (Sid. 193)

Lösning

a) $f(x) = e^x$; $P_3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ (Sats 2, s. 413).

b) $R_4(x) = \frac{e^{\theta x}}{24}x^4$, $0<\theta<1$.

c) $|x| \leq 0{,}1 \Rightarrow 0 \leq R_4(x) \leq \frac{e^{0,1}}{24}\cdot 0{,}1^4 = 0{,}046\cdot 10^{-4} \leq 5\cdot 10^{-6}$

d) $e^{0,1} = 1{,}105170918$, $P_3(0{,}1) = 1{,}105166667$.

$e^{0,1} \approx P_4(0{,}1) \Rightarrow e^{0,1} \approx P_4(0) = 1{,}10517$ (5 decimaler).

e) $|R_4(x)| \leq \frac{1}{4}e^{0,1}x^4 = 0{,}046x^4$, $|x| \leq 0{,}1$.

f) $|e^x - P_3(x)| = \frac{1}{24}e^{\theta x}\cdot x^4 \leq 0{,}046x^4 < 0{,}125x^4 < \frac{1}{8}x^4$, $|x| \leq 0{,}1$.

Restterm på Lagranges form

Övning 9.11 (Sid. 193)

Lösning

Fullständigt löst på sidan 204.

Övning 9.12 (Sid. 193)

Lösning

$f(x) = \ln(1+x) \Rightarrow f'(x) = (1+x)^{-1} \Rightarrow f''(x) = -(1+x)^{-2} \Rightarrow f''(x) =$

$= 2(1+x)^{-3}$, $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f''(\theta x) = 2(1+\theta x)^{-3}$,

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3(1+\theta x)^3}x^4 \Rightarrow |\ln(1+x) - x + \frac{1}{2}x^2| =$

$= \frac{1}{3}\frac{1}{|1+\theta x|^3}|x|^3 \overset{!}{\leq} \frac{1}{3}\frac{1}{(1-1/2)^3}\cdot|x|^3 = \frac{8}{3}|x|^3.$

Anm. I $\overset{!}{\leq}$ underförstås följande:

$|x| \leq \frac{1}{2} \Leftrightarrow -\frac{1}{2} \leq -x \leq \frac{1}{2} \Leftrightarrow \frac{1}{2} \leq 1-x \leq \frac{3}{2} \Leftrightarrow \frac{2}{3} \leq \frac{1}{1-x} \leq 2 \Leftrightarrow$

$\Leftrightarrow (\frac{2}{3})^3 \leq \frac{1}{(1-x)^3} \leq 2^3 \Leftrightarrow \frac{1}{(1-x)^3} \leq 8.$

Övning 9.13 (Sid. 177)

Lösning

$f(x) = \arctan x = x + \frac{f''(\theta x)}{2!}x^2, \quad f''(x) = -\frac{2x}{(1+x^2)^2};$

$|x| \leq 0{,}1 \Rightarrow |f''(\theta x)| \leq \frac{2\cdot 0{,}1}{(1+0)^2} = 0{,}2 \Rightarrow |R_2(x)| < \frac{0{,}2}{2}\cdot 10^{-2} = 10^{-3}.$

Övning 9.14 (Sid. 178)

Lösning

$f(x) = \tan x, \ -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}.$

$f'(x) = 1 + \tan^2 x, \ f''(x) = 2\tan x + 2\tan^3 x, \ f'''(x) = 2 +$

$+ 8\tan^2 x + 6\tan^4 x$ (Övning 9.6).

$|x| \leq \frac{\pi}{4} \Rightarrow |f'''(x)| \leq 2+8+6 = 16 \Rightarrow |R_3(x)| \leq \frac{16}{6}|x|^3 < 3|x|^3.$

Övning 9.15 (Sid. 178)

Lösning

Övning 9.15 (Sid. 194)

Lösning

$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}, \ |x| \leq 1.$ (Se sidan 121 i boken).

$f'(x) = \sinh x, \ f''(x) = \cosh x, \ f'''(x) = \sinh x, \ f^{(4)}(x) = \cosh x.$

$f(0) = 1, \ f'(0) = 0, \ f''(0) = 1, \ f'''(0) = 0, \ f^{(4)}(\theta x) = \cosh(\theta x);$

$f(x) = \cosh x = 1 + \frac{1}{2}x^2 + \frac{\cosh(\theta x)}{24}x^4;$

$|\cosh x - 1 - \frac{1}{2}x^2| = \frac{\cosh(\theta x)}{24}x^4 \Leftrightarrow |e^x + e^{-x} - 2 - x^2| \leq$

$\leq \frac{\cosh 1}{12}\cdot|x|^4 = 0{,}12859\,x^4 < \frac{1}{6}x^4.$

Övning 9.16 (Sid. 194)

Lösning

$f(x) = \sin x = x - \frac{1}{6}x^3 + \frac{1}{120}\cos(\theta x)\cdot x^5, \quad 0<\theta<1;$

$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{\cos(\theta x)}{120}x^4 \Rightarrow |\frac{\sin x}{x} - 1 + \frac{1}{6}x^2| \overset{!}{\leq} \frac{1}{120}x^4.$

I $\overset{!}{\leq}$ underförstås $|\cos\theta x| \leq 1$.

Övning 9.17 (Sid. 194)

Lösning: Fullständigt löst på sidan 204.

Övning 9.18 (Sid. 194)

Lösning

Se nästa sida.

$g(u) = \ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3(1+\theta u)^3}u^3, \quad 0<\theta<1;$

$f(x) = \ln(1-x^2) = -x^2 - \frac{1}{2}x^4 - \frac{1}{3(1-\theta x^2)^3}x^6, \quad 0<\theta<1.$

a) $P_4(x) = -x^2 - \frac{1}{2}x^4.$

b) $|x| \le \frac{1}{4} \Rightarrow |\ln(1-x^2) + x^2 + \frac{1}{2}x^4| = \frac{1}{3}\frac{1}{|1-\theta x^2|^3}x^6 \le (!) \le$

$\le \frac{1}{3}\frac{1}{(1-1/4)^3}x^6 = \frac{1}{3}\frac{3^3}{4^3}x^6 \le \frac{3^2}{4^3}\cdot\frac{1}{4^6} = \frac{3^2}{2^{18}} = 3{,}4\cdot 10^{-5} < 10^{-4}.$

I (!) underförstås den omvända triangelolikheten $|x-y| \ge ||x|-|y||$ (Se (6) s.46 i boken).

Övning 9.19 (Sid. 194)

Lösning: Fullständigt löst på sidan 205.

Övning 9.20 (Sid. 194)

Lösning

Jag hänvisar till Ö. 9.16.

$$\int_0^1 \frac{\sin x}{x}dx = \int_0^1 (1 - \frac{1}{6}x^2 + \frac{1}{120}x^4)dx - \frac{\cos\xi}{5040}\int_0^1 x^6 dx =$$

$$= \left[x - \frac{1}{18}x^3 + \frac{1}{600}x^5\right]_0^1 - \frac{\cos\xi}{5040}\left[\frac{x^7}{7}\right]_0^1 =$$

$$= 1 - \frac{1}{18} + \frac{1}{600} - \frac{\cos\xi}{5040}\cdot\frac{1}{7} = \frac{1703}{1800} - \frac{\cos\xi}{35280} \Leftrightarrow$$

$$\Leftrightarrow \left|\int_0^1 \frac{\sin x}{x}dx - \frac{1703}{1800}\right| = \frac{\cos\xi}{35280} < 3\cdot 10^{-5};$$

Resultat: $\int_0^1 \frac{\sin x}{x}dx \approx 0{,}9461 \approx Si(1) = 0{,}946083.$

Entydighet i Maclaurinutvecklingar

Övning 9.21 (Sid. 194)

Lösning

a) $e^u = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + u^4 B_1(u);$

$e^{3x} = 1 + 3x + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + (3x)^4 B_1(3x) =$

$= 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + x^4 B_2(x).$

b) $e^{-x} = 1 + (-x) + \frac{1}{2}(-x)^2 + \frac{9}{2}(-x)^3 + (-x)^4 B\ (-x)$

$= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + x^4 B_3(x).$

c) $\cos t = 1 - \frac{1}{2}t^2 + t^4 B_1(t);$

$\cos\frac{x}{2} = 1 - \frac{1}{2}(\frac{x}{2})^2 + (\frac{x}{2})^4 B_1(\frac{x}{2}) = 1 - \frac{1}{8}x^2 + x^4 B_2(x).$

d) $\ln(1+v) = v + v^2 B_1(v);$

$\ln(1+x^2) = x^2 + (x^2)^2 B_1(x^2) = x^2 + x^4 B_2(x).$

e) $\ln(1-x^2) = -x^2 + (-x^2)^2 B_1(-x^2) = -x^2 + x^4 B_2(x).$

Övning 9.22 (Sid. 195)

Lösning

$(1+u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2 + u^3 B(u);$

a) $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + x^3 B_1(x);$

b) $\frac{1}{1+x}=(1+x)^{-1}=1-x+x^2+x^3B_2(x);$

c) $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+x^3B_3(x);$

d) $\sqrt{1-\frac{x}{2}}=(1+(-\frac{x}{2}))^{1/2}=1+(-\frac{x}{2})-\frac{1}{8}(-\frac{x}{2})^2+(-\frac{x}{2})^3B_1(-\frac{x}{2})=$

$=1-\frac{x}{2}-\frac{x^2}{32}+x^3B_2(x).$

e) $(1+x^2)^{1/3}=(\text{Se c}))=1+\frac{1}{3}x^2-\frac{1}{9}x^4+x^6B_3(x).=$

$=1+\frac{1}{3}x^2+x^4B_4(x).$

Övning 9.23 (Sid. 195)

Lösning

a) $e^x(1-x^2)=(1+x+\frac{1}{2}x^2+x^3B_1(x))(1-x^2)=$

$=1+x+\frac{1}{2}x^2-x^2+x^3B_2(x)=1+x-\frac{1}{2}x^2+x^3B_3(x).$

b) $x\cdot\sin x=x(x-\frac{1}{6}x^3+x^5B(x))=x^2-\frac{1}{6}x^4+x^6B(x).$

c) $x(\cos x-1)=x(1-\frac{1}{2}x^2+x^4B(x)-1)=-\frac{1}{2}x^3+x^5B(x).$

Övning 9.24 (Sid. 195)

Lösning

$e^{x^2}\cos x=(1+x^2+\frac{1}{2}x^4+x^6B_1(x))(1-\frac{1}{2}x^2+\frac{1}{24}x^4+x^6B_2(x))=$

$=1-\frac{1}{2}x^2+\frac{1}{24}x^4+x^2-\frac{1}{2}x^4+\frac{1}{2}x^4+x^6B_2(x)=$

$=1+\frac{1}{2}x^2+\frac{1}{24}x^4+x^6B_2(x).$

Övning 9.25 (Sid. 195)

Lösning

$\sin x\cdot\arctan x=(x-\frac{1}{6}x^3+x^5B_1(x))\cdot(x-\frac{1}{3}x^3+x^5B_2(x))=$

$=x^2-\frac{1}{3}x^4-\frac{1}{6}x^4+x^6B_3(x)=x^2-\frac{1}{2}x^4+x^6B_3(x).$

Övning 9.26 (Sid. 195)

Lösning

$\ln(1+\cos x)=\ln(1+1-\frac{1}{2}x^2+\frac{1}{24}x^4+2x^6B_1(x))=$

$=\ln 2(1-\frac{1}{4}x^2+\frac{1}{48}x^4+x^6B_1(x))=$

$=\ln 2+\ln(1+(-\frac{x^2}{4}+\frac{x^4}{48})+x^6B_1(x))=$

$=\ln 2+(-\frac{x^2}{4}+\frac{x^4}{48})-\frac{1}{2}(-\frac{x^2}{4}+\frac{x^4}{48})^2+x^6B_2(x)=$

$=\ln 2-\frac{x^2}{4}+\frac{x^4}{48}-\frac{1}{2}\cdot\frac{x^4}{16}+x^6B_3(x)=\ln 2-\frac{x^2}{4}-\frac{x^4}{196}+x^6B_3(x).$

Övning 9.27 (Sid. 195)

Lösning

a) $\exp\{\sin x\}=\exp\{x-\frac{1}{6}x^3+x^5B_1(x)\}=1+(x-\frac{1}{6}x^3+$

$+x^5B_1(x))+\frac{1}{2}(x-\frac{1}{6}x^3+x^5B_1(x))^2+\frac{1}{6}(x+x^3B_2(x))+$

$+\frac{1}{24}(x+B_2(x))^4=1+x-\frac{1}{6}x^3+\frac{1}{2}(x^2-\frac{1}{3}x^4)+\frac{1}{6}x^3+\frac{1}{24}x^4+$

$+x^5B_3(x)=1+x+\frac{1}{2}x^2-\frac{1}{8}x^4+x^5B_3(x).$

b) $\exp\{\cos x\} = \exp\{1-\frac{1}{2}x^2+\frac{1}{24}x^4+x^6B_1(x)\} =$

$= e\cdot\exp\{-\frac{1}{2}x^2+\frac{1}{24}x^4+x^6B_1(x)\} =$

$= e\cdot(1+(-\frac{1}{2}x^2+\frac{1}{24}x^4)+\frac{1}{2}(-\frac{1}{2}x^2+x^4B_2(x))^2) =$

$= e\cdot(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{8}x^4+x^6B_3(x)) =$

$= e\cdot(1-\frac{1}{2}x^2+\frac{1}{6}x^4+x^6B_3(x)) = e-\frac{e}{2}x^2+\frac{e}{6}x^4+x^6B_4(x).$

Resttermen av formen $x^nB(x)$

Övning 9.28 (Sid. 195)

Lösning

a) $\lim_{x\to0}\frac{1-\cos x}{\ln(1+x)-x} = (\frac{0}{0}) = \lim_{x\to0}\frac{1-(1-x^2/2+x^4B_1(x))}{x-x^2/2-x+x^3B_2(x)} =$

$= \lim_{x\to0}\frac{x^2/2+x^4B_1(x)}{-x^2/2+x^3B_2(x)} = \lim_{x\to0}\frac{(\cancel{x^2/2})(1+x^2B_3(x))}{(\cancel{x^2/2})(-1+xB_4(x))} =$

$= \lim_{x\to0}\frac{1+xB_3(x)}{-1+x^2B_4(x)} = \frac{1}{-1} = \underline{-1}.$

b) $\lim_{x\to0}\frac{\ln(1-x)+x}{1-\sqrt{1-x^2}} = (\frac{0}{0}) = \lim_{x\to0}\frac{(x+\ln(1-x))(1+\sqrt{1-x^2})}{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})} =$

$= \lim_{x\to0}\frac{(x+\ln(1-x))(1+\sqrt{1-x^2})}{1-(1-x^2)} = \lim_{x\to0}\frac{1}{x^2}(1+\sqrt{1-x^2})(x+\ln(1-x)) =$

$= \lim_{x\to0}\frac{1}{x^2}(1+1-\frac{1}{2}x^2+x^3B_1(x))(x-x-\frac{1}{2}x^2+x^3B_2(x)) =$

$= \lim_{x\to0}\frac{1}{x^2}(2-\frac{1}{2}x^2+x^3B_1(x))(-\frac{1}{2}x^2+x^3B_2(x)) =$

$= \lim_{x\to0}(-1+x\cdot B_3(x)) = -1.$

Övning 9.29 (Sid. 195)

Lösning

a) $\lim_{x\to0}\frac{(1+x)^{1/x}-e}{x} = (\frac{0}{0}) = \lim_{x\to0}\frac{e^{\ln(1+x)^{1/x}}-e}{x} =$

$= \lim_{x\to0}\frac{e^{(\ln(1+x))/x}-e}{x} = \lim_{x\to0}\frac{e^{(x-x^2/2+x^3B_1(x))/x}-e}{x} =$

$= \lim_{x\to0}\frac{e^{(1-x/2+x^2B_1(x)}-e}{x} = \lim_{x\to0}\frac{e\cdot e^{-x/2+x^2B_1(x)}-e}{x} =$

$= \lim_{x\to0}\frac{e(1-x/2+x^2B_2(x)-1)}{x} = e\cdot\lim_{x\to0}(-\frac{1}{2}+xB_2(x)) = -\frac{e}{2}.$

b) För små $|x|$ har vi $\sin^2 x = x^2$, så att

$\lim_{x\to0}(1+\sin^2x)^{-2/x^2} = \lim_{x\to0}(1+x^2)^{-2/x^2} = [u=1/x^2] =$

$= \lim_{u\to\infty}(1+\frac{1}{u})^{-2u} = (\lim_{u\to\infty}(1+\frac{1}{u})^u)^{-2} = \underline{e^{-2}}.$

Övning 9.30 (Sid. 195)

Lösning

a) $\lim_{x\to0}(\frac{1}{\sin^2x}-\frac{1}{x^2}) = (\infty-\infty) = \lim_{x\to0}\frac{x^2-\sin^2x}{x^2\sin^2x} = (\frac{0}{0}) =$

$= \lim_{x\to0}\frac{(x-\sin x)(x+\sin x)}{x^2\sin^2x} = \lim_{x\to0}\frac{(x^3/6+x^5B_1(x))(2x+x^3B_2(x))}{(x(x+x^3B_3(x))^2} =$

$= \lim_{x\to0}\frac{x^4/3+x^6B_4(x)}{x^4(1+x^2B_3(x))} = \lim_{x\to0}\frac{\cancel{x^4}(1/3+x^2B_4(x))}{\cancel{x^4}(1+x^2B_5(x))} =$

$= \lim_{x\to0}\frac{1/3+x^2B_4(x)}{1+x^2B_5(x)} = \frac{1}{3}.$

forts.

b) $\lim_{x\to 0}\frac{\sin x-\arctan x}{x(\cos 2x-1)}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{x-x^3/6-(x-x^3/3)+x^5B_1(x)}{x(-2x^2+x^4B_2(x))}=$

$=\lim_{x\to 0}\frac{x^3/6+x^5B_1(x)}{-2x^3+x^5B_2(x)}=\lim_{x\to 0}\frac{\cancel{x^3}(1/6+x^2B_1(x))}{\cancel{x^3}(-2+x^2B_2(x))}=\frac{1/6}{-2}=-\frac{1}{12}.$

Övning 9.31 (Sid. 195)

Lösning

a) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=(\infty-\infty)=\lim_{x\to 1}\frac{x\ln x-(x-1)}{(x-1)\ln x}\ [u=x-1]=$

$=\lim_{u\to 0}\frac{(u+1)\ln(1+u)-u}{u\ln(1+u)}=\lim_{u\to 0}\frac{(1+u)(u-u^2/2+u^3B_1(u))-u}{u(u+u^2B_2(u))}=$

$=\lim_{u\to 0}\frac{u+u^2-u^2/2+u^3B_3(u)-u}{u^2(1+uB_2(u))}=\lim_{u\to 0}\frac{u^2/2+u^3B_3(u)}{u^2(1+uB_2(u))}=$

$=\lim_{u\to 0}\frac{\cancel{u^2}(1/2+uB_3(u))}{\cancel{u^2}(1+uB_2(u))}=\lim_{u\to 0}\frac{1/2+uB_3(u)}{1+uB_2(u)}=\frac{1/2}{1}=\frac{1}{2}.$

b) $\lim_{n\to\infty}\left[(1+\tfrac{1}{n})^n\right]^n e^{-n}=\lim_{n\to\infty}e^{\ln[(1+\frac{1}{n})^n]^n}\cdot e^{-n}=$

$=\lim_{n\to\infty}e^{n\cdot\ln(1+\frac{1}{n})^n}\cdot e^{-n}=\lim_{n\to\infty}e^{n^2\ln(1+\frac{1}{n})-n}=$

$=\lim_{n\to\infty}e^{n^2(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{n^3}B_1(\frac{1}{n}))-n}=\lim_{n\to\infty}e^{-\frac{1}{2}+\frac{1}{n}B_1(\frac{1}{n})}=$

$=e^{-1/2+\lim_{n\to\infty}\frac{1}{n}B_1(\frac{1}{n})}=e^{-1/2}.$

c) $\lim_{x\to\infty}(x^2\sqrt[3]{1+x^3}-x^3)=(\infty-\infty)=\lim_{x\to\infty}x^3(\sqrt[3]{1+1/x^3}-1)=$

$=\lim_{x\to\infty}x^3\left(1+\frac{1}{3x^3}+\frac{1}{x^6}B(\tfrac{1}{x})-1\right)=\lim_{x\to\infty}\left(\frac{1}{3}+\frac{1}{x^3}B(\tfrac{1}{x})\right)=\frac{1}{3}.$

(Man kan även sätta $u=1/x$...)

Övning 9.32 (Sid. 195)

Lösning

$\frac{\sin x}{x}=\frac{1}{x}(x-\frac{1}{6}x^3+x^5B(x))=1-\frac{1}{6}x^2+x^4B(x)\Rightarrow f'(0)=0$

(ty x-termen fattas i utvecklingen.)

Övning 9.33 (Sid. 196)

Lösning

$\frac{e^x-\cos x}{x}=\frac{1+x-1+x^2B(x)}{x}=\frac{x+x^2B(x)}{x}=1+xB(x)\xrightarrow[x\to 0]{}1=f(0).$

Svar: $a=1$. (Se Def. 1 på sidan 478.)

Övning 9.34 (Sid. 196)

Lösning

$\sqrt{a+2x}-\sqrt{a+x}=\sqrt{a}\,(\sqrt{1+2x/a}-\sqrt{1+x/a})=\sqrt{a}\,(1+\frac{x}{a}-$

$-\frac{x^2}{4a^2}-(1+\frac{x}{2a}-\frac{x^2}{8a^2}))+x^3B(x)=\sqrt{a}\,(\frac{x}{2a}-\frac{x^2}{8a^2})+x^3B(x)\Rightarrow$

$\sqrt{a+2x}-\sqrt{a+x}-x=(\frac{1}{2\sqrt{a}}-1)x-\frac{\cdot}{8a\sqrt{a}}+x^3B(x)\Rightarrow$

$\Rightarrow\lim_{x\to 0}\frac{\sqrt{a+2x}-\sqrt{a+x}-x}{x^2}=(a=\tfrac{1}{4})=\lim_{x\to 0}\frac{-3x^2+x^3B_1(x)}{x^2}=$

$=\lim_{x\to 0}(-3+xB_1(x))=-3.$

Resultat: För $a=1/4$ existerar gränsvärdet och är lika med -3. (Läs även bokens lösning).

Övning 9.35 (Sid. 196)

Lösning

$\sin ax = ax - \frac{a^3x^3}{6} + x^5 B_1(x)$;

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^4 B_2(x)$;

$1-\cos ax = 1-(1-\frac{1}{2}a^2x^2 + x^4 B_3(x)) = \frac{1}{2}a^2x^2 + x^4 B_4(x)$;

$\sin ax - \ln(1+x) = (a-1)x + \frac{1}{2}x^2 + x^4 B_5(x)$;

$$\lim_{x\to 0}\frac{\sin ax - \ln(1+x)}{1-\cos ax} = \lim_{x\to 0}\frac{(a-1)x + x^2/2 + x^4 B_5(x)}{a^2x^2/2 + x^4 B_4(x)} = (a=1) =$$

$$= \lim_{x\to 0}\frac{x^2/2 + x^4 C(x)}{x^2/2 + x^4 D(x)} = \lim_{x\to 0}\frac{\cancel{x^2}(1/2 + x^2 C(x))}{\cancel{x^2}(1/2 + x^2 D(x))} = \lim_{x\to 0}\frac{\frac{1}{2}+x^2 C}{\frac{1}{2}+x^2 D} = 1.$$

Svar: För $a=1$ blir gränsvärdet 1.

Övning 9.36 (Sid. 196)

Lösning

$(1+\frac{1}{n})^{n+x} = e \Leftrightarrow e^{(n+x)\ln(1+1/n)} = e \Leftrightarrow (n+x)\ln(1+\frac{1}{n}) = 1$

$\Leftrightarrow (n+x)(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{n^3}B(\frac{1}{n})) = 1 \Leftrightarrow 1 - \frac{1}{2n} + \frac{1}{n^2}B(\frac{1}{n}) + \frac{x}{n} - \frac{x}{2n^2} +$

$+ \frac{x}{n^3}B(\frac{1}{n}) = 1 \Leftrightarrow (x-\frac{1}{2})\cdot\frac{1}{n} - \frac{x}{2n^2} + \frac{x}{n^3}B(\frac{1}{n}) \Leftrightarrow x - \frac{1}{2} - \frac{x}{2n} +$

$+ \frac{x}{n^2}B(\frac{1}{n}) = 0 \overset{!}{\Rightarrow} (n\to\infty \Rightarrow x \to \frac{1}{2})$; $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Anm. I $\overset{!}{\Rightarrow}$ underförstås att $\lim_{n\to\infty} x_n$ är ändligt så $\lim_{n\to\infty}(x/n) = \lim_{n\to\infty}(x/n^2) = 0$.

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Övning 9.37 (Sid. 196)

Lösning

$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \Rightarrow \sum_{k=0}^{\infty}\frac{1}{k!} = e.$

Övning 9.38 (Sid. 196)

Lösning

a) $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow \sum_{n=0}^{\infty}\frac{2^n}{n!} = e^2.$

b) $\arctan x = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}x^{2k-1} \Rightarrow \sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{2k-1} = \arctan 1 = \frac{\pi}{4}.$

c) $\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} \Rightarrow \sum_{k=0}^{\infty}(-1)^k\frac{\pi^{2k}}{(2k)!} = \cos\pi = -1.$

d) $\sin x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \Rightarrow \sum_{k=0}^{\infty}\frac{(-1)^k\pi^{2k+1}}{6^{2k+1}(2k+1)!} = \sin\frac{\pi}{6} = \frac{1}{2}.$

e) $\ln(1+x) = \sum_{k=0}^{\infty}\frac{(-1)^{k-1}}{k}x^k \Rightarrow \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} = \ln 2$

Övning 9.39 (Sid. 196)

Lösning

a) $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \Rightarrow \sum_{k=0}^{\infty}\frac{3^k}{k!} = e^3.$

b) $\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^k \Rightarrow \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\cdot 2^k} = \ln(1+\frac{1}{2}) = \ln\frac{3}{2}.$

Blandade problem

Övning 9.40 (Sid. 197)

Lösning

$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + x^3 B_1(x)$;

$e^{x/3} = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + x^3 B_2(x) \Rightarrow (1+x)^{1/3} - e^{x/3} = \frac{-x^2}{6} + x^3 B_3$

$\Rightarrow x((1+x)^{1/3} - e^{x/3}) = -\frac{1}{6}x^3 + x^4 B_3(x) = x^3(-\frac{1}{6} + x B_3(x))$;

$\arctan x - \sin x = x - \frac{1}{3}x^3 - (x - \frac{1}{6}x^3) + x^5 B_4(x) = -\frac{1}{6}x^3 +$

$+ x^5 B_4(x) = x^3(-\frac{1}{6} + x^2 B_4(x))$;

$$\lim_{x\to 0} \frac{\arctan x - \sin x}{x((1+x)^{1/3} - e^{x/3})} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{-1/6 + x^2 B_4(x)}{-1/6 + x^2 B_3(x)} = \frac{-1/6}{-1/6} = 1$$

Övning 9.41 (Sid. 197)

Lösning

$\ln \frac{1+2x}{(1+x)^2} = \ln(1+2x) - 2\ln(1+x) = 2x - 2x^2 - (2x - x^2) + x^3 B_1(x)$

$= -x^2 + x^3 B_1(x) = x^2(-1 + x B_1(x))$;

$1 - \cos 2x = 1 - (1 - \frac{1}{2}(2x)^2) + x^4 B_2(x) = 2x^2 + x^4 B_2(x)$;

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2\ln(1+x)}{1 - \cos 2x} = \lim_{x\to 0} \frac{-1 + x B_1(x)}{2 + x^2 B_2(x)} = -\frac{1}{2}.$$

Övning 9.42 (Sid. 197)

Övning 9.42 (Sid. 197)

Lösning

a) Sats 1 på sidan 411 i grundboken.

b) $\cos x - (1-x^2)^{1/2} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - (1 - \frac{x^2}{2} - \frac{x^4}{8}) + x^6 B_1(x) =$

$= \frac{x^4}{6} + x^6 B_1(x) = x^4(\frac{1}{6} + x^2 B_1(x))$;

$\ln(1+x^2) - x\sin x = x^2 - \frac{1}{2}x^4 - x(x - \frac{1}{6}x^3) + x^6 B_2(x) =$

$= x^2 - \frac{1}{2}x^4 - x^2 + \frac{1}{6}x^3 + x^6 B_2(x) =$

$= -\frac{1}{3}x^4 + x^6 B_2(x) = x^4(-\frac{1}{3} + x^2 B_2(x))$;

$$\lim_{x\to 0} \frac{\cos x - \sqrt{1-x^2}}{\ln(1+x^2) - x\sin x} = \lim_{x\to 0} \frac{1/6 + x^2 B_1(x)}{-1/3 + x^2 B_2(x)} = \frac{1/6}{-1/3} = -\frac{1}{2}.$$

Övning 9.43 (Sid. 197)

Lösning

$f(x) = (1+2x)^{1/3} \Rightarrow f'(x) = \frac{2}{3}(1+2x)^{-2/3} \Rightarrow f''(x) = -\frac{8}{9}(1+2x)^{-5/3}$

$\Rightarrow f'''(x) = \frac{80}{27}(1+2x)^{-8/3}$; $f(0) = 1$, $f'(0) = \frac{2}{3}$, $f''(0) = -\frac{8}{9}$;

$f(x) = 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}(1 + 2\theta x)^{-8/3} \cdot x^3$;

$|f(x) - 1 - \frac{2}{3}x + \frac{4}{9}x^2| = \frac{40}{81}(1 + \theta \cdot 2x)^{-8/3} \cdot x^3 \leq \frac{40}{80} \cdot 10^{-3} < 10^{-3}$.

Resultat: $f(x) \approx 1 + \frac{2}{3}x - \frac{4}{9}x^2$.

Övning 9.45 (Sid. 197)

Lösning

$S(x)=\int_0^x \ln(\cos x)\,dx$;

$S'(x)=\ln(\cos x)$;

$S''(x)=-\tan x$;

$S'''(x)=-\frac{1}{\cos^2 x}=-1-\tan^2 x$

$S^{(4)}(x)=-2\tan x(1+\tan^2 x)=-2\tan x-2\tan^3 x$;

$S^{(5)}(x)=-2(1+\tan^2 x+3\tan^2 x(1+\tan^2 x))=$

$=-2(1+4\tan^2 x+3\tan^4 x)$;

$S(0)=S'(0)=S''(0)=0,\ S'''(0)=-1,\ S^{(4)}(0)=0.$

$S(x)=-\frac{1}{6}x^3-\frac{1}{60}(1+4\tan^2\xi+3\tan^4\xi)x^5,\quad |x|\leq\frac{\pi}{4}$;

$|S(x)+\frac{1}{6}x^3|=\frac{1}{60}(1+4\tan^2\xi+3\tan^4\xi)|x|^5\leq\frac{8}{60}|x|^5<\frac{1}{3}|x|^5.$

Övning 9.46 (Sid. 198)

Lösning

a) $\gamma=\frac{v}{c}\ \Rightarrow\ T(v)=mc^2-m_0c^2=(m-m_0)c^2=\left(\frac{1}{\sqrt{1-\gamma^2}}-1\right)m_0c^2=$

$=((1-\gamma^2)^{-1/2}-1)m_0c^2=(1+\frac{1}{2}\gamma^2-\frac{3}{8}\gamma^4+\ldots-1)m_0c^2=$

$=(\frac{1}{2}\gamma^2-\frac{3}{8}\gamma^4+\ldots)m_0c^2=(\frac{1}{2}\frac{v^2}{c^2}-\frac{3}{8}\frac{v^4}{c^4}+\ldots)m_0c^2=\frac{1}{2}m_0v^2-$

$-\frac{3}{8}m_0\frac{v^4}{c^2}+\ldots\ =T_{cl}+\Delta T\ \Rightarrow\ \Delta T=-\frac{3}{8}m_0\frac{v^4}{c^2}$;

T_{cl} = den klassiska kinetiska energin.

b) $\Delta T=T(v)-T_{cl}(v)=-\frac{3}{8}m_0\frac{v^4}{c^2}\Rightarrow\left|\frac{\Delta T}{T}\right|=\frac{3}{4}\left(\frac{v}{c}\right)^2\leq\frac{3}{4}\cdot10^{-6}<10^{-6}.$

Övning 9.47 (Sid. 198)

Lösning

$\lim_{x\to0}\frac{\arctan x-x}{x(\cos 2x-1)}=\left(\frac{0}{0}\right)=\lim_{x\to0}\frac{x-x^3/3-x+x^5B_1(x)}{x(1-2x^2+x^4B_2(x)-)}=$

$=\lim_{x\to0}\frac{-x^3/3+x^5B_1(x)}{-2x^3+x^5B_2(x)}=\lim_{x\to0}\frac{-1/3+x^2B_1(x)}{-2+x^2B_2(x)}=\frac{-1/3}{-2}=\frac{1}{6}.$

Övning 9.48 (Sid. 198)

Lösning

$f(x)=\ln(1+2\sin x)=\ln(1+2x-\frac{1}{3}x^3+x^5B_1(x))=$

$=2x-\frac{1}{3}x^3-\frac{1}{2}(2x)^2+\frac{1}{3}(2x)^3+x^4B_2(x)=$

$=2x-\frac{1}{3}x^3-2x^2+\frac{8}{3}x^3+x^4B_2(x)=$

$=2x-2x^2+\frac{7}{3}x^3+x^4B_2(x).$

Resultat: $P_3(x)=2x-2x^2+\frac{7}{3}x^3.$

Appendix A: Komplexa tal

Övning A.1 (Sid. 209)

Lösning

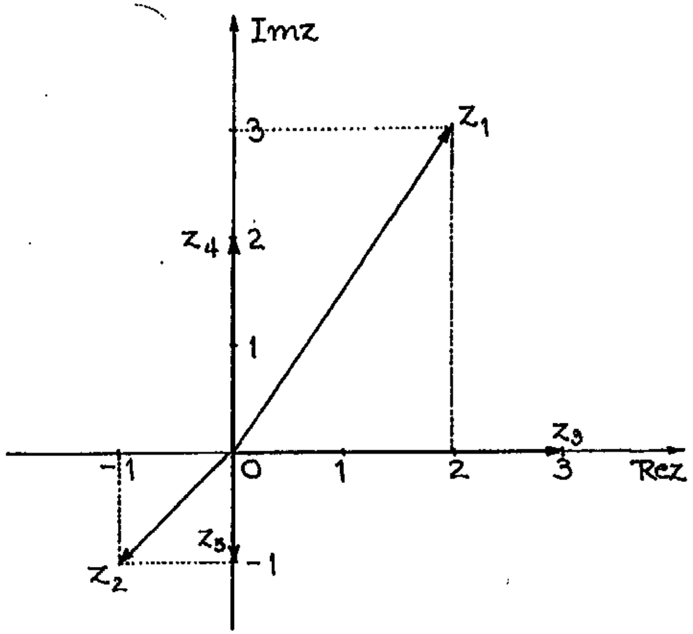
a) $z_1 = 2+3i \Rightarrow \underline{\mathrm{Re}\,z_1 = 2} \wedge \underline{\mathrm{Im}\,z_1 = 3}.$

b) $z_2 = -1-i = -1+(-1)i \Rightarrow \underline{\mathrm{Re}\,z_2 = -1 = \mathrm{Im}\,z_2}.$

c) $z_3 = 3 = 3+0\cdot i \Rightarrow \underline{\mathrm{Re}\,z_3 = 3} \wedge \underline{\mathrm{Im}\,z_3 = 0}.$

d) $z_4 = 2i = 0+2i \Rightarrow \underline{\mathrm{Re}\,z_4 = 0} \wedge \underline{\mathrm{Im}\,z_4 = 2}.$

e) $z_5 = -i = 0+(-1)i \Rightarrow \underline{\mathrm{Re}\,z_5 = 0} \wedge \underline{\mathrm{Im}\,z_5 = -1}.$



Axelgraderingen sker med reella tal.

Övning A.2 (Sid. 209)

Lösning

a) $(1+i)+(-3-2i) = 1+i-3-2i = 1-3+i-2i = \underline{-2-i}.$

b) $(1+i)-(3-4i) = 1+i-3+4i = 1-3+i+4i = \underline{-2+5i}.$

c) $(1+i)\cdot(3-4i) = 1\cdot(3-4i)+i(3-4i) = 3-4i+3i+4 = \underline{7-i}.$

d) $(1-i)^2 = 1-2\cdot 1\cdot i+i^2 = 1-2i-1 = \underline{-2i}.$

e) $(5-2i)^3 = 5^3+3\cdot 5^2\cdot(-2i)+3\cdot 5\cdot(-2i)^2+(-2i)^3 = 125-150i-$

$-60+8i = 125-60-150i+8i = \underline{65-142i}.$

f) $(1-i)^4 = ((1-i)^2)^2 = (-2i)^2 = 4i^2 = \underline{-4}.$

g) $(1+i)(1-i) = 1^2-i^2 = 1+1 = \underline{2}.$

Övning A.3 (Sid. 209)

Lösning

a) $\overline{1+i} = 1-i$

b) $\overline{3-5i} = 3+5i$

c) $\overline{-7} = -7$

d) $(1+i)\overline{(1+i)} = (1+i)(1-i) = \underline{2}.$

e) $|1+i| = \sqrt{1^2+1^2} = \underline{\sqrt{2}}.$

f) $|i| = \sqrt{0^2+1^2} = \underline{1}.$

g) $|3-2i| = \sqrt{3^2+2^2} = \underline{\sqrt{13}}.$

h) $|-5i| = \sqrt{0^2+5^2} = \underline{5}.$

Övning A.4 (Sid. 209)

Lösning nästa sida.

Lösning

a) $\frac{1}{1+i}=\frac{1-i}{(1+i)(1-i)}=\frac{1-i}{2}=\frac{1}{2}+\frac{1}{2}i.$ (Se även Ö A.2g).

b) $\frac{1}{3-4i}=\frac{3+4i}{(3-4i)(3+4i)}=\frac{3+4i}{3^2+4^2}=\frac{3+4i}{25}=\frac{3}{25}+\frac{4}{25}i.$

c) $\frac{3-4i}{1+i}=\frac{(3-4i)(1-i)}{(1+i)(1-i)}=\frac{3-3i-4i-4}{2}=\frac{-1-7i}{2}=-\frac{1}{2}-\frac{7}{2}i.$

d) $\frac{1-i}{1+i}=\frac{(1-i)^2}{(1+i)(1-i)}=\frac{-2i}{2}=-i.$

e) $(1+i)^{-2}=\frac{1}{(1+i)^2}=\frac{1}{2i}=\frac{-i}{2}=-\frac{1}{2}i.$

f) $\frac{1}{i}=\frac{-i}{i(-i)}=\frac{-i}{1}=-i.$

Övning A.5 (Sid. 209)

Lösning

a) $|(1-i)^{14}|=|(-2i)^7|=|(-2)^7\cdot i^7|=|-128\cdot(-i)|=|128i|=128.$

b) $\left|\frac{3+i}{4+3i}\right|=\frac{|3+i|}{|4+3i|}=\frac{\sqrt{3^2+1^2}}{\sqrt{4^2+3^2}}=\frac{\sqrt{10}}{5}.$

Övning A.6 (Sid. 209)

Lösning

$$\left|\frac{(1+2i)(7+\sqrt{3}i)^2}{(5+i)^2}\right|=\frac{|(1+2i)(7+\sqrt{3}i)^2|}{|(5+i)^2|}=\frac{|1+2i|\cdot|(7+\sqrt{3}i)^2|}{|(5+i)^2|}=$$
$$=\frac{|1+2i|\cdot|7+\sqrt{3}i|^2}{|5+i|^2}=\frac{\sqrt{1^2+2^2}\,(\sqrt{7^2+(\sqrt{3})^2})^2}{5^2+1^2}=\frac{\sqrt{5}.52}{26}=2\sqrt{5}.$$

Övning A.7 (Sid. 209)

Lösning

$\underline{z=x+iy}\Rightarrow VL=z+2\bar{z}=x+iy+2(x-iy)=x+iy+$
$+2x-2iy=3x-iy=2-i=HL\Leftrightarrow 3x=2\ \wedge\ -y=-1\Leftrightarrow$
$\Leftrightarrow x=\frac{2}{3}\wedge y=1\Leftrightarrow z=\frac{2}{3}+i.$

Övning A.8 (Sid. 209)

Lösning

a) $z=x+iy\Rightarrow VL=3z-i\bar{z}=3(x+iy)-i(x-iy)=3x+3iy-$
$-ix+i^2y=3x-y+i(-x+3y)=7-5i=HL$ (identifieras)
$\Leftrightarrow\begin{cases}3x-y=7\\-x+3y=-5 \quad 3\end{cases}\Leftrightarrow\begin{cases}8y=-8\\x=3y+5\end{cases}\Leftrightarrow\begin{cases}y=-1\\x=2\end{cases}\Leftrightarrow z=2-i.$

c) $z\cdot 2\bar{z}=1+i\Leftrightarrow 2|z|^2=1+i$; lösning(ar) saknas, ty VL är reellt medan HL inte är det.

Övning A.9 (Sid. 209)

Lösning

$z=x+iy\Rightarrow \mathrm{Re}z+\mathrm{Im}z=x+y$;
$\mathrm{Re}z+\mathrm{Im}z=1\Leftrightarrow x+y=1\Leftrightarrow y=-x+1$; en rät linje med riktningskoefficienten $k=-1$ och $m=1$.

Övning A.10 (Sid. 209)

Lösning

$z = x + iy$ i hela övningen.

a) $\mathrm{Re}\,z = x = 3$; en rät linje genom $(3,0)$ parallell med y-axeln (den imaginära axeln).

b) $\mathrm{Im}\,z = y = -1$; en rät linje genom $(0,-1)$ parallell med x-axeln (Re-axeln).

c) $\mathrm{Im}\,z = y > 0$; det övre komplexa halvplanet.

d) $z + \bar{z} = 0 \Leftrightarrow \mathrm{Re}\,z = 0 \Rightarrow$ den imaginära axeln.

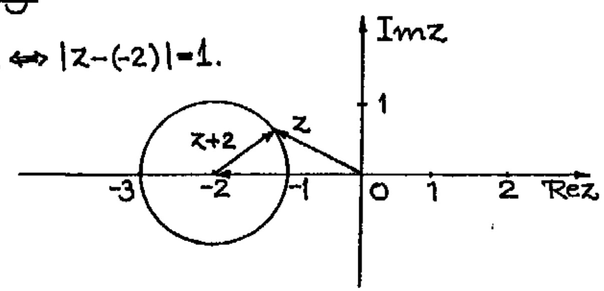
e) $z = \bar{z} \Leftrightarrow \mathrm{Im}\,z = \frac{z-\bar{z}}{2i} = 0 \Leftrightarrow z \in \mathbb{R}$; den reella axeln.

Att illustrera ovanstående punktmängder tillhör normalt E-kursen i gymnasiet.

Övning A.11 (Sid. 209)

Lösning

$|z+2| = 1 \Leftrightarrow |z-(-2)| = 1.$

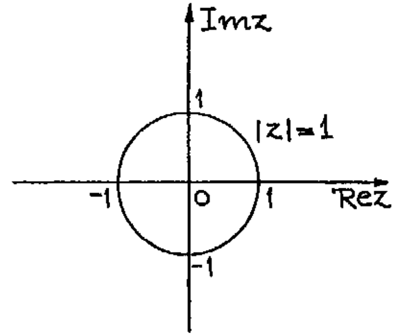


Avståndet från z till -2 är 1; cirkeln ovan.

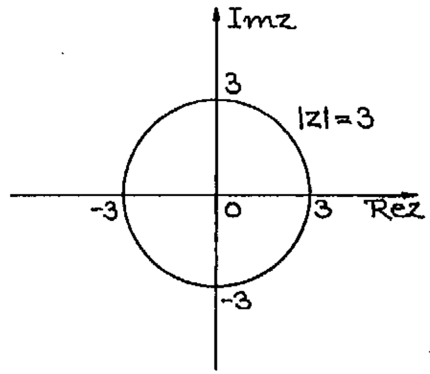
Övning A.12 (Sid. 210)

Lösning

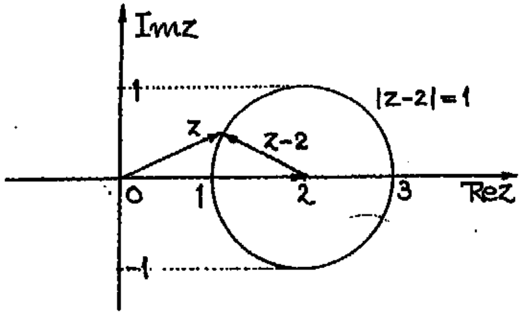
a) $|z| = 1 \Leftrightarrow |z-0| = 1$; avståndet från z till origo är konstant 1; denna punktmängd går under namnet "enhetscirkeln i det komplexa planet.



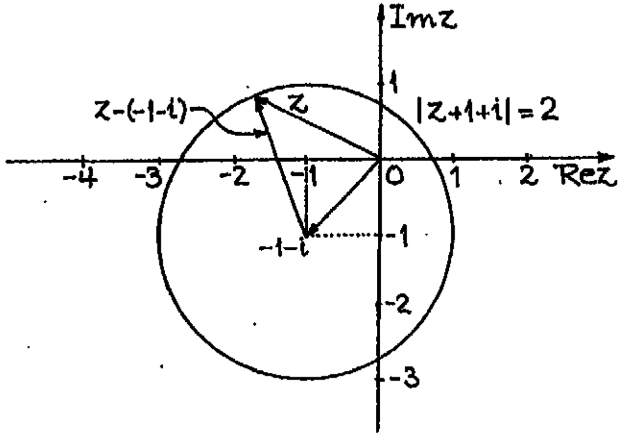
b) $|z| = 3 \Leftrightarrow |z-0| = 3$; avståndet från z till origo är konstant 3; cirkeln nedan.



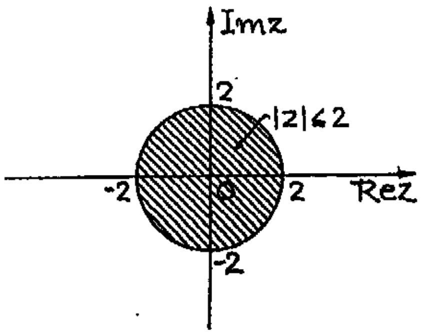
c) $|z-2| = 1$; avståndet från z till 2 är konstant 1;



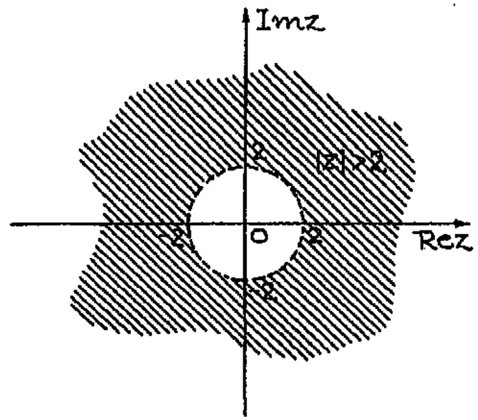
d) $|z+1+i|=2 \Leftrightarrow |z-(-1-i)|=2$, avståndet från z till $-1-i$ är konstant 2; cirkel igen. Se nedan.



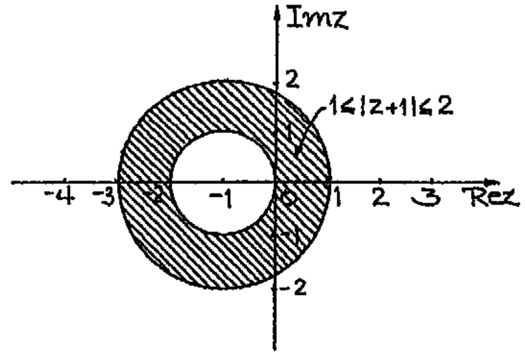
e) $|z|\leq 2$, en disk med centrum 0 och radien 2.



f) $|z|>2$, avståndet från z till origo är större än 2.



g) $1\leq|z+1|\leq 2$; avståndet från z till -1 är lägst 1 och högst 2; det är "ringen" nedan.



Övning A.13 (Sid. 210)

Lösning

$$\begin{cases} |z-3i|=2 \\ z+\bar{z}=2 \\ z=x+iy \end{cases} \Leftrightarrow \begin{cases} |1+i(y-3)|=2 \\ x=1 \\ z=1+iy \end{cases} \Rightarrow 1+(y-3)^2=4 \Leftrightarrow y=3\pm\sqrt{3};$$

Resultat: $z_1 = 1+i(3+\sqrt{3})$, $z_2 = 1+i(3-\sqrt{3})$.

Övning A.14 (Sid. 210)

Lösning

$|z-1| = |z+1| \Leftrightarrow |1+x+iy| = |-1+x+iy| \Leftrightarrow (x+1)^2+y^2 =$

$= (x-1)^2+y^2 \Leftrightarrow x+1 = -(x-1) \Leftrightarrow 2x=0 \Leftrightarrow \underline{\operatorname{Re} z = 0}.$

Avståndet från z till 1 är lika med avståndet från z till −1; detta uppfylls av alla rent imaginära talen. (Läs även författarnas lösning).

Övning A.15 (Sid. 210)

Lösning

$|z-1| = 2|z+1| \Leftrightarrow |x-1+iy| = 2|x+1+iy| \Leftrightarrow (x-1)^2+y^2 =$

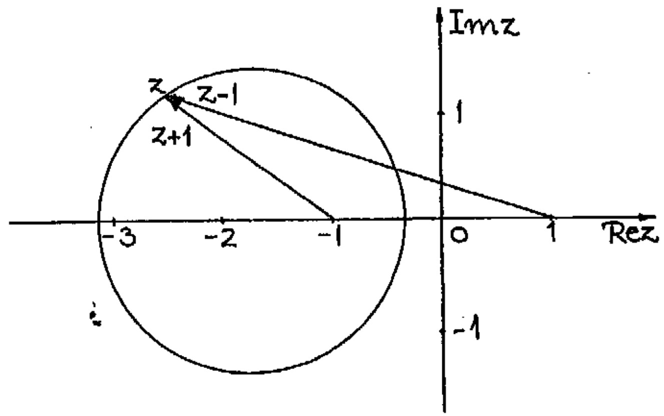
$= 4((x+1)^2+y^2) \Leftrightarrow x^2-2x+1+y^2 = 4(x^2+2x+1+y^2) \Leftrightarrow$

$\Leftrightarrow x^2-2x+1+y^2 = 4x^2+8x+4+4y^2 \Leftrightarrow 3x^2+3y^2+10x+3=0$

$\Leftrightarrow x^2+y^2+\frac{10}{3}x+1=0 \Leftrightarrow (x+\frac{5}{3})^2+y^2 = \frac{25}{9}-1 = \frac{16}{9} = (\frac{4}{3})^2 \Leftrightarrow$

$\Leftrightarrow |z+\frac{5}{3}|^2 = (\frac{4}{3})^2 \Leftrightarrow \underline{|z+\frac{5}{3}| = \frac{4}{3}}.$

Avståndet från z till $-\frac{5}{3}$ är konstant $\frac{4}{3}$; detta är en cirkel med centrum i $-\frac{5}{3}$ och radien $\frac{4}{3}$.



Övning A.16 (Sid. 210)

Lösning

$|\frac{1}{z}-\frac{1}{4}| = \frac{1}{4} \Leftrightarrow |\frac{4-z}{4z}| = \frac{1}{4} \Leftrightarrow |4-z| = |z| \Leftrightarrow |4-x-iy| = |x+iy|$

$\Leftrightarrow (x-4)^2+y^2 = x^2+y^2 \Leftrightarrow 8x-16=0 \Leftrightarrow x = \underline{\operatorname{Re} z = 2}.$

Övning A.17 (Sid. 210)

Lösning

$z = x+iy \Rightarrow z+\frac{1}{z} = x+iy+\frac{1}{x+iy} = x+iy+\frac{x-iy}{x^2+y^2} = x+$

$+\frac{x}{x^2+y^2}+i(y-\frac{y}{x^2+y^2})$; $z+\frac{1}{z} \in \mathbb{R} \Rightarrow y(1-\frac{1}{x^2+y^2}) = 0 \Leftrightarrow$

$\Leftrightarrow y=0 \vee x^2+y^2=1 \Leftrightarrow \underline{\operatorname{Im} z = 0} \vee \underline{|z|=1}.$

Övning A.18 (Sid. 210)

Lösning

a) $z = \sqrt{2}\angle\pi/4 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \underline{\underline{1+i}}.$

Anm. Beteckningen $r\angle\theta$ är ... elektrikernas. Samma folk använder ~~beteckningen~~ $j=\sqrt{-1}$ i stället för i; detta är reserverat för variabla strömstyrkor.

b) $z = 1\angle\pi = \cos\pi + i\cdot\sin\pi = -1+i\cdot 0 = \underline{\underline{-1}}.$

c) $z = \sqrt{2}\angle 9\pi/4 = \sqrt{2}\left(\cos\frac{9\pi}{4} + i\cdot\sin\frac{9\pi}{4}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) =$
$= \sqrt{2}\angle\pi/4 = 1+i$ (Se under a)).

d) $z = 1\angle\pi/2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i\cdot 1 = \underline{\underline{i}}.$

e) $z = 1\angle 2\pi = \cos 2\pi + i\sin 2\pi = 1 + i\cdot 0 = \underline{\underline{1}}.$

f) $z = 1/\sqrt{2}\angle -\pi/4 = \frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \underline{\underline{\frac{1-i}{2}}}.$

g) $z = 1\angle -100\pi = 1\angle 0+(-50)\cdot 2\pi = 1\angle 0 = \cos 0 + i\sin 0 = \underline{\underline{1}}.$

Övning A.10 (Sid. 210)

Lösning

Ur figuren på nästföljande sida avläses:

a) $z_1 = 1 = 1\angle 0.$

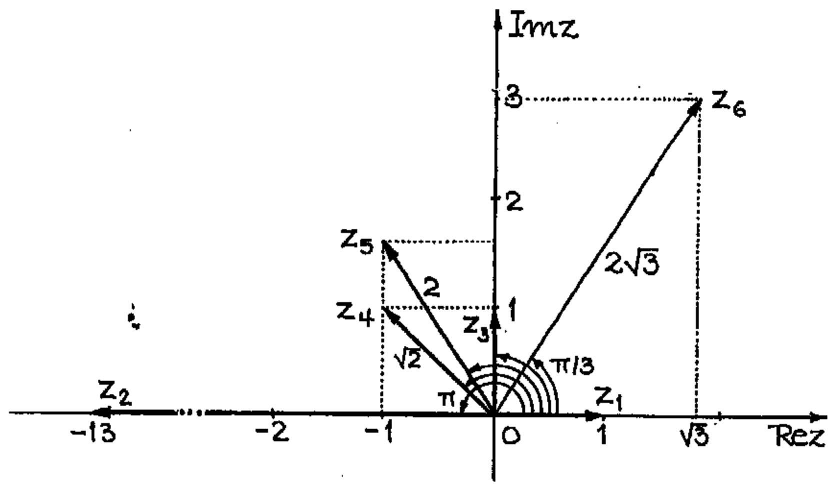
b) $z_2 = -13 = 13\angle\pi.$

c) $z_3 = i = 1\angle\pi/2.$

d) $z_4 = -1+i = \sqrt{2}\angle 3\pi/4.$

e) $z_5 = -1+i\sqrt{3} = 2\angle 2\pi/3.$

f) $z_6 = \sqrt{3}+3i = 2\sqrt{3}\angle\pi/3.$



Övning A.20 (Sid. 210)

Lösning

$\forall\theta\in\mathbb{R}: |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$

Svar: a) 1, b) 1 och c) 1.

Anm. $\forall\theta\in\mathbb{R}$: utläses "för alla reella theta gäller att". (Se även Appendix B.)

Övning A.21 (Sid. 210)

Lösning

$\forall\theta\in\mathbb{R}: |e^{i\theta}| = |\cos\theta + i\sin\theta| = 1$ (enl. A.20).

Svar: a) 1, b) 1 och c) 1.

Anm. $e^{i\theta} = \cos\theta + i\sin\theta = \mathrm{cis}\,\theta = 1\angle\theta.$

Övning A.22 (Sid. 210)

Lösning

$$\begin{cases} \arg z = \frac{\pi}{3} \\ \arg w = \frac{\pi}{4} \end{cases} \Rightarrow \begin{cases} \arg\{zw\} = \arg z + \arg w = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}. \\ \arg\{\frac{z}{w}\} = \arg z - \arg w = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}. \end{cases}$$

Svar: a) och b) se ovan. c) Nej.

Övning A.23 (Sid. 210)

Lösning

$\arg z = \frac{\pi}{3} \Rightarrow \arg\{z^{2000}\} = 2000 \arg z = \frac{2000}{3}\pi =$

$= (666 + \frac{2}{3})\pi = \frac{2\pi}{3} + 333 \cdot 2\pi$;

Svar: Det sökta argumentet är $\frac{2\pi}{3}$.

Övning A.24 (Sid. 210)

Lösning

$z = \frac{1+i\sqrt{3}}{(2-2i)^3} \Rightarrow \arg z = \arg\{1+i\sqrt{3}\} - \arg\{(2-2i)^3\} =$

$= \arg\{1+i\sqrt{3}\} - 3\arg\{2-2i\} = \frac{\pi}{3} - 3(-\frac{\pi}{4}) = \frac{\pi}{3} + \frac{3\pi}{4} = \frac{13\pi}{12}$;

Samtliga argument är $\frac{13\pi}{12} + k \cdot 2\pi$, k heltal.

Övning A.26 (Sid. 211)

Lösning

$z = \frac{(2+2i)(1+i\sqrt{3})}{3i(\sqrt{12}-2i)} = \frac{\cancel{2}(1+i)(1+i\sqrt{3})}{3i \cdot \cancel{2}(\sqrt{3}-i)} = \frac{(1+i)(1+i\sqrt{3})}{3(1+i\sqrt{3})} = \frac{1+i}{3}$;

$\arg z = \arg\{1+i\} = \frac{\pi}{4}$.

Samtliga argument ges av $\frac{\pi}{4} + n \cdot 2\pi$, $n \in \mathbb{Z}$.

Övning A.26 (Sid. 211)

Lösning

Jag förutsätter hela vägen $\omega > 0$.

a) $\arg\{1+i2\omega\} = \underline{\arctan 2\omega}$; $1+2i\omega$ ligger i den första kvadranten.

b) $-1+i2\omega$ ligger i den andra kvadranten så att

$\arg\{-1+i2\omega\} = \arg\{(-1)(1-i2\omega)\} = \arg\{-1\} + \arg\{1-2i\omega\} =$

$= \underline{\pi - \arctan 2\omega}$.

c) $\arg\{\frac{1}{1+2i\omega}\} = -\arg\{1+i2\omega\} = \underline{-\arctan 2\omega}$.

d) $\arg\{\frac{1}{-1+2i\omega}\} = -\arg\{-1+i2\omega\} = \underline{\arctan 2\omega - \pi}$. (Se b)).

e) $\arg\{\frac{e^{i\omega}}{(1+i2\omega)^2}\} = \arg\{e^{i\omega}\} - \arg\{(1+i2\omega)^2\} = \omega -$

$- 2\arg\{1+i2\omega\} = \underline{\omega - 2\arctan 2\omega}$.

Övning A.27 (Sid. 211)

Lösning

Se nästföljande sida.

$$\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)^{100}=\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right)^{99}\cdot\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=(e^{i\pi/3})^{99}\cdot\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=$$

$$=((e^{i\pi/3})^{33}\cdot\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=(e^{i\pi})^{33}\cdot\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\right)^{3}=(-1)^{99}\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=$$

$$=-1\cdot\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=\underline{-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i}.$$

Övning A.28 (Sid. 211)

Lösning

$\underline{(z+w)^4=z^4+4z^3w+6z^2w^2+4zw^3+w^4,\quad z,w\in\mathbb{C}.}$

$$\cos 4\theta+i\sin 4\theta=e^{i4\theta}=(e^{i\theta})^4=(\cos\theta+i\sin\theta)^4=\cos^4\theta+$$

$$+4\cos^3\theta(i\sin\theta)+6\cos^2\theta(i\sin\theta)^2+4\cos\theta(i\sin\theta)^3+(i\sin\theta)^4=$$

$$=\cos^4\theta+\sin^4\theta-6\cos^2\theta\sin^2\theta+i(4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta).$$

Identifiering av real- resp. imaginärdelarna ger

$$\begin{cases}\cos 4\theta=\cos^4\theta+\sin^4\theta-6\cos^2\theta\sin^2\theta\\ \sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta\end{cases}$$

Övning A.29 (Sid. 211)

Lösning

$$\cos\alpha\sin\beta=\frac{e^{i\alpha}+e^{-i\alpha}}{2}\cdot\frac{e^{i\beta}-e^{-i\beta}}{2i}=\frac{1}{4i}(e^{i\alpha}\cdot e^{i\beta}-e^{i\alpha}\cdot e^{-i\beta}+$$

$$+e^{-i\alpha}\cdot e^{i\beta}-e^{-i\alpha}\cdot e^{-i\beta})=\frac{1}{4i}(e^{i(\alpha+\beta)}-e^{-i(\alpha+\beta)}-e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)})=$$

$$=\frac{1}{2}\left(\frac{e^{i(\alpha+\beta)}-e^{-i(\alpha+\beta)}}{2i}-\frac{e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}}{2i}\right)=\frac{1}{2}(\sin(\alpha+\beta)-\sin(\alpha-\beta)).$$

Övning A.30 (Sid. 211)

Lösning

Metod 1

$$\sin^4\theta=(\sin^2\theta)^2=\left(\frac{1-\cos 2\theta}{2}\right)^2=\frac{1}{4}(1-2\cos 2\theta+\cos^2 2\theta)=$$

$$=\frac{1}{4}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)=\frac{1}{4}\left(1-2\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)=$$

$$=\frac{1}{4}\left(\frac{3}{2}-2\cos 2\theta+\frac{1}{2}\cos 4\theta\right)=\frac{3}{8}-\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta.$$

Metod 2

$$\sin^4\theta=(\sin\theta)^4=\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^4=\left(\frac{e^{-i\theta}(e^{2i\theta}-1)}{2i}\right)^4=$$

$$=\frac{1}{(2i)^4}(e^{-i\theta})^4\cdot(e^{2i\theta}-1)^4=(\text{Se A.28})=\frac{1}{16}e^{-i4\theta}(e^{2i\theta}-1)^4=$$

$$=\frac{1}{16}e^{-4i\theta}((e^{2i\theta})^4-4(e^{2i\theta})^3+6(e^{2i\theta})^2-4(e^{2i\theta})+1)=$$

$$=\frac{1}{16}e^{-i4\theta}(e^{8i\theta}-4e^{6i\theta}+6e^{4i\theta}-4e^{2i\theta}+1)=$$

$$=\frac{1}{16}(e^{-i4\theta}e^{i8\theta}-4e^{-i4\theta}\cdot e^{i6\theta}+6e^{-i4\theta}\cdot e^{i4\theta}-4e^{-i4\theta}e^{2i\theta}+e^{-i4\theta})=$$

$$=\frac{1}{16}(e^{4i\theta}-4e^{2i\theta}+6-4e^{-2i\theta}-e^{-4i\theta})=\frac{1}{16}(e^{4i\theta}+e^{-4i\theta}-$$

$$-4(e^{2i\theta}+e^{-2i\theta})+6)=\frac{1}{8}\frac{e^{4i\theta}+e^{-4i\theta}}{2}-\frac{1}{2}\frac{e^{2i\theta}+e^{-2i\theta}}{2}+\frac{3}{8}=$$

$$=\frac{1}{8}\cos 4\theta-\frac{1}{2}\cos 2\theta+\frac{3}{8}.$$

Anm. Det är "Metod 2" som ~~bör~~ föredras.

Övning A.31 (Sid. 211)

$\arg z+\frac{\pi}{2}=\arg\{iz\}$; multiplikation med $i=e^{i\pi/2}$.

a) $z=1 \Rightarrow w=iz=\underline{i}$. (1 övergår i $i=\sqrt{-1}$)

b) $z=-3+2i \Rightarrow w=iz=i(-3+2i)=\underline{-2-3i}$.

Övning A.32 (Sid. 211)

Lösning

Transformationsfaktorn är $3e^{i5\pi/6}=3\cdot\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=$

$=\frac{3}{2}(-\sqrt{3}+i)$, varav följer att $1 \longmapsto \frac{3}{2}(-\sqrt{3}+i)$ och

$(-1+i) \longmapsto \frac{3}{2}(-\sqrt{3}+i)(-1+i)=\frac{3}{2}(\sqrt{3}-1-i(\sqrt{3}+1))$.

Anm. Pilen $\longmapsto$ utläses "avbildas på".

Övning A.33 (Sid. 211)

Lösning

Antag att transformationsfaktorn är λ, $\lambda \in \mathbb{C}$.

$2\cdot\lambda=7+i \Leftrightarrow \lambda=\frac{7+i}{2}$. Vi får i tur och ordning:

$0 \longmapsto 0$, $2 \longmapsto 7+i$, $2+i \longmapsto \frac{13}{2}+\frac{7}{2}i$ resp. $i \longmapsto -\frac{1}{2}+\frac{7}{2}i$.

Övning A.34 (Sid. 211)

Lösning

a) $e^0=\cos 0+i\sin 0=1$.

b) $e^{i\pi/2}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i\cdot 1=i$.

c) $\exp\{\frac{1}{2}\ln 2+i\frac{\pi}{4}\}=e^{\frac{1}{2}\ln 2}\cdot e^{i\pi/4}=e^{\ln\sqrt{2}}\cdot e^{i\pi/4}=\sqrt{2}e^{i\pi/4}=$

$=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\underline{1+i}$.

Anm $a^z=\exp_a(z)$, $a>0$; $e^z=\exp z$.

d) $e^{i\pi}=\cos\pi+i\sin\pi=-1+0\cdot i=\underline{-1}$.

e) $e^{3-i}=e^3\cdot e^{-i}=e^3(\cos(-1)+i\sin(-1))=\underline{e^3\cos 1-ie^3\sin 1}$.

Övning A.35 (Sid. 211)

Lösning

a) $\frac{1+ix}{1-ix}=\frac{i(x-i)}{-i(x+i)}=-\frac{x-i}{x+i}=-\frac{(x-i)^2}{(x+i)(x-i)}=-\frac{x^2-1-i2x}{x^2+1}=\frac{1-x^2}{1+x^2}+$

$+i\frac{2x}{1+x^2}=\mathrm{Re}\{\frac{1+ix}{1-ix}\}+i\cdot\mathrm{Im}\{\frac{2x}{1+x^2}\}$

b) $e^{(-1+i)x}=e^{-x+ix}=e^{-x}\cdot e^{ix}=e^{-x}(\cos x+i\cdot\sin x)=e^{-x}\cos x+$

$+ie^{-x}\sin x=\mathrm{Re}\{e^{(-1+i)x}\}+i\cdot\mathrm{Im}\{e^{(-1+i)x}\}$.

Resultat: a) $\frac{1-x^2}{1+x^2}$ resp. $\frac{2x}{1+x^2}$; b) $e^{-x}\cos x$ resp. $e^{-x}\sin x$.

Övning A.36 (Sid. 211)

Lösning

$|e^z|=|e^{x+iy}|=|e^x e^{iy}|=e^x|e^{iy}|=e^x\cdot 1=e^x$. (Se A.20).

$e^z=e^{x+iy}=e^x\cdot e^{iy}=|e^z|\cdot e^{i\arg\{e^z\}} \Rightarrow \arg e^z=y+2k\pi$.

Anm. $e^{x+i(y+2k\pi)}=e^x(\cos y+i\sin y)$.

Övning A.37 (Sid. 211)

Lösning

a) $z^2=5+12i \Leftrightarrow (x+iy)^2=5+12i \Leftrightarrow x^2-y^2+i2xy=5+12i$

$$\Leftrightarrow \begin{cases} x^2-y^2=5 \\ 2xy=12 \\ |(x+iy)^2|=|5+12i| \end{cases} \Leftrightarrow \begin{cases} x^2-y^2=5 \\ x^2+y^2=13 \\ xy=6 \end{cases} \Leftrightarrow \begin{cases} x^2=9 \\ y^2=4 \\ xy=6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=\pm 3 \\ y=\pm 2 \\ xy=6 \end{cases} \Leftrightarrow \begin{cases} x=3 \\ y=2 \end{cases} \vee \begin{cases} x=-3 \\ y=-2 \end{cases} \Leftrightarrow \underline{z=\pm(3+2i)}.$$

b) $z^2-(2+2i)z-5-10i=0 \Leftrightarrow z^2-2(1+i)z=5+10i \Leftrightarrow$

$\Leftrightarrow z^2-2(1+i)z+(1+i)^2=5+10i+(1+i)^2=5+12i \Leftrightarrow$

$\Leftrightarrow (z-(1+i))^2=(3+2i)^2 \Leftrightarrow z-(1+i)=\pm(3+2i) \Leftrightarrow$

$\Leftrightarrow z=1+i+3+2i=3+3i \vee z=1+i-3-2i=-2-i.$

Resultat: a) $z_1=3+2i$, $z_2=-3-2i$; b) $z_1=3+3i$, $z_2=-2-i$.

Övning A.38 (Sid. 211)

Lösning

a) $z^2=-i \Leftrightarrow (x+iy)^2=-i \Leftrightarrow x^2-y^2+i2xy=0+(-1)i \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x^2-y^2=0 \\ 2xy=-1 \end{cases} \Leftrightarrow \begin{cases} y=\pm x \\ 2xy=-1 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ 2xy=-1 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ 2x^2=1 \end{cases} \Leftrightarrow$$

(x och y måste ha motsatt tecken.)

$$\Leftrightarrow \begin{cases} y=-x \\ x=\pm\frac{1}{\sqrt{2}} \end{cases} \Leftrightarrow \begin{cases} x=-\frac{1}{\sqrt{2}} \\ y=\frac{1}{\sqrt{2}} \end{cases} \vee \begin{cases} x=\frac{1}{\sqrt{2}} \\ y=-\frac{1}{\sqrt{2}} \end{cases} \Rightarrow z=\pm\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right).$$

Anm. Antag att α är komplext och $\neq 0$.

$$z^2=\alpha^2 \Leftrightarrow z^2-\alpha^2=(z-\alpha)(z+\alpha)=0 \Leftrightarrow \underline{z=\pm\alpha}.$$

b) $z^2=1+i\sqrt{3}=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2e^{i\pi/3}=(\sqrt{2}\,e^{i\pi/6})^2 \Leftrightarrow$

$\Leftrightarrow z=\pm\sqrt{2}e^{i\pi/6} \Leftrightarrow z=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i \vee z=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i.$

c) $z^2=3+4i=4+4i+i^2=(2+i)^2 \Leftrightarrow z=2+i \vee z=-2-i.$

Övning A.39 (Sid. 211)

Lösning

a) $z^2+2iz-1+2i \Leftrightarrow z^2+2iz+i^2=2i=(1+i)^2 \Leftrightarrow (z+i)^2=$

$=(1+i)^2 \Leftrightarrow z+i=\pm(1+i) \Leftrightarrow \underline{z=-1} \vee \underline{z=1-2i}.$

b) $z^2+2(1-i)z-3-6i=0 \Leftrightarrow z^2+2(1-i)z=3+6i \Leftrightarrow$

$\Leftrightarrow z^2+2(1-i)z+(1-i)^2=3+6i+(1-i)^2=3+4i=(2+i)^2 \Leftrightarrow$

$\Leftrightarrow (z+1-i)^2=(2+i)^2 \Leftrightarrow z+1-i=\pm(2+i) \Leftrightarrow z=-1+i\pm(2+i)$

$\Leftrightarrow \underline{z=-3} \vee \underline{z=1+2i}.$

Svar: a) $z_1=-1$, $z_2=1-2i$; b) $z_1=-3$, $z_2=1+2i$.

Anm. $a\cdot i=\frac{a}{2}(2i)=\frac{a}{2}(1+i)^2=\left(\sqrt{\frac{a}{2}}(1+i)\right)^2$, för $a>0$.

Denna lilla anmärkning är bra å kunna.

Övning A.40 (Sid. 212)

Lösning

$(2+i)z^2+(1-7i)z-5=0$ (mult. med $2-i$).

$(2-i)(2+i)z^2+(2-i)(1-7i)z-5(2-i)=0$ (hyfsa)

$5z^2-(5+15i)z-5(2-i)=0 \Leftrightarrow z^2-(1+3i)z-2+i=0$

$\Leftrightarrow z=\frac{1+3i}{2}\pm\sqrt{\left(\frac{1+3i}{2}\right)^2+2-i}=\frac{1+3i}{2}\pm\frac{\sqrt{2i}}{2}\overset{\mathrm{I}}{=}\frac{1+3\pm(1+i)}{2}\Leftrightarrow$

$\Leftrightarrow \underline{z=i} \vee \underline{z=1+2i}.$

Anm I $\overset{\mathrm{I}}{=}$ utnyttjas anmärkningen i A.39.

Övning A.41 (Sid. 212)

Lösning

a) $z^3=i \Rightarrow |z^3|=|i| \Leftrightarrow |z|^3=1 \Leftrightarrow |z|=1;$

$\arg\{z^3\}=\arg\{i\} \Leftrightarrow 3\arg z=\frac{\pi}{2}+k\cdot 2\pi \Leftrightarrow \arg z=\frac{4k+1}{6}\pi;$

$z_{k+1}=\exp\{i\frac{4k+1}{6}\pi\}$, $k=0,1,2.$

$z_1=e^{i\pi/6}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i.$

$z_2=e^{i5\pi/6}=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i.$

$z_3=e^{i3\pi/2}=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i.$

b) $z^3=1+i \Rightarrow |z^3|=|1+i| \Leftrightarrow |z|^3=\sqrt{2} \Leftrightarrow |z|=\sqrt[6]{2};$

$\arg\{z^3\}=\arg\{1+i\} \Leftrightarrow 3\arg z=\frac{\pi}{4}+k\cdot 2\pi=\frac{8k+1}{4}\pi \Leftrightarrow$

$\Leftrightarrow \arg z=\frac{8k+1}{12}\pi \Rightarrow z_{k+1}=2^{1/6}\exp\{i\frac{8k+1}{12}\pi\}$, $k=0,1,2.$

$z_1=2^{1/6}e^{i\pi/12}$, $z_2=2^{1/6}e^{i3\pi/4}$, $z_3=2^{1/6}e^{i17\pi/12}.$

c) $z^4=16 \Leftrightarrow z^2=\pm 4 \Leftrightarrow z=\pm 2 \vee z=\pm 2i.$

$\underline{z_1=2,\ z_2=2i,\ z_3=-2,\ z_4=-2i.}$

d) $z^3=-1+i\sqrt{3}=2(-\frac{1}{2}+i\frac{\sqrt{3}}{2})\cdot 1=2e^{i2\pi/3}\cdot e^{i2k\pi}$, $k\in\mathbb{Z}$,

$\Leftrightarrow z_{k+1}=2^{1/3}\cdot e^{i2\pi/9}\cdot e^{i2k\pi/3}=2^{1/3}e^{i(2+6k)\pi/9}$, $k=0,1,2.$

$\underline{z_1=2^{1/3}e^{i2\pi/9}}$, $\underline{z_2=2^{1/3}e^{i8\pi/9}}$, $\underline{z_3=2^{1/3}e^{i14\pi/9}}.$

e) $z^5=4i=2^2e^{i\pi/2}\cdot e^{i2k\pi}=2^2\cdot e^{i(4k+1)\pi/2}$, $k\in\mathbb{Z}$, $\Leftrightarrow$

$\Leftrightarrow z_{k+1}=2^{2/5}e^{i(4k+1)\pi/10}$, $k=0,1,2.$

$\underline{z_1=2^{2/5}e^{i\pi/10}}$, $\underline{z_2=2^{2/5}e^{i\pi/2}}$, $\underline{z_3=2^{2/5}e^{i9\pi/10}}.$

f) $z^4=-1 \Leftrightarrow z^2=i \vee z^2=-i \Leftrightarrow z^2=\frac{(1+i)^2}{2} \vee z^2=\frac{(1-i)^2}{2}$

$\Leftrightarrow \underline{z=\pm\frac{1+i}{\sqrt{2}}} \vee \underline{z=\pm\frac{1-i}{\sqrt{2}}}.$ (Se Anm. i Ö. A.38-39).

Övning A.42 (Sid. 212)

Lösning

$z^6-2z^3+2=0 \Leftrightarrow z^3=1+i \vee z^3=1-i;$

(i) $z^3=1+i \Rightarrow$ Se Ö A.41 b).

(ii) $z^3=1-i \Rightarrow \bar{z}^3=1+i \Rightarrow$ de föregåendes konjugat.

Övning A.43 (Sid. 212)

Lösning

$(1+z^2)^3 = -8 = 8\cdot(-1)\cdot 1 = 2^3\cdot e^{i\pi}\cdot e^{i2k\pi} = 2^3 e^{i(2k+1)\pi} \Leftrightarrow$

$\Leftrightarrow\ 1+z^2 = 2e^{i(2k+1)\pi/3} \Leftrightarrow z^2 = -3 \vee z^2 = \pm\sqrt{3}i \Leftrightarrow$

$\Leftrightarrow\ z = \pm\sqrt{3}i \vee z^2 = \frac{\sqrt{3}}{2}(1\pm i)^2 \Leftrightarrow z = \pm\sqrt{3}i \vee z = \pm\sqrt[4]{\frac{3}{4}}(1\pm i)$

Svar: $z_1 = \sqrt{3}i,\ z_2 = -\sqrt{3}i,\ z_3 = \sqrt[4]{\frac{3}{4}}(1+i),\ z_4 = -\sqrt[4]{\frac{3}{4}}(1+i),$
$z_5 = \sqrt[4]{\frac{3}{4}}(1-i)$ och $z_6 = -\sqrt[4]{\frac{3}{4}}(1-i)$.

Övning A.44 (Sid. 212)

Lösning

$x-1$ faktor i $p(x)$ innebär att $p(1)=0$, enligt faktorsatsen; $p(1) = 1-2-19+a = 0 \Leftrightarrow a = 2$.

Divisionsalgoritmen ger

$$\begin{array}{l|l}
\quad x^2 - x - 20 & \\
\hline
x^3 - 2x^2 - 19x + 20 & \underline{x-1} \\
\underline{x^3 - x^2 + 0x + 0}\ (-) & \\
-x^2 - 19x + 20 & \\
\underline{-x^2 + 1\cdot x + 0}\ (-) & \\
-20x + 20 & \\
\underline{-20x + 20}\ (-) & \\
0 &
\end{array}$$

$x^2 - x - 20 = 0 \Leftrightarrow x = \frac{1}{2} \pm \frac{9}{2} \Leftrightarrow x = -4 \vee x = 5$.

Resultat: $a = 20$; $p(x) = (x-1)(x+4)(x-5)$.

Övning A.45 (Sid. 212)

Lösning

$p(z) = z^4 - 2z^3 + 2z^2 - 10z + 25$; $p(z) \in \mathbb{C}[z]$.

$z = 2+i$ rot $\Leftrightarrow z = 2+i$ nollställe till $p \Leftrightarrow z = 2-i$ också nollställe, enligt Sats 10 på sid. 465.

På samma sätt visas att $z = -1+2i$ är rot.

Svar: Ekvationen har rötterna $2\pm i$ och $-1\pm 2i$.

Övning A.46 (Sid. 212)

Lösning

$2-i$ nollställe $\Rightarrow 2+i$ nollstället. Det innebär att $z-2+i$ och $z-2-i$ är faktorer i $p(z)$, kalla det så. Efter hopmultiplikation av dessa två faktorer fås den kvadratiska faktorn z^2-4z+5.

Om $-i$ är dubbelt nollställe, så är även i det.

$(z+i)^2(z-i)^2 = ((z-i)(z+i))^2 = (z^2+1)^2 = z^4+2z^2+1$ är således faktor i $p(z)$ och vi får efter hopmultiplikationen av faktorerna z^2-4z+5 och z^4+2z^2+1 polynomet $p(z)=z^6-4z^5+7z^4-8z^3+11z^2-4z+5$.

Övning A.47 (Sid. 212)

Lösning

a) $z^2-7z+10=(z-2)(z-5)=0 \Leftrightarrow z=2 \vee z=5 \Rightarrow \begin{cases} z_1+z_2=7 \\ z_1\cdot z_2=10 \end{cases}$.

b) $3z^2-21z+30=3(z^2-7z+10)=0 \Leftrightarrow z^2-7z+10=0$ (se a)).

c) $z^3-7z^2+10z=z(z^2-7z+10)=0 \Leftrightarrow z=0 \vee z=2 \vee z=5 \Rightarrow$

$\Rightarrow z_1+z_2+z_3=7 \wedge z_1\cdot z_2\cdot z_3=0$.

Övning A.48 (Sid. 212)

Lösning

Kalla rötterna $z_1, z_2, \dots, z_7$; ekvationen har 7 rötter, enligt algebras fundamentalsats. Enligt faktorsatsen kan ekvationens VL skrivas som en produkt av 7 linjära faktorer $z-z_j$, $j=1,\dots,7$. Det innebär efter hopmultiplikation att

$$(z-z_1)(z-z_2)\cdots(z-z_7)=z^7-(z_1+z_2+\dots+z_7)z^6+\dots-z_1\cdot z_2\cdots z_7 \equiv$$

$$\equiv z^7+(3-i)z^6+\pi z^3+e=0 \Rightarrow \sum_{j=1}^{7} z_j=-3+i \wedge \prod_{j=1}^{7} z_j=-e.$$

Övning A.49 (Sid. 212)

Lösning (Se sidan 224 i övningsboken).

Övning A.50 (Sid. 213)

Lösning

x^2+2x+2 har nollställena $-1\pm i$. Faktorsatsen ger

$p(-1+i)=(-1+i)^4+2(-1+i)^3+3(-1+i)^2+a(-1+i)+2=\dots=$

$=-2i-a(1-i)+2=0 \Leftrightarrow a=2 \Rightarrow p(x)=x^4+2x^3+3x^2+2x+2$.

$$\begin{array}{l|l} \multicolumn{1}{c}{x^2 \qquad +1} & \\ \hline x^4+2x^3+3x^2+2x+2 & x^2+2x+2 \\ \underline{x^4+2x^3+2x^2+0x+0} \; (-) & \\ \qquad x^2+2x+2 & \\ \qquad \underline{x^2+2x+2} \; (-) & \\ \qquad\qquad 0 & \end{array}$$

$p(x)=(x^2+2x+2)(x^2+1)=0 \Leftrightarrow x=-1\pm i \vee x=\pm i$.

Övning A.51 (Sid. 213)

Lösning

$VL=p(z)=z^4-2z^3+12z^2-14z+35$; $z=1+ia$ rot.

$p(1+ia)=(1+ia)^4-2(1+ia)^3+12(1+ia)^2-14(1+ia)+35=$

$=a^4-6a^2+1+i(4a-4a^3)-2(1-3a^2+i(3a-a^3))+$

$+12(1-a^2+2ai)-14(1+ai)+35=a^4-6a^2+1+6a^2-2+12-$

$-12a^2-14+35+i(4a-4a^3+2a^3-6a+24a-14a) =$

$= a^4-12a^2+32+i(8a-2a^3) = 0 = 0+i\cdot 0 = HL \Leftrightarrow$

$\Leftrightarrow a^4-12a^2+32 = 0 = 8a-2a^3 \Leftrightarrow a = \pm 2 \Rightarrow \underline{1\pm 2i \text{ rötter}}$

$\Rightarrow (z-1-2i)(z-1+2i) = (z-1)^2-4 = z^2-2z+5$ faktor i VL.

$$\begin{array}{rl}
 z^2 \qquad\quad + 7 & \\
\hline
z^4-2z^3+12z^2-14z+35 & \boxed{z^2-2z+5} \\
(-)\ \underline{z^4-2z^3+5z^2+0z+0} & \\
7z^2-14z+35 & \\
(-)\ \underline{7z^2-14z+35} & \\
0 &
\end{array}$$

$P(z) = (z^2-2z+5)(z^2+7) = 0 \Leftrightarrow \underline{z = 1\pm 2i} \vee \underline{z = \pm\sqrt{7}i}.$

Övning A.52 (Sid. 213)

Lösning

a) $x^3-x = x(x^2-1) = \underline{x(x+1)(x-1)}.$

b) $x^3+x = \underline{x(x^2+1)}$; faktor x^2+1 irreducibelt i $\mathbb{R}[x]$.

c) $x^4-1 = (x^2)^2-1^2 = (x^2-1)(x^2+1) = \underline{(x-1)(x+1)(x^2+1)}.$

d) $x^4+1 = (x^4+2x^2+1)-2x^2 = (x^2+1)^2-(\sqrt{2}x)^2 =$ (konj/regel)

$= (x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x) = \underline{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}.$

Övning A.53 (Sid. 213)

$P(x) = x^5-x^4+4x-4 = x^4(x-1)+4(x-1) = (x-1)(x^4+4) =$

$= (x-1)((x^4+4x^2+4)-4x^2) = (x-1)((x^2+2)^2-(2x)^2) =$

$= (x-1)(x^2+2-2x)(x^2+2+2x) = \underline{(x-1)(x^2+2x+2)(x^2-2x+2)}.$

Övning A.54 (Sid. 213)

Lösning

$\underline{a^3+b^3 = (a+b)(a^2-ab+b^2)}$; $\underline{a^3-b^3 = (a-b)(a^2+ab+b^2)}.$

$P(x) = x^6-8 = (x^3)^2-(\sqrt{8})^2 = (x^3-\sqrt{8})(x^3+\sqrt{8}) = \dots =$

$= (x^3-(\sqrt{2})^3)(x^3+(\sqrt{2})^3) =$

$= (x-\sqrt{2})(x+\sqrt{2})(x^2+\sqrt{2}x+2)(x^2-\sqrt{2}x+2)$

Blandade problem

Övning A.55 (Sid. 213)

Lösnin

a) $(1+i\sqrt{3})^5 = \sum_{k=0}^{5}\binom{5}{k}(i\sqrt{3})^k = \binom{5}{0}\cdot(i\sqrt{3})^0+\binom{5}{1}(i\sqrt{3})^1+$

$+\binom{5}{2}(i\sqrt{3})^2+\binom{5}{3}(i\sqrt{3})^3+\binom{5}{4}(i\sqrt{3})^4+\binom{5}{5}(i\sqrt{3})^5 =$

$= 1+5(i\sqrt{3})+10(-3)+10(-i3\sqrt{3})+5\cdot 3^2+3^2 i\sqrt{3} =$

$= 1+i5\sqrt{3}-30-i30\sqrt{3}+45+i9\sqrt{3} = \underline{16-16\sqrt{3}i}.$

b) $1+i\sqrt{3} = 2e^{i\pi/3} \Rightarrow (1+i\sqrt{3})^5 = (2e^{i\pi/3})^5 = 2^5e^{i5\pi/3} =$

$= 32\cdot(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}) = 32\cdot(\frac{1}{2}-\frac{\sqrt{3}}{2}i) = \underline{16-16\sqrt{3}i}.$

Övning A.56 (Sid. 213)

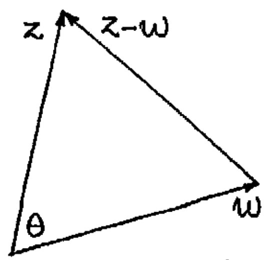
Lösning

$1+e^{i\pi/2}=1+i=|1+i|e^{i\arg\{1+i\}}=\underline{\sqrt{2}e^{i\pi/4}}$

Övning A.57 (Sid. 213)

Lösning

a) $\underline{z=5+14i}$, $\underline{w=2+3i}$; $\underline{z-w=3+11i}$



Cos-satsen ger $|z-w|^2=|z|^2+|w|^2-2\cdot|z|\cdot|w|\cos\theta \Rightarrow$

$\Rightarrow |3+11i|^2=|5+14i|^2+|2+3i|^2-2\,|5+14i|\cdot|2+3i|\cos\theta \Leftrightarrow$

$\Leftrightarrow 130=221+13-2\sqrt{221}\cdot\sqrt{13}\cos\theta \Leftrightarrow 2\cdot 13\sqrt{17}\cos\theta=104$

$\Leftrightarrow \cos\theta=\frac{104}{26\sqrt{17}}\approx 0{,}970 \Leftrightarrow \underline{\theta\approx 14{,}03^\circ}$.

b) $\arg\{5+14i\}+\frac{\pi}{4}=\arg\{(5+14i)\cdot t(1+i)\}\overset{t>0}{=}\arg\{-9+19i\}$

$\arg\{5+14i\}-\frac{\pi}{4}=\arg\{(5+14i)\cdot u(1-i)\}\overset{t>0}{=}\arg\{19+9i\}$.

Svar: $z=s\cdot(-9+19i)$, $s\in\mathbb{R}$ och $z=t(19+9i)$, $t\in\mathbb{R}$.

Det finns två räta "linjer" genom origo.

Övning A.58 (Sid. 213)

Lösning

$HL=2\,\frac{e^{i(x+y)/2}-e^{-i(x+y)/2}}{2i}\cdot\frac{e^{i(x-y)/2}+e^{-i(x-y)/2}}{2}=$

$=\frac{1}{2i}\left(e^{i(x+y)/2}\cdot e^{i(x-y)/2}-e^{-i(x+y)/2}\cdot e^{i(x-y)/2}+\right.$

$\left.+e^{i(x+y)/2}\cdot e^{-i(x-y)/2}-e^{-i(x+y)/2}\cdot e^{-i(x-y)/2}\right)=$

$=\frac{1}{2i}(e^{ix}-e^{-iy}+e^{iy}-e^{-ix})=\frac{1}{2i}(e^{ix}-e^{-ix}+e^{iy}-e^{-iy})=$

$=\frac{e^{ix}-e^{-ix}}{2i}+\frac{e^{iy}-e^{-iy}}{2i}=\sin x+\sin y=VL.$

Övning A.59 (Sid. 213)

Lösning

a) $(1+2i)\longmapsto e^{i\pi/2}(1+2i)=i(1+2i)=-2+i.\Leftrightarrow \underline{(1,2)\mapsto(-2,1)}.$

b) $(1+2i)\longmapsto e^{i\pi/4}(1+2i)=\frac{1}{\sqrt{2}}(1+i)(1+2i)=\frac{-1+3i}{\sqrt{2}}=\left(-\frac{1}{\sqrt{2}},\frac{3}{\sqrt{2}}\right).$

c) $(1+2i)\longmapsto e^{i\pi}(1+2i)=(-1)(1+2i)=-1-2i=(-1,-2).$

d) $(1+2i)\longmapsto e^{i\theta}(1+2i)=(\cos\theta-2\sin\theta,\ \sin\theta+2\cos\theta).$

Övning A.60 (Sid. 213)

Lösning

a) $(x,y)=x+iy\longmapsto(x+iy)e^{i\pi/2}=(x+iy)\cdot i=-y+ix=(-y,x).$

b) $(x,y)=x+iy\longmapsto(x+iy)e^{i\pi/4}=(x+iy)\cdot\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=$

$= \frac{1}{\sqrt{2}}(x-y+i(x+y)) = \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right).$

c) $(x,y) = x+iy \longmapsto (x+iy)e^{i\pi} = (x+iy)(-1) = -x-iy = (-x,-y).$

d) $(x,y) = x+iy \longmapsto (x+iy)\cdot e^{i\theta} = (x+iy)(\cos\theta + i\sin\theta) =$

$= x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta) =$

$= (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$

Övning A.61 (Sid. 214)

Lösning

$z^3 = \pm 7i \Leftrightarrow (z^3)^2 = (\pm 7i)^2 \Leftrightarrow z^6 = -49 \Leftrightarrow \underline{z^6 + 49 = 0.}$

Övning A.62 (Sid. 214)

Lösning

$$\left.\begin{aligned} VL &= \sum_{k=0}^{8}\binom{8}{k}(z^3)^k = 1 + 8z^3 + \dots + 8z^{21} + z^{24} \\ HL &= \sum_{k=0}^{6}\binom{6}{k}(z^4)^k = 1 + 6z^4 + \dots + 6z^{20} + z^{24} \end{aligned}\right\} \Rightarrow VL - HL =$$

$= 8z^{21} - 6z^{20} + \dots - 6z^4 + 8z^3 \Rightarrow \text{grad}((1+z^3)^8 - (1+z^4)^6) = 21.$

Övning A.63 (Sid. 214)

Lösning

$A\sin(\omega t + \delta) = \text{Im}\{Ae^{i(\omega t + \delta)}\} = \text{Im}\{Ae^{i\delta}e^{i\omega t}\}.$

b) $f(t) = 3\sin 100\pi t + \sqrt{3}\cos 100\pi t =$

$= \sqrt{3}(\sqrt{3}\sin 100\pi t + \cos 100\pi t) =$

$= 2\sqrt{3}(\sin 100\pi t \cos\frac{\pi}{6} + \cos 100\pi t \cdot \sin\frac{\pi}{6}) =$

$= 2\sqrt{3}\sin(100\pi t + \frac{\pi}{6}) = \text{Im}\{2\sqrt{3}e^{i\pi/6}e^{i100\pi t}\} =$

$= \text{Im}\{(3+i\sqrt{3})e^{i100\pi t}\} \Rightarrow \underline{E = 3 + i\sqrt{3}.}$

Övning A.64 (Sid. 214)

Lösning

$z^4 - 2z^3 - (5+i)z^2 + (6+i)z + 6i = 0.$

Eventuella reella rötter är delare till den konstanta termen; prövning visar att -2 och 3 är rötter, vilket innebär att VL är delbart med $z+2$ och $z-3$, dvs med $(z+2)(z-3) = z^2 - z - 6$.

$$\begin{array}{rl}
z^2 - z - i & \\
\hline
z^4 - 2z^3 - (5+i)z^2 + (6+i)z + 6i & \big|\ \underline{z^2 - z - 6} \\
(-)\ \underline{z^4 - z^3 - 6z^2} & \\
-z^3 + (1-i)z^2 + (6+i)z + 6i & \\
(-)\ \underline{-z^3 + z^2 + 6z} & \\
-iz^2 + iz + 6i & \\
(-)\ \underline{-iz^2 + iz + 6i} & \\
0 &
\end{array}$$

$z^2 - z - i = 0 \Leftrightarrow z = \frac{1}{2} \pm \sqrt{\frac{1}{4} + i} = \frac{1 \pm \sqrt{1+4i}}{2};$

$w = u+iv = \sqrt{1+4i} \Rightarrow (u+iv)^2 = u^2-v^2+i2uv = 1+4i$

$\Leftrightarrow \begin{cases} u^2-v^2=1 \\ u^2+v^2=\sqrt{17} \\ 2uv=4 \end{cases} \Leftrightarrow \begin{cases} u^2=(\sqrt{17}+1)/2 \\ v^2=(\sqrt{17}-1)/2 \\ 2uv=4 \end{cases} \Leftrightarrow \begin{cases} u=\sqrt{(\sqrt{17}+1)/2} \\ v=\sqrt{(\sqrt{17}-1)/2} \end{cases} \vee$

$\vee \begin{cases} u=-\sqrt{(\sqrt{17}+1)/2} \\ v=-\sqrt{(\sqrt{17}-1)/2} \end{cases} \Rightarrow z = \frac{1}{2}(1 \pm (\sqrt{(\sqrt{17}+1)/2} + i\sqrt{(\sqrt{17}-1)/2})$

Resultat: Ekvationen har de reella rötterna -2 och 3; de övriga rötterna ges här ovan.

Övning A.65 (Sid. 214)

Lösning

$z=2i$ rot $\Leftrightarrow z-2i$ faktor i VL. Divisionen ger

$$\begin{array}{rl}
z^2 - 3iz - 3 - i & \\
\hline
z^3-5iz^2-(9+i)z-2+6i & \big| \; z-2i \\
(-)\; \underline{z^3-2iz^2} & \\
-3iz^2-(9+i)z & \\
(-)\; \underline{-3iz^2 - 6z} & \\
-(3+i)z-2+6i & \\
(-)\; \underline{-(3+i)z-2+6i} & \\
0 &
\end{array}$$

De andra rötterna är lösningar till ekvationen $z^2-3iz-3-i=0$ och är $z=\frac{3i}{2} \pm \sqrt{-\frac{9}{4}+3+i} =$
$= \frac{3i}{2} \pm \frac{\sqrt{3+4i}}{2} = \frac{3i \pm \sqrt{(2+i)^2}}{2} = \frac{3i \pm (2+i)}{2} \Leftrightarrow \begin{cases} z_1 = 1+2i \\ z_2 = -1+i \end{cases}$

Övning A.66 (Sid. 214)

Lösning

$z^2+(2-i)z+3-i=0 \Leftrightarrow z=\frac{-2+i}{2} \pm \sqrt{\frac{3-4i}{4}-3+i} = -\frac{2-i}{2} \pm$
$\pm \frac{\sqrt{-9}}{2} = \frac{-2+i \pm 3i}{2} \Leftrightarrow \underline{z=-1+2i} \vee \underline{z=-1-i}.$

$p(z) = 2z^3+3z^2+2z-2 \Rightarrow p(-1-i) = 2(-1-i)^3+3(1+i)^2+$
$+2(-1-i)-2 = 4-4i+6i-2-2i-2=0 \Rightarrow z=-1-i$ rot $\Rightarrow$
$\Rightarrow z=-1+i$ också rot, ty koefficienterna reella

$2z^3+3z^2+2z-2=0 \Leftrightarrow z^3+\frac{3}{2}z^2+z-1=0$; rötternas produkt är lika med den konstanta termen med ombytt tecken; den tredje roten är reell, så $z_0 \cdot (-1-i)(-1+i) = +1 \Leftrightarrow 2z_0 = 1 \Leftrightarrow z_0 = \frac{1}{2}$.

Svar: Rötterna är $-1+2i, -1-i$ resp. $\frac{1}{2}, -1 \pm i$.

Övning A.67 (Sid. 214)

Lösning

a) Koefficienterna är reella, så även $1-i\sqrt{2}$ är rot; det ger 0,2 poäng.

b) Enligt faktorsatsen är vänsterledet delbart med både $z-1-i\sqrt{2}$ och $z-1+i\sqrt{2}$, dvs. med